EXPONENTIAL ATTRACTORS FOR MODIFIED SWIFT-HOHENBERG EQUATION IN $\mathbb{R}^N$

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Abstract. A Cauchy problem for a modification of the Swift-Hohenberg equation in $\mathbb{R}^N$ with a mildly integrable potential is considered. Existence of exponential attractors containing a finite dimensional global attractor in $H^2(\mathbb{R}^N)$ is shown under the dissipative mechanism of fourth order parabolic equations in unbounded domains.

1. Introduction

We consider the Cauchy problem for the fourth order parabolic equation

$$u_t + \Delta^2 u + \gamma\Delta u + \delta u = f(x,u),\ t>0,\ x\in\mathbb{R}^N, \tag{1.1}$$

together with the initial condition

$$u(0) = u_0, \tag{1.2}$$

where $\gamma,\delta \geqslant 0$ and f belongs to the class of functions of the form

$$f(x,s) = g(x) + m(x)s + f_0(x,s),\ x\in\mathbb{R}^N,\ s\in\mathbb{R}, \tag{1.3}$$

with mildly integrable potential $m\colon \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\|m\|_{L^r_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|m\|_{L^r(B(y,1))} < \infty \ \text{ for some } \ \max\left\{\tfrac{N}{4},1\right\} < r \leqslant \infty, \tag{1.4}$$

$$g \in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),$$

and $f_0\colon \mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ such that

$$f_0(x,0) = 0,\ x\in\mathbb{R}^N, \tag{1.5}$$

$$|f_0(x,s_1) - f_0(x,s_2)| \leqslant c_0|s_1-s_2|(1+|s_1|^{\rho-1}+|s_2|^{\rho-1}),\ x\in\mathbb{R}^N,\ s_1,s_2\in\mathbb{R}, \tag{1.6}$$

where c_0 is a certain positive constant and the exponent

$$\rho \geqslant 1 \text{ is arbitrarily large for } N\leqslant 4 \text{ and } 1\leqslant\rho\leqslant \tfrac{N}{N-4} \ \text{ for } \ N\geqslant 5. \tag{1.7}$$

Equation (1.1) is a modification of the Swift-Hohenberg equation

$$u_t + (q_0^2 I + \Delta)^2 u = \alpha u - u^3$$

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introduced to model the pattern formation of cells in the Rayleigh-Bénard convection (see [25, 16]). The long-time behavior of solutions to this equation, and to its various generalizations, was mostly investigated in the case when the problem is considered in a bounded domain under given boundary conditions, mainly of the Dirichlet type. For parabolic problems in bounded domains, including the Swift-Hohenberg type equations, the existence of a global attractor and its finite fractal dimensionality is thus well understood (see e.g. [20, 21, 23, 17, 14, 18]). The case of unbounded domains or $\mathbb{R}^N$ still constitutes a challenging task. Investigating the asymptotic behavior of solutions in this situation is much more difficult, mainly due to the lack of compactness of Sobolev embeddings. To avoid this hardship such problems are usually studied in weighted Sobolev spaces or a very weak type of an attractor is considered in locally uniform spaces or spaces of bounded and uniformly continuous functions. For the Swift-Hohenberg type equations this was done e.g. in [19, 13, 12].

For classical Sobolev spaces another approach can be taken in specific situations. A dissipative mechanism for nonlinear reaction-diffusion equations in unbounded domains exploiting exponentially decaying linear semigroup generated by $\Delta - V(x)I$ with weakly integrable Schrödinger potential V was introduced by Arrieta et al. in [1]. They showed that the interplay between diffusion and reaction terms guarantees existence of compact global attractors in Bessel potential spaces and thus in classical Sobolev spaces for a particular choice of parameters. Later Cholewa and Rodríguez-Bernal in [4, 5] formulated a similar dissipative mechanism for semilinear fourth order parabolic equations of the form

$$u_t + \Delta^2 u = f(x,u),\ x \in \mathbb{R}^N,\ t > 0, \tag{1.8}$$

to prove existence of global attractors in $H^2(\mathbb{R}^N)$ for semigroups generated by (1.8). This approach was also successfully applied to the modification of the Swift-Hohenberg equation of the form (1.1) in our paper [9].

We recall that the dissipativity mechanism of fourth order equations in unbounded domains for the problem (1.1)–(1.3) is based on a structure condition

$$sf(x,s) \leqslant C(x)s^2 + D(x)\,|s|\,,\ x \in \mathbb{R}^N,\ s \in \mathbb{R}, \tag{1.9}$$

with some functions $C, D\colon \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\|C\|_{L^r_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|C\|_{L^r(B(y,1))} < \infty \quad \text{for some} \quad \max\left\{\tfrac{N}{4}, 1\right\} < r \leqslant \infty, \tag{1.10}$$

$$0 \leqslant D \in L^q(\mathbb{R}^N) \text{ for some } \max\left\{\tfrac{2N}{N+4}, 1\right\} \leqslant q \leqslant 2 \quad (q > 1 \text{ if } N = 4),$$

and such that solutions of the linear problem

$$\begin{cases} w_t + \Delta^2 w = C(x)w,\ x \in \mathbb{R}^N,\ t > 0,\\ w(0) = w_0 \in L^2(\mathbb{R}^N) \end{cases}$$

decay exponentially as $t \to \infty$. The latter requirement is equivalent to the existence of $\omega_C > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\Delta\phi|^2 - C(x)\phi^2\right) \geqslant \omega_C \,\|\phi\|^2_{L^2(\mathbb{R}^N)} \quad \text{for} \quad \phi \in H^2(\mathbb{R}^N). \tag{1.11}$$

For simplification of further references, we introduce the following assumption.

Assumption 1. The function f satisfies conditions (1.3)-(1.7) and the dissipativity mechanism (1.9)-(1.11) holds.

The asymptotic behavior of the solutions to the problem (1.1)–(1.2) under Assumption 1 was investigated in [9] by treating it as an abstract Cauchy problem

$$u_t + \mathcal{A}u = \mathcal{F}(u) \tag{1.12}$$

in the sense of [15, 6] with the main linear operator $\mathcal{A}$ being, for a sufficiently large $\mu > 0$, a positive definite self-adjoint operator

$$\mathcal{A} = \Delta^2 - m(\cdot)I + \mu I \tag{1.13}$$

in $L^2(\mathbb{R}^N)$ with the domain $D(\mathcal{A})$ dense in $H^2(\mathbb{R}^N)$, whereas

$$\mathcal{F}(u)(x) = -\gamma\Delta u(x) + (\mu - \delta)u(x) + g(x) + f_0(x, u(x)),\ u \in H^2(\mathbb{R}^N),\ x \in \mathbb{R}^N.$$

The initial data for the abstract Cauchy problem was taken from $X^\alpha = D(\mathcal{A}^\alpha)$, $\alpha \in [\frac{1}{2}, 1)$, being the fractional power space corresponding to $\mathcal{A}$. Note that $X^{\frac{1}{2}}$ coincides up to the equivalence of norms with $H^2(\mathbb{R}^N)$ endowed with the equivalent norm $\|\phi\|^2_{H^2(\mathbb{R}^N)} = \|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} + \|\phi\|^2_{L^2(\mathbb{R}^N)}$.

In order to precisely recall the result on the existence of a regular global attractor from [9, Theorem 1.1, Theorem 5.4], we let

$$\nu_C = \frac{\omega_C}{2(\omega_C + \zeta_C) + 1}, \tag{1.14}$$

where ω_C comes from (1.11) and $\zeta_C > 0$ is such that

$$\int_{\mathbb{R}^N} C(x)\phi^2 \leqslant \frac{1}{4}\|\phi\|^2_{H^2(\mathbb{R}^N)} + \zeta_C \|\phi\|^2_{L^2(\mathbb{R}^N)},\ \phi \in H^2(\mathbb{R}^N), \tag{1.15}$$

holds, see [4, (2.24)]. Note that the constant $\nu_C > 0$ depends exclusively on the properties of $C(\cdot)$ from the structure condition (1.9).

Theorem 1.1. *Let Assumption 1 hold. If*

$$0 \leqslant \gamma \leqslant \sqrt{\nu_C \delta}, \tag{1.16}$$

then the Cauchy problem (1.1), (1.2) *defines a* C^0 *semigroup* $\{S(t)\colon t \geqslant 0\}$ *of global solutions in* X^α, $\alpha \in [\frac{1}{2}, 1)$, *possessing a compact global attractor* $\mathbf{A}$ *in* X^α. *More precisely, we have*

$$\mathbf{A} = W^u(\mathcal{E}),$$

where $W^u(\mathcal{E})$ *denotes the unstable manifold of the set of stationary solutions of* (1.1). *Moreover,* $\mathbf{A}$ *is bounded in* $L^\infty(\mathbb{R}^N)$ *and is contained in a positively invariant bounded absorbing set* $\mathbf{B}$ *from* $X^\alpha \cap L^\infty(\mathbb{R}^N)$.

The question of finite fractal dimension of this global attractor requires a special attention, since our problem is set in an infite-dimensional space of functions defined in $\mathbb{R}^N$ for which the compactness of Sobolev embeddings is lacking. In [2], based on the quasi-stability method developed by Lasiecka and Chueshov ([8, 7]), it was proved that the global attractor for the problem (1.8) has a finite fractal dimension and is contained in an exponential attractor. This modern approach to show finite fractal dimension of invariant sets is more flexible than squeezing property or smoothing property methods used before e.g. in [11] or [3] for problems in bounded domains. Moreover, the quasi-stability of a semigroup on a positively invariant absorbing set (cp. Definition 2.4 and (1.21), (1.22) below) is a crucial step to prove existence of an exponential attractor, which not only exponentially attracts initial data from bounded sets, but is also more robust under perturbations of the system.

The aim of this note is to further study the semigroup from Theorem 1.1 in the case of $\alpha = \frac{1}{2}$ in order to show that the global attractor $\mathbf{A}$ has finite fractal dimension in $X^{\frac{1}{2}} = H^2(\mathbb{R}^N)$. Moreover, following the ideas of [2], we will embed $\mathbf{A}$ into an exponential attractor of a finite fractal dimension, which attracts all bounded subsets of $X^{\frac{1}{2}}$ uniformly exponentially fast. To this end, we strengthen the conditions of Theorem 1.1 introducing the following assumption.

Assumption 2. Let Assumption 1 hold and let the function f_0 in (1.3), besides (1.5) and (1.6), satisfy

$$\frac{\partial f_0}{\partial s}(x,s) \to \frac{\partial f_0}{\partial s}(x,0) = 0 \ \text{ as } s \to 0 \ \text{ uniformly for } \ x \in \mathbb{R}^N. \tag{1.17}$$

Moreover, let the function D in (1.9) satisfy one of the conditions:

$$D \in L^p(\mathbb{R}^N) \ \text{ for some } \ p \geqslant \max\left\{\tfrac{N}{4}, 1\right\} \quad (p > 1 \text{ if } N = 4) \tag{1.18}$$

or

$$D(x) \to 0 \quad \text{ as } \quad |x| \to \infty. \tag{1.19}$$

The estimate of the fractal dimension of the global attractor $\mathbf{A}$ and the exponential attractors will be expessed in terms of the maximal cardinality of distinguishable subsets of the closed unit ball in the space

$$Z(T_1,T_2) = C^1([T_1,T_2]; L^2(\mathbb{R}^N)) \cap C([T_1,T_2]; H^2(\mathbb{R}^N))$$

equipped with the norm $\|\cdot\|_{Z(T_1,T_2)} = \|\cdot\|_{H^1(T_1,T_2;L^2(\mathbb{R}^N))} + \|\cdot\|_{L^2(T_1,T_2;H^2(\mathbb{R}^N))}$ with respect to the compact seminorm

$$\mathfrak{n}_{Z(T_1,T_2),R,\mu}(z) = \mu \, \|z|_{B_R}\|_{L^2(T_1,T_2;L^{2r'}(B_R))}\,, \ \ z \in Z(T_1,T_2), \tag{1.20}$$

where r' denotes Hölder's conjugate to r from (1.4), $B_R = \{x \in \mathbb{R}^N \colon \ |x| < R\}$ and positive constants R, μ and $T_2 > T_1$ are given. For the proof of the compactness of the seminorm $\mathfrak{n}_{Z(T_1,T_2),R,\mu}$, we refer the reader to [2, Lemma 13.8].

Then the main theorem of this paper is the following.

Theorem 1.2. *Let Assumption 2 and* (1.16) *be satisfied and let* $\{S(t)\colon t \geqslant 0\}$ *be the semigroup of global* $X^{\frac{1}{2}}$*-solutions of* (1.1)-(1.2) *possessing the global attractor* $\mathbf{A}$ *contained in the absorbing set* $\mathbf{B}$ *from Theorem 1.1. Then*

(i) There exist positive constants $R^*, T^* > 0$, $T > T^*$, $\eta_T \in (0,1)$, $\kappa_T, \mu_T > 0$ *such that the semigroup* $\{S(t)\colon t \geqslant 0\}$ *is quasi-stable on* $\mathbf{B}$ *with respect to the compact seminorm* $\mathfrak{n}_{Z(T^*,T)} = \mathfrak{n}_{Z(T^*,T),R^*,\mu_T}$ *defined in* (1.20) *and parameters* (η_T, κ_T), *that is, for any* $u_0, v_0 \in \mathbf{B}$ *we have*

$$\|S(\cdot)u_0 - S(\cdot)v_0\|_{Z(T^*,T)} \leqslant \kappa_T \|u_0 - v_0\|_{H^2(\mathbb{R}^N)}, \tag{1.21}$$

and

$$\|S(T)u_0 - S(T)v_0\|_{H^2(\mathbb{R}^N)} \leqslant \eta_T \|u_0 - v_0\|_{H^2(\mathbb{R}^N)} + \mathfrak{n}_{Z(T^*,T)}(S(\cdot)u_0 - S(\cdot)v_0), \quad (1.22)$$

(ii) For any $\sigma \in (0, 1-\eta_T)$ there exist a $T-$weak exponential attractor $\mathbf{M_0} \subset \mathbf{B}$ and an exponential attractor $\mathbf{M} \subset \mathbf{B}$ for the semigroup (see Definition 2.1), both with rate of attraction $\xi \in (0, \frac{1}{T}\ln\frac{1}{\eta_T+\sigma})$, containing the global attractor $\mathbf{A}$,

(iii) The fractal dimensions of $\mathbf{A}$, $\mathbf{M_0}$ and $\mathbf{M}$ are estimated by

$$\dim_f^{H^2(\mathbb{R}^N)}(\mathbf{A}) \leqslant \dim_f^{H^2(\mathbb{R}^N)}(\mathbf{M_0}) \leqslant \log_{\frac{1}{\eta_T+\sigma}} \mathfrak{m}_{Z(T^*,T)}\left(2\kappa_T\sigma^{-1}\right)$$

and

$$\dim_f^{H^2(\mathbb{R}^N)}(\mathbf{A}) \leqslant \dim_f^{H^2(\mathbb{R}^N)}(\mathbf{M}) \leqslant 1 + \log_{\frac{1}{\eta_T+\sigma}} \mathfrak{m}_{Z(T^*,T)}\left(2\kappa_T\sigma^{-1}\right), \quad (1.23)$$

where $\mathfrak{m}_{Z(T^*,T)}(2\kappa_T\sigma^{-1})$ is the maximal number of z_j in

$$\overline{B}^{Z(T^*,T)}(0,1) = \{z \in Z(T^*,T)\colon \|z\|_{Z(T^*,T)} \leqslant 1\}$$

having the property that $\mathfrak{n}_{Z(T^*,T)}(z_j - z_l) \geqslant \frac{\sigma}{2\kappa_T}$ for $j \neq l$.

The paper is organized as follows. In Section 2 we formulate the existence results concerning exponential attractors, which emphasize the role of quasi-stable semigroups among dissipative semigroups possessing finite-dimensional attractors. In Section 3, using the Assumption 2, we prove that the semigroup generated by the modified Swift-Hohenberg equation (1.1) is quasi-stable on the absorbing set $\mathbf{B}$ from Theorem 1.1. Furthermore, we show that the semigroup is Hölder continuous with respect to time, uniformly on the set $\mathbf{B}$. Hence the abstract results from Section 2 apply and yield Theorem 1.2.

2. Exponential attractors via quasi-stability

In this section we review the abstract results concerning the existence of exponential attractors. Here and subsequently, given a subset G of a metric space V and $\varepsilon > 0$, by $N^V(G,\varepsilon)$ we denote the minimal number of open $\varepsilon-$balls centered at points from G necessary to cover the set G. Moreover, $\Lambda^V(G)$ stands for the $\omega-$limit set of G, that is, $\Lambda^V(G) = \bigcap_{s\geqslant 0} \mathrm{cl}_V \bigcup_{t\geqslant s} S(t)G$.

Let us first recall the notion of an exponential attractor and its weaker counterpart for a semigroup $\{S(t)\colon t \geqslant 0\}$ on a metric space V, that is, a family of maps $S(t)\colon V \to V$, $t \geqslant 0$, such that $S(t)S(s) = S(t+s)$, $t,s \geqslant 0$, with $S(0)$ being an identity map on V.

Definition 2.1. An *exponential attractor* for a semigroup $\{S(t)\colon t \geqslant 0\}$ on a metric space (V,d) is a nonempty compact set $\mathbf{M} \subset V$ such that

(i) $\mathbf{M}$ is positively invariant under the semigroup, i.e., $S(t)\mathbf{M} \subset \mathbf{M}$ for $t \geqslant 0$,

(ii) the fractal dimension of $\mathbf{M}$ in V is finite with a given bound $\chi \geqslant 0$, i.e.,

$$\dim_f^V(\mathbf{M}) = \limsup_{\varepsilon\to 0^+} \log_{\frac{1}{\varepsilon}} N^V(\mathbf{M},\varepsilon) \leqslant \chi < \infty,$$

(iii) there is $\xi > 0$ such that for every bounded subset G of V we have

$$\lim_{t\to\infty} e^{\xi t}\operatorname{dist}^V(S(t)G,\mathbf{M}) = \lim_{t\to\infty} e^{\xi t}\sup_{x\in G}\inf_{y\in\mathbf{M}} d(S(t)x,y) = 0.$$

If instead of (i) we consider a weaker requirement that

(i') there exists a positive number $T > 0$ such that $S(T)\mathbf{M} \subset \mathbf{M}$,

then we call $\mathbf{M}$ a $T-$*weak exponential attractor* for the semigroup.

Before we give an equivalent condition for the existence of a weak exponential attractor for a semigroup on a complete metric space, based on [22, Theorem 2.1], we recall the notion of an asymptotically closed semigroup.

Definition 2.2. A semigroup $\{S(t)\colon t \geqslant 0\}$ on a metric space (V,d) is called *asymptotically closed* if for any $t \geqslant 0$, $t_k \geqslant 0$, $t_k \to \infty$ and any bounded sequence $x_k \in V$ the following implication holds:

$$\text{if } S(t_k)x_k \to x \text{ and } S(t+t_k)x_k \to y \text{ with } x,y \in V, \text{ then } S(t)x = y.$$

Theorem 2.3. *Let $\{S(t)\colon t \geqslant 0\}$ be an asymptotically closed semigroup on a complete metric space (V,d) and let $T > 0$. Then, the following statements are equivalent:*

(1) There exists a $T-$weak exponential attractor $\mathbf{M_0}$ in V for the semigroup.

(2) There exists a nonempty bounded (optionally also positively invariant) absorbing set $\mathbf{B} \subset V$ for the semigroup such that

$$N^V(S(kT)\mathbf{B}, aq^k) \leqslant bh^k,\ k \in \mathbb{N},\ k \geqslant k_0, \tag{2.1}$$

holds for some $k_0 \in \mathbb{N}$, $a,b > 0$, $q \in (0,1)$ and $h \geqslant 1$.

Moreover, if (2) holds, then

$$\mathbf{M_0} = \mathbf{A} \cup \mathbf{E_0} = \mathrm{cl}_V\,\mathbf{E_0} \subset \mathbf{B},$$

with $\mathbf{E_0}$ being a certain countable subset of $\mathbf{B}$ and $\mathbf{A} = \Lambda^V(\mathrm{cl}_V\,\mathbf{E_0})$ being the global attractor for the semigroup, is a $T-$weak exponential attractor with rate of attraction $\xi \in (0, \frac{1}{T}\ln\frac{1}{q})$, and its fractal dimension is estimated by

$$\dim_f^V(\mathbf{M_0}) \leqslant \log_{\frac{1}{q}} h.$$

The most versatile method to verify (2.1) is based on the quasi-stability of a semigroup, which was introduced by I. Chueshov in [7, Definition 3.4.1] (see also [8]).

Definition 2.4. We say that a semigroup $\{S(t)\colon t \geqslant 0\}$ on a metric space (V,d) is *quasi-stable on a set $B \subset V$ at positive time $T > 0$ with respect to a compact seminorm $\mathfrak{n}_Z$ and parameters (η,κ)* if there exist constants $\eta \in [0,1)$, $\kappa > 0$ and a map $K\colon B \to Z$ into some auxiliary normed space Z such that

$$\|Kx - Ky\|_Z \leqslant \kappa d(x,y),\ x,y \in B, \tag{2.2}$$

$$d(S(T)x, S(T)y) \leqslant \eta d(x,y) + \mathfrak{n}_Z(Kx - Ky),\ x,y \in B, \tag{2.3}$$

hold, where $\mathfrak{n}_Z\colon Z \to [0,\infty)$ is some compact seminorm on Z, which means that for any bounded sequence $z_k \in Z$ there exists a Cauchy subsequence z_{k_j} with respect to $\mathfrak{n}_Z$, that is, $\mathfrak{n}_Z(z_{k_j} - z_{k_l}) \to 0$ as $j,l \to \infty$.

Following the lines of the proof of [7, Theorem 3.1.21], we formulate the result, which shows that the quasi-stability of a semigroup on a positively invariant bounded absorbing set $\mathbf{B}$ implies the covering condition of the form (2.1).

Theorem 2.5. *Let $\{S(t)\colon t \geqslant 0\}$ be a semigroup on a metric space (V,d), $T > 0$ and let $\mathbf{B}$ be a positively invariant bounded absorbing set for the semigroup. If the semigroup is quasi-stable on $\mathbf{B}$ at time T with respect to a compact seminorm $\mathfrak{n}_Z$ and parameters (η,κ), then there exists $a > 0$ such that for any $\sigma \in (0, 1-\eta)$*

$$N^V(S(kT)\mathbf{B}, a(\eta+\sigma)^k) \leqslant \left(\mathfrak{m}_Z\left(2\kappa\sigma^{-1}\right)\right)^k, \ k \in \mathbb{N}.$$

Thus (2.1) is satisfied with $q = \eta + \sigma$ and $h = \mathfrak{m}_Z(2\kappa\sigma^{-1})$, where $\mathfrak{m}_Z(2\kappa\sigma^{-1})$ is the maximal number of elements z_j in $\overline{B}^Z(0,1) = \{z \in Z\colon \|z\|_Z \leqslant 1\}$ having the property that $\mathfrak{n}_Z(z_j - z_l) \geqslant \frac{\sigma}{2\kappa}$ for $j \neq l$.

Combining the above result with Theorem 2.3, we get the following corollary.

Corollary 2.6. *Let $\{S(t)\colon t \geqslant 0\}$ be an asymptotically closed semigroup on a complete metric space (V,d), $T > 0$ and let $\mathbf{B}$ be a bounded absorbing set for the semigroup. If the semigroup is quasi-stable on $\mathbf{B}$ at time T with respect to a compact seminorm $\mathfrak{n}_Z$ and parameters (η,κ), then for any $\sigma \in (0,1-\eta)$ there exists a $T-$weak exponential attractor $\mathbf{M_0} \subset \mathbf{B}$ in V for the semigroup with rate of attraction $\xi \in (0, \frac{1}{T}\ln\frac{1}{\eta+\sigma})$, and its fractal dimension is estimated by*

$$\dim_f^V(\mathbf{M_0}) \leqslant \log_{\frac{1}{\eta+\sigma}} \mathfrak{m}_Z\left(2\kappa\sigma^{-1}\right).$$

Moreover, the semigroup has a global attractor $\mathbf{A}$ contained in $\mathbf{M_0}$.

Assuming further that the semigroup is Hölder continuous in time, uniformly on $\mathbf{B}$, one can show that the global attractor $\mathbf{A}$ is in fact contained in an exponential attractor $\mathbf{M}$ (cp. [24, Theorem 3.4]).

Theorem 2.7. *Let $\{S(t)\colon t \geqslant 0\}$ be an asymptotically closed semigroup on a complete metric space (V,d), $T > 0$ and let $\mathbf{B}$ be a bounded absorbing set for the semigroup. If the semigroup is quasi-stable on $\mathbf{B}$ at time T with respect to a compact seminorm $\mathfrak{n}_Z$ and parameters (η,κ) and there exist $T_2 > T_1 \geqslant 0$, $\zeta > 0$ and $\nu \in (0,1]$ such that*

$$d(S(t_1)x, S(t_2)x) \leqslant \zeta\,|t_1 - t_2|^\nu, \ t_1, t_2 \in [T_1, T_2], \ x \in \mathbf{B},$$

then for any $\sigma \in (0,1-\eta)$ there exists an exponential attractor $\mathbf{M} \subset \mathbf{B}$ in V (independent of ν, ζ, T_2) for the semigroup with rate of attraction $\xi \in (0, \frac{1}{T}\ln\frac{1}{\eta+\sigma})$, and its fractal dimension is estimated by

$$\dim_f^V(\mathbf{M}) \leqslant \tfrac{1}{\nu} + \log_{\frac{1}{\eta+\sigma}} \mathfrak{m}_Z\left(2\kappa\sigma^{-1}\right).$$

We also have $\mathbf{M} = \mathbf{A} \cup \mathbf{E} = \mathrm{cl}_V\,\mathbf{E} \subset \mathbf{B}$, where $\mathbf{E}$ is a certain subset of $\mathbf{B}$.

For more details of the proofs of results in this section we refer the reader to [10].

3. Existence of exponential attractors

In this section we show that the global attractor $\mathbf{A}$ for the semigroup $\{S(t)\colon t \geqslant 0\}$ of global $X^{\frac{1}{2}}$ solutions to (1.12), $S(t)u_0 = u(t)$, $t \geqslant 0$, is contained in a finite dimensional exponential attractor. Recall from [9] that the $X^{\frac{1}{2}}$ solutions of the abstract Cauchy problem (1.12) satisfy the Duhamel formula

$$u(t) = e^{-\mathcal{A}t}u_0 + \int_0^t e^{-\mathcal{A}(t-s)}\mathcal{F}(u(s))ds, \ t \geqslant 0 \tag{3.1}$$

and

$$\left\|e^{-\mathcal{A}t}\right\|_{\mathcal{L}(H^{4\beta_1}(\mathbb{R}^N),H^{4\beta_2}(\mathbb{R}^N))} \leqslant M\frac{e^{-\omega t}}{t^{\beta_2-\beta_1}}, \ t > 0, \ -\beta^* \leqslant \beta_1 \leqslant \beta_2 \leqslant \beta^* \tag{3.2}$$

with

$$\beta^* = 1 + \min\left\{\left(\tfrac{N}{8} - \tfrac{N}{4r}\right), 0\right\} \in \left(\tfrac{1}{2}, 1\right].$$

In what follows, given $R > 0$, we denote

$$B_R = \{x \in \mathbb{R}^N : |x| < R\} \quad \text{and} \quad B_R^c = \mathbb{R}^N \setminus B_R,$$

whereas $\mathbf{B}$ is the absorbing set from Theorem 1.1. Since by Theorem 1.1 the values of elements of $\mathbf{B}$ belong to some interval $I_{\mathbf{B}} \subset \mathbb{R}$, note that if we let $u, v \in I_{\mathbf{B}}$ and use (1.6), then

$$|f_0(x,u) - f_0(x,v)| \leqslant L_{\mathbf{B}}\,|u-v| \quad \text{for} \ \ x \in \mathbb{R}^N, \ u, v \in I_{\mathbf{B}}, \tag{3.3}$$

where $L_{\mathbf{B}}$ is a positive constant depending only on $I_{\mathbf{B}}$ and c_0, ρ from (1.6).

Lemma 3.1. *Under Assumption 1 for arbitrarily fixed $q \in [2,\infty)$ and each $\varepsilon > 0$ there exist certain $t_\varepsilon > 0$ and $R_\varepsilon > 0$ such that*

$$\|S(t)u_0\|_{L^q(B^c_{R_\varepsilon})} < \varepsilon \ \ \textit{for any} \ \ t \geqslant t_\varepsilon, \ u_0 \in \mathbf{B}. \tag{3.4}$$

Proof. For $q = 2$ it was proved in [9, Lemma 4.1], whereas for $q > 2$ it follows from [9, Proposition 5.3], since there exists $R_{\mathbf{B}} > 0$ such that

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^N)} \leqslant R_{\mathbf{B}}, \ t \geqslant \tau, \ u_0 \in \mathbf{B}$$

for any $\tau > 0$ small enough. □

Lemma 3.2. *Let Assumption 1 hold. If $u_0, v_0 \in \mathbf{B}$ and $u = S(\cdot)u_0$, $v = S(\cdot)v_0$, then $U = u - v$ satisfies*

$$\sup_{t\in[0,T]} \|U(t)\|_{H^2(\mathbb{R}^N)} \leqslant c_T\,\|U(0)\|_{H^2(\mathbb{R}^N)}\,, \ T \geqslant 0, \tag{3.5}$$

$$\sup_{t\in[0,T]} t^{\frac{1}{2}}\,\|U(t)\|_{H^2(\mathbb{R}^N)} \leqslant c_T\,\|U(0)\|_{L^2(\mathbb{R}^N)}\,, \ T \geqslant 0, \tag{3.6}$$

for some positive constant c_T. In particular, we have

$$\|U\|_{L^2(0,T;H^2(\mathbb{R}^N))} \leqslant c_T T^{\frac{1}{2}}\,\|U(0)\|_{H^2(\mathbb{R}^N)}\,, \ T > 0. \tag{3.7}$$

Proof. Given $u_0, v_0 \in \mathbf{B}$ and using Duhamel's formula (3.1) we have

$$U(t) = e^{-\mathcal{A}t}U(0) + \int_0^t e^{-\mathcal{A}(t-s)}\big[-\gamma\Delta U(s) + (\mu-\delta)U(s) + f_0(\cdot,u) - f_0(\cdot,v)\big]ds,\ t \geqslant 0. \tag{3.8}$$

From (3.3) we infer that

$$\|f_0(\cdot,u) - f_0(\cdot,v)\|_{L^2(\mathbb{R}^N)} \leqslant L_{\mathbf{B}}\,\|U(t)\|_{L^2(\mathbb{R}^N)} \leqslant L_{\mathbf{B}}\,\|U(t)\|_{H^2(\mathbb{R}^N)}\,, \tag{3.9}$$

whereas (3.2) yields in particular with some $M_T > 0$

$$\begin{aligned} &\left\|e^{-\mathcal{A}t}\right\|_{\mathcal{L}(H^2(\mathbb{R}^N))} \leqslant M_T,\ t \in [0,T],\\ &\left\|e^{-\mathcal{A}t}\right\|_{\mathcal{L}(L^2(\mathbb{R}^N),H^2(\mathbb{R}^N))} \leqslant M_T t^{-\frac{1}{2}},\ t \in (0,T]. \end{aligned} \tag{3.10}$$

Joining (3.8)-(3.10), we obtain with $\widetilde{L}_{\mathbf{B}} = \gamma + \mu + \delta + L_{\mathbf{B}}$

$$\|U(t)\|_{H^2(\mathbb{R}^N)} \leqslant M_T\,\|U(0)\|_{H^2(\mathbb{R}^N)} + M_T\widetilde{L}_{\mathbf{B}}\int_0^t (t-s)^{-\frac{1}{2}}\,\|U(s)\|_{H^2(\mathbb{R}^N)}\,ds,\ t \in (0,T],$$

$$\|U(t)\|_{H^2(\mathbb{R}^N)} \leqslant M_T t^{-\frac{1}{2}}\,\|U(0)\|_{L^2(\mathbb{R}^N)} + M_T\widetilde{L}_{\mathbf{B}}\int_0^t (t-s)^{-\frac{1}{2}}\,\|U(s)\|_{H^2(\mathbb{R}^N)}\,ds,\ t \in (0,T],$$

and the claims follow applying the Volterra type inequality [6, Lemma 1.2.9]. □

If $f_0\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ in (1.3) satisfies additionally (1.17), then the structure condition (1.9) and the growth condition (1.6) have the following two implications (for their proofs, see [2, Lemmas 13.3 and 13.4]).

Lemma 3.3. *Under Assumption 1 and* (1.17) *for each* $\varepsilon > 0$ *there exist* $\alpha_\varepsilon > 0$ *and* $c_\varepsilon > 0$ *such that*

$$m(x) - C(x) \leqslant \varepsilon + \alpha_\varepsilon D(x) \quad \text{for} \quad x \in \mathbb{R}^N, \tag{3.11}$$

$$|f_0(x,u) - f_0(x,v)| \leqslant \big(\varepsilon + c_\varepsilon(|u|^{\rho-1} + |v|^{\rho-1})\big)\,|u-v|,\ u,v \in \mathbb{R},\ x \in \mathbb{R}^N \tag{3.12}$$

with $\rho \geqslant 1$ *as in* (1.7).

For the purpose of showing that the semigroup is quasi-stable, we will further impose on D in (1.9) one of the conditions: (1.18) or (1.19).

Lemma 3.4. *Let Assumption 2 hold. Then there exist constants* $a, b, R^*, T^* > 0$ *such that for any* $u_0, v_0 \in \mathbf{B}$ *the function* $U(t) = S(t)u_0 - S(t)v_0$, $t \geqslant 0$, *satisfies the estimate*

$$\|U(t)\|_{L^2(\mathbb{R}^N)} \leqslant e^{-a(t-T^*)}\,\|U(T^*)\|_{L^2(\mathbb{R}^N)} + b\,\|U\|_{L^2(T^*,t;L^{2r'}(B_{R^*}))}\,,\ t \geqslant T^*, \tag{3.13}$$

where r' *denotes the Hölder conjugate to* r *from* (1.10).

Proof. Setting $u = S(\cdot)u_0$, $v = S(\cdot)v_0$ for $u_0, v_0 \in \mathbf{B}$ and $U = u - v$, we have

$$U_t + \Delta^2 U + \gamma\Delta U + \delta U = m(x)U + f_0(x,u) - f_0(x,v).$$

Multiplying the above equation by U in $L^2(\mathbb{R}^N)$ we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\,\|U\|^2_{L^2(\mathbb{R}^N)} &+ \int_{\mathbb{R}^N}(|\Delta U|^2 - C(x)U^2)\,dx - \int_{\mathbb{R}^N}(m(x) - C(x))U^2\,dx\\ &- \gamma\|\nabla U\|^2_{L^2(\mathbb{R}^N)} + \delta\,\|U\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N}(f_0(x,u) - f_0(x,v))U\,dx. \end{aligned}$$

Using the Cauchy inequality with $\nu_C \in \left(0, \frac{1}{2}\right)$ defined in (1.14) we have

$$\gamma \|\nabla U\|^2_{L^2(\mathbb{R}^N)} \leqslant \gamma \|\Delta U\|_{L^2(\mathbb{R}^N)} \|U\|_{L^2(\mathbb{R}^N)} \leqslant \frac{\nu_C}{4} \|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \frac{\gamma^2}{\nu_C} \|U\|^2_{L^2(\mathbb{R}^N)}$$

and thus (1.11), (1.14) and (1.15) imply

$$\begin{aligned}\int_{\mathbb{R}^N} (|\Delta U|^2 - C(x)U^2)\, dx - \gamma \|\nabla U\|^2_{L^2(\mathbb{R}^N)} &\geqslant +(1-2\nu_C) \int_{\mathbb{R}^N} (|\Delta U|^2 - C(x)U^2)\, dx \\ &+ \frac{5\nu_C}{4} \|\Delta U\|^2_{L^2(\mathbb{R}^N)} - \left(2\nu_C \zeta_C + \frac{\nu_C}{2} + \frac{\gamma^2}{\nu_C}\right) \|U\|^2_{L^2(\mathbb{R}^N)} \\ &\geqslant \frac{5\nu_C}{4} \|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \left(\frac{\nu_C}{2} - \frac{\gamma^2}{\nu_C}\right) \|U\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Consequently, applying (1.16) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \|U\|^2_{L^2(\mathbb{R}^N)} + \frac{5\nu_C}{4} \|\Delta U\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (m(x) - C(x))U^2\, dx + \frac{\nu_C}{2} \|U\|^2_{L^2(\mathbb{R}^N)} \\ \leqslant \int_{\mathbb{R}^N} (f_0(x,u) - f_0(x,v))U\, dx.\end{aligned} \tag{3.14}$$

We first consider the case when D satisfies (1.18). Splitting the first integral on B_R and B_R^c for arbitrary $R > 0$ and using (3.11) we get with some $\alpha_C > 0$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \|U\|^2 + \frac{5\nu_C}{4} \|\Delta U\|^2 - \int_{B_R} (m(x) - C(x))U^2\, dx - \alpha_C \int_{B_R^c} D(x)U^2\, dx \\ + \frac{\nu_C}{4} \|U\|^2 \leqslant \int_{\mathbb{R}^N} (f_0(x,u) - f_0(x,v))U\, dx.\end{aligned} \tag{3.15}$$

Note that

$$\int_{B_R^c} D(x)U^2\, dx \leqslant \|D\|_{L^p(B_R^c)} \|U\|^2_{L^{2p'}(\mathbb{R}^N)} \leqslant b_p \|D\|_{L^p(B_R^c)} (\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \|U\|^2_{L^2(\mathbb{R}^N)}),$$

where the constant $b_p > 0$ is such that

$$\|U\|^2_{L^{2p'}(\mathbb{R}^N)} \leqslant b_p(\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \|U\|^2_{L^2(\mathbb{R}^N)})$$

with p as in (1.18) and p' denoting its Hölder conjugate. Moreover, due to (1.4) and (1.10), there exists a certain positive constant k_R such that

$$\int_{B_R} (m(x) - C(x))U^2\, dx \leqslant \|m - C\|_{L^r(B_R)} \|U\|^2_{L^{2r'}(B_R)} \leqslant k_R \|U\|^2_{L^{2r'}(B_R)}.$$

Combining these estimates with (3.15), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \|U\|^2_{L^2(\mathbb{R}^N)} + \left(\frac{\nu_C}{4} - \alpha_C b_p \|D\|_{L^p(B_R^c)}\right)(\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \|U\|^2_{L^2(\mathbb{R}^N)}) \\ - k_R \|U\|^2_{L^{2r'}(B_R)} \leqslant \int_{\mathbb{R}^N} (f_0(x,u) - f_0(x,v))U\, dx.\end{aligned} \tag{3.16}$$

Due to (3.3) and the embedding $L^{2r'}(B_R) \hookrightarrow L^2(B_R)$ we have

$$\int_{B_R} (f_0(x,u) - f_0(x,v))U\,dx \leqslant L_{\mathbf{B}} \|U\|^2_{L^2(B_R)} \leqslant L_{\mathbf{B}} b_{r,R} \|U\|^2_{L^{2r'}(B_R)}$$

with some $b_{r,R} > 0$. By (3.12) there exists $c_* > 0$ such that for $\rho > 1$ and $q > \max\{\frac{N}{4}, \frac{2}{\rho-1}, 1\}$ we have

$$\int_{B_R^c} (f_0(x,u) - f_0(x,v))U\,dx \leqslant \int_{B_R^c} \left(\frac{\nu_C}{16} + c_*(|u|^{\rho-1} + |v|^{\rho-1})\right) U^2$$

$$\leqslant \frac{\nu_C}{16} \|U\|^2_{L^2(B_R^c)} + c_* \left(\|u\|^{\rho-1}_{L^{q(\rho-1)}(B_R^c)} + \|v\|^{\rho-1}_{L^{q(\rho-1)}(B_R^c)}\right) \|U\|^2_{L^{2q'}(\mathbb{R}^N)}$$

$$\leqslant \frac{\nu_C}{16} \|U\|^2_{L^2(B_R^c)} + c_* b_q \left(\|u\|^{\rho-1}_{L^{q(\rho-1)}(B_R^c)} + \|v\|^{\rho-1}_{L^{q(\rho-1)}(B_R^c)}\right) (\|\Delta U\|^2_{L^2(\mathbb{R}^N)} + \|U\|^2_{L^2(\mathbb{R}^N)}).$$

Due to (1.18) and (3.4) we choose $R = R^* > 0$ and $T^* \geqslant 0$ such that

$$\alpha_C b_p \|D\|_{L^p(B_{R^*}^c)} + c_* b_q \left(\|u\|^{\rho-1}_{L^{q(\rho-1)}(B_{R^*}^c)} + \|v\|^{\rho-1}_{L^{q(\rho-1)}(B_{R^*}^c)}\right) < \frac{\nu_C}{16} \quad \text{for} \quad t \geqslant T^*.$$

From (3.16) and the above estimates we obtain

$$\frac{1}{2}\frac{d}{dt} \|U\|^2_{L^2(\mathbb{R}^N)} + \frac{\nu_C}{8} \|U\|^2_{L^2(\mathbb{R}^N)} \leqslant (k_{R^*} + L_{\mathbf{B}} b_{r,R^*}) \|U\|^2_{L^{2r'}(B_{R^*})}, \ t \geqslant T^*. \tag{3.17}$$

Note that it is easy to prove an analogous estimate for $\rho = 1$. Moreover, if D satisfies (1.19) instead of (1.18), then instead of directly applying (3.11), we use in (3.14) its consequence

$$m(x) - C(x) \leqslant \frac{\nu_C}{4} \quad \text{for} \quad |x| \geqslant R$$

with a sufficiently large R, to get from (3.14)

$$\frac{1}{2}\frac{d}{dt} \|U\|^2 + \frac{5\nu_C}{4} \|\Delta U\|^2 - \int_{B_R} (m(x) - C(x))U^2\,dx + \frac{\nu_C}{4} \|U\|^2$$
$$\leqslant \int_{\mathbb{R}^N} (f_0(x,u) - f_0(x,v))U\,dx.$$

Since the other arguments of the previous part of the proof carry over to this case as if α_C would be equal to 0, we obtain (3.17) again. Application of the Gronwall inequality to (3.17) leads to (3.13) with $a = \frac{\sqrt{\nu_C}}{2}$ and $b = \sqrt{2(k_{R^*} + L_{\mathbf{B}} b_{r,R^*})}$. □

Lemma 3.5. *Let Assumption 2 hold. For every $\tau > 0$ there exist positive constants $a, b, R^*, T^*, c_\tau, c_{T^*}$ such that for any $u_0, v_0 \in \mathbf{B}$ the function $U(t) = S(t)u_0 - S(t)v_0$, $t \geqslant 0$, satisfies for all $t \geqslant T^*$ the estimate*

$$\|U(t+\tau)\|_{H^2(\mathbb{R}^N)} \leqslant \frac{c_\tau c_{T^*}}{\tau^{\frac{1}{2}}} e^{-a(t-T^*)} \|U(0)\|_{H^2(\mathbb{R}^N)} + \frac{b c_\tau}{\tau^{\frac{1}{2}}} \|U\|_{L^2(T^*,t;L^{2r'}(B_{R^*}))}, \tag{3.18}$$

where r' denote Hölder conjugate to r from (1.10).

Proof. Since $\mathbf{B}$ is positively invariant and $u_0, v_0 \in \mathbf{B}$, we have $S(t)u_0, S(t)v_0 \in \mathbf{B}$ for every $t \geqslant 0$. Using the semigroup property and (3.6) we see that for $\tau > 0$ there exists a positive constant c_τ such that

$$\|U(t+\tau)\|_{H^2(\mathbb{R}^N)} = \|S(\tau)U(t)\|_{H^2(\mathbb{R}^N)} \leqslant c_\tau \tau^{-\frac{1}{2}} \|U(t)\|_{L^2(\mathbb{R}^N)}, \ t \geqslant 0.$$

Next it follows from Lemma 3.4 that there are positive constants a, b, T^*, R^* such that for $t \geqslant T^*$

$$\|U(t+\tau)\|_{H^2(\mathbb{R}^N)} \leqslant c_\tau \tau^{-\frac{1}{2}} \left(e^{-a(t-T^*)} \|U(T^*)\|_{L^2(\mathbb{R}^N)} + b\, \|U\|_{L^2(T^*,t;L^{2r'}(B_{R^*}))} \right).$$

Combining the above estimate with (3.5), due to imbedding $H^2(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, we obtain (3.18). □

Note that applying Lemma 3.5 with $\tau = 1$ for all $t \geqslant T^*$, we get

$$\|U(t+1)\|_{H^2(\mathbb{R}^N)} \leqslant c_1 c_{T^*} e^{-a(t-T^*)} \|U(0)\|_{H^2(\mathbb{R}^N)} + bc_1 \|U\|_{L^2(T^*,t;L^{2r'}(B_{R^*}))}.$$

Taking t so large that $c_1 c_{T^*} e^{-a(t-T^*)} < 1$, we obtain the estimate (1.22), i.e., for $u_0, v_0 \in \mathbf{B}$ we have

$$\|S(T)u_0 - S(T)v_0\|_{H^2(\mathbb{R}^N)} \leqslant \eta_T \|u_0 - v_0\|_{H^2(\mathbb{R}^N)} + \mu_T \|S(\cdot)u_0 - S(\cdot)v_0\|_{L^2(T^*,T;L^{2r'}(B_R))},$$

with $T = t+1$, $\eta_T = c_1 c_{T^*} e^{-a(T-T^*-1)} < 1$ and $\mu_T = bc_1$.

Additionally, the estimate (1.21) in Theorem 1.2 is a direct consequence of the following lemma.

Lemma 3.6. *Let Assumption 2 be satisfied,* $u_0, v_0 \in \mathbf{B}$ *and* $U(t) = S(t)u_0 - S(t)v_0$. *Then*

$$\|U\|_{H^1(0,T;L^2(\mathbb{R}^N))} + \|U\|_{L^2(0,T;H^2(\mathbb{R}^N))} \leqslant \kappa_T \|U(0)\|_{H^2(\mathbb{R}^N)}, \ T > 0, \tag{3.19}$$

for some positive constant κ_T.

Proof. Setting $u = S(\cdot)u_0$, $v = S(\cdot)v_0$ for $u_0, v_0 \in \mathbf{B}$ and $U = u - v$, we have

$$U_t + \mathcal{A}U = (\mu - \delta)U - \gamma \Delta U + f_0(x,u) - f_0(x,v), \tag{3.20}$$

where $\mathcal{A}$ is the positive definite self-adjoint operator in $L^2(\mathbb{R}^N)$ defined in (1.13). Multiplying the above equation by $\mathcal{A}U$ in $L^2(\mathbb{R}^N)$ and using the Cauchy inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\mathcal{A}^{\frac{1}{2}}U\right\|^2 + \frac{1}{2}\|\mathcal{A}U\|^2 \leqslant (\mu-\delta)\left\|\mathcal{A}^{\frac{1}{2}}U\right\|^2 + \gamma^2\|\Delta U\|^2 + \|f_0(x,u) - f_0(x,v)\|^2.$$

Since $D(\mathcal{A}^{\frac{1}{2}}) = H^2(\mathbb{R}^N)$ (see [4, Corollary 2.7]), due to (3.9) and (3.5), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\mathcal{A}^{\frac{1}{2}}U\right\|^2 + \frac{1}{2}\|\mathcal{A}U\|^2 &\leqslant (\mu-\delta)\left\|\mathcal{A}^{\frac{1}{2}}U\right\|^2 + \gamma^2\|\Delta U\|^2 + L_{\mathbf{B}}^2\|U\|_{H^2(\mathbb{R}^N)}^2 \\ &\leqslant c_T^2(L_{\mathbf{B}}^2 + \gamma^2 + c\mu)\|U(0)\|_{H^2(\mathbb{R}^N)}^2, \ t \in (0,T). \end{aligned}$$

Integrating over $(0,T)$ we obtain with some $k_T > 0$

$$\|\mathcal{A}U\|_{L^2(0,T;L^2(\mathbb{R}^N))} \leqslant k_T \|U(0)\|_{H^2(\mathbb{R}^N)}.$$

Applying this estimate and Lemma 3.2 to (3.20) yields

$$\begin{aligned} \|U_t\|_{L^2(0,T;L^2(\mathbb{R}^N))} &\leqslant \|\mathcal{A}U\|_{L^2(0,T;L^2(\mathbb{R}^N))} + (\mu+\delta+\gamma+L_{\mathbf{B}})\|U\|_{L^2(0,T;H^2(\mathbb{R}^N))} \\ &\leqslant (k_T + T^{\frac{1}{2}}c_T(\mu+\delta+\gamma+L_{\mathbf{B}}))\|U(0)\|_{H^2(\mathbb{R}^N)}, \end{aligned}$$

which together with (3.7) gives (3.19). □

Before we give a proof of Theorem 1.2 we need one more property of the semigroup.

Lemma 3.7. *Let $u_0 \in \mathbf{B}$. For every $\nu \in (0,1)$ and $T > 0$ there exists a positive constant $\zeta_{T,\nu}$ such that the function $S(\cdot)u_0$ satisfies the following local Hölder condition with exponent ν*

$$\|S(t_1)u_0 - S(t_2)u_0)\|_{H^2(\mathbb{R}^N)} \leqslant \zeta_{T,\nu}|t_1 - t_2|^\nu, \ t_1, t_2 \in [T, 2T]. \tag{3.21}$$

Proof. Since f_0 satisfies the Lipschitz condition (3.3) and $S(\cdot)u_0$ the integral equation (3.1), using (3.2) and reasoning as in the proof of [6, (2.2.3)], we obtain the desired conclusion. □

Proof of Theorem 1.2. Having established (1.21) and (1.22) from part (i) as consequences of Lemmas 3.5 and 3.6, we conclude that the semigroup is quasi-stable on $\mathbf{B}$ in the sense of Definition 2.4. Note that estimates (2.2) and (2.3) are satisfied with the map $K\colon \mathbf{B} \to Z(T^*, T)$ defined as $K(u_0) = S(\cdot)u_0$.

Since $\{S(t)\colon t \geqslant 0\}$, as a C^0 semigroup, is asymptotically closed on $H^2(\mathbb{R}^N)$ and is Hölder continuous in time by Lemma 3.7, the parts (ii) and (iii) are direct consequences of Corollary 2.6 and Theorem 2.7. Because $\mathbf{M}$ does not depend on $\nu \in (0,1)$, which was arbitrary in (3.21), the estimate (1.23) follows by passing to the limit with $\nu \to 1$. □

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