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Dissipative mechanism and global attractor for modified Swift–Hohenberg equation in $\mathbb{R}^N$

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Abstract: A Cauchy problem for a modification of the Swift–Hohenberg equation in $\mathbb{R}^N$ with a mildly integrable potential is considered. Applying the dissipative mechanism of fourth order parabolic equations in unbounded domains, it is shown that the equation generates a semigroup of global solutions possessing a global attractor in the scale of Bessel potential spaces and in $H^2(\mathbb{R}^N)$ in particular.

Key words: Initial value problems for higher order parabolic equations, Swift–Hohenberg equation, semilinear parabolic equations, dissipative mechanism, global attractor

1. Introduction

Dissipative mechanism for nonlinear reaction-diffusion equations in unbounded domains exploiting exponentially decaying linear semigroup generated by $\Delta - V(x)I$ with weakly integrable Schrödinger potential V was introduced by Arrieta et al. in [2]. They showed that the interplay between diffusion and reaction terms guarantees existence of compact global attractors in Bessel potential spaces and thus in classical Sobolev spaces for a particular choice of parameters. These compact sets are used to describe long-time dynamics of dissipative infinite-dimensional dynamical systems appearing in models from mathematical physics (see, e.g., [3, 5, 13, 25, 28]).

Later Cholewa and Rodríguez-Bernal in [6, 7] formulated a similar dissipative mechanism for semilinear fourth order parabolic equations of the form

$$u_t + \Delta^2 u = f(x,u),\ x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

to prove existence of global attractors in $H^2(\mathbb{R}^N)$ for semigroups generated by (1.1).

In [4], based on the quasistability method, it was further proved that under slightly strengthened assumptions these attractors have finite fractal dimension and are contained in exponential attractors. In the case of fourth order problems, the dissipative mechanism is based on a structure condition

$$sf(x,s) \leqslant C(x)s^2 + D(x)\,|s|\,,\ x \in \mathbb{R}^N,\ s \in \mathbb{R}, \tag{1.2}$$

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with some functions $C, D\colon \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\|C\|_{L^r_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|C\|_{L^r(B(y,1))} < \infty \quad \text{for some } r \text{ satisfying } \max\left\{\tfrac{N}{4}, 1\right\} < r \leqslant \infty, \tag{1.3}$$

with $B(y,1) \subset \mathbb{R}^N$ denoting a unit ball centered at y,

$$0 \leqslant D \in L^q(\mathbb{R}^N) \text{ for some } q \text{ satisfying } \max\left\{\tfrac{2N}{N+4}, 1\right\} \leqslant q \leqslant 2 \quad (q > 1 \text{ if } N = 4), \tag{1.4}$$

and such that solutions of the linear problem

$$\begin{cases} w_t + \Delta^2 w = C(x)w, \ x \in \mathbb{R}^N, \ t > 0, \\ w(0) = w_0 \in L^2(\mathbb{R}^N) \end{cases} \tag{1.5}$$

decay exponentially as $t \to \infty$. Another application of this mechanism appeared in [8] in the context of suitable perturbations of the Cahn–Hilliard equation in $\mathbb{R}^N$, which define asymptotically compact semigroups in $H^1(\mathbb{R}^N)$ and in consequence admit global attractors in this space.

In this paper, following the above approach, we study the long-time behavior of solutions in Bessel potential spaces (in particular in the standard Sobolev space $H^2(\mathbb{R}^N)$) to the Cauchy problem for the fourth order parabolic equation

$$u_t + \Delta^2 u + \gamma\Delta u + \delta u = f(x,u), \ t > 0, \tag{1.6}$$

subject to the initial condition

$$u(0) = u_0, \tag{1.7}$$

where γ, δ are nonnegative and x varies in the whole $\mathbb{R}^N$.

Such a form of perturbation of the main linear operator is motivated by the Swift–Hohenberg equation ([15, 27]) and its generalizations like

$$u_t + \left(q_0^2 I + \Delta\right)^2 u + h(u) = g(x)$$

with h being a cubic polynomial in the original model from 1977 by Swift and Hohenberg.

The Swift–Hohenberg type equations play a significant role in models with pattern formation (see, e.g., [9, 22] and the references therein). Having been introduced in connection with the investigation of Rayleigh–Bénard convection cells, they also appear in a variety of other problems such as wavelength selection in cellular flows [24] or the study of large aspect ratio lasers [19].

As concerns the long-time behavior of solutions in terms of attractors, the Swift–Hohenberg equation and its modifications were mainly considered in bounded domains, see, e.g., [12, 17, 18, 21, 23, 26], under the boundary conditions

$$u = \Delta u = 0 \text{ on } \partial\Omega. \tag{1.8}$$

Polat, in [23], showed that the equation

$$u_t + (I + \Delta)^2 u + u^3 + \alpha u^2 - \kappa u + b|\nabla u|^2 = 0, \tag{1.9}$$

with $\alpha = 0$ and $\kappa, b \in \mathbb{R}$, considered in a bounded domain Ω in $\mathbb{R}^2$, generates a semigroup of mild solutions and has a global attractor in $H_0^2(\Omega)$. Song et al. generalized this result in [26] showing that Polat's problem has a global attractor in any Sobolev space $H^k(\Omega)$. In [12], Giorgini considered a modification of the Swift–Hohenberg equation in bounded $\Omega \subset \mathbb{R}^3$

$$u_t + \Delta^2 u + 2\Delta u + h(u) = 0,\ x \in \Omega,\ t > 0,$$

with the nonlinearity $h \in C^2(\mathbb{R})$ satisfying for some positive δ, K

$$h(s)s \geqslant (1+\delta)s^2 - K, \quad h(0) = 0.$$

Under these assumptions, the existence of a global attractor in $L^2(\Omega)$, bounded in $H^4(\Omega)$, and of an exponential attractor in a certain subspace of $L^2(\Omega)$ was established in [12]. Next, Giorgini's problem was studied in [18] with $h \in C^1(\mathbb{R})$ satisfying

$$\liminf_{|s|\to\infty}(h(s)s - s^2) > 0,$$

where Khanmamedov proved the well-posedness of the problem and uniform global boundedness of solutions w.r.t. the initial data. Moreover, he showed that the semigroup has the global attractor in $H^2(\Omega) \cap H_0^1(\Omega)$, which is a bounded subset of $H^5(\Omega)$.

The problem (1.6)–(1.8) in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \leqslant 7$, was also considered in [17] with positive parameters γ, δ and the nonlinear term $f(x,u) = -h(u)$ with $h \in C^1(\mathbb{R})$ satisfying $h(0) = 0$ and certain growth conditions. It was proved there that the semigroup generated by (1.6)–(1.8) possesses a global attractor, which coincides with the unstable manifold of the set of stationary solutions in $H^2(\Omega) \cap H_0^1(\Omega)$ provided that

$$0 < \gamma < \min\left\{\mu_1^D, 2\sqrt{\delta}\right\}, \tag{1.10}$$

where μ_1^D denotes the smallest positive eigenvalue of $-\Delta$ on Ω with the Dirichlet boundary condition.

Due to the lack of compactness of Sobolev embeddings the Swift–Hohenberg type equations in unbounded domains were investigated in locally uniform spaces and weighted Sobolev spaces. On the one hand, it allowed to maintain in the considered space interesting stationary solutions of these equations (consult, e.g., [22]); on the other hand, it significantly weakened the notion of attractor or made it dependent on the particular choice of class of weights. The problem of existence of such weak attractors for Swift–Hohenberg type equations in unbounded domains was addressed, e.g., in [10, 11, 16, 20]. In [20], Mielke and Schneider proved the existence of the global attractor for a modified Swift–Hohenberg equation on the whole real line in a weighted Sobolev space $H_\rho^1(\mathbb{R})$. Precisely, they showed that the equation

$$u_t + (I + \partial_{xx})^2 u = \kappa u - \lambda u^3 + \beta u u_x,$$

with $\kappa, \lambda > 0$ and $\beta \in \mathbb{R}$, defines a semigroup on $H_{l,u}^1(\mathbb{R})$, that is the subspace of $H_\rho^1(\mathbb{R})$ consisting of translation continuous elements, with a global $(H_{l,u}^1(\mathbb{R}), H_\rho^1(\mathbb{R}))$-attractor, which is translation invariant and attracts bounded subsets of $H_{l,u}^1(\mathbb{R})$ w.r.t. the metric of H_ρ^1. Ion in [16] studied the two-dimensional equation (1.9) with $b = 0$ in bounded as well as unbounded domains, showing the existence of a global $(L^2(\mathbb{R}^2), L_\rho^2(\mathbb{R}^2))$-attractor for a certain class of weights ρ. Efendiev and Peletier in [10, 11] considered

$$u_t + \Delta^2 u + \gamma\Delta u + h(u) = g(x) \ \text{ in } \mathbb{R}^3$$

with $g \in L^2_U(\mathbb{R}^3)$ and the nonlinearity $h \in C^2(\mathbb{R})$ such that

$$h(s)s \geqslant -c_1 + c_2|s|^{2+\epsilon} \quad \text{for} \quad |s| \gg 1 \qquad \text{and} \qquad |h(s)s| \leqslant c_3|s| \quad \text{for} \quad |s| \ll 1,$$

$$h'(s) \geqslant -c_4 \quad \text{for } s \in \mathbb{R},$$

with positive constants c_1, c_2, c_3, c_4 and ϵ. Under these conditions, the semigroup of global solutions was generated in $W^{4,2}_U(\mathbb{R}^3)$, where

$$W^{l,p}_U(\mathbb{R}^N) = \{u \in \mathcal{D}'(\mathbb{R}^N)\colon \|u\|_{W^{l,p}_U(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \|u\|_{W^{l,p}(B(y,1))} < \infty\},$$

but its attractor, called there locally compact attractor, was shown to be bounded in $W^{4,2}_U(\mathbb{R}^3)$, compact only in $W^{4,2}_{loc}(\mathbb{R}^3)$ and attracting in the local topology of $W^{4,2}_{loc}(\mathbb{R}^3)$. The upper bound for Kolmogorov's $\varepsilon-$entropy of this object in the space $W^{4,2}(B(y,R))$ was provided and it was observed that in some situations such an attractor can have infinite fractal dimension.

Note that the modification (1.6) of the Swift–Hohenberg equation considered here contains the term $\gamma\Delta u$, which counteracts the dissipation of energy for $\gamma > 0$. We can expect that (1.6) will have nicer properties if we make the term $\gamma\Delta u$ subordinate to the terms $\Delta^2 u$ and δu. Therefore, we will show that in general the equation (1.6) generates a dissipative semigroup (e.g., in $H^2(\mathbb{R}^N)$) only in a certain region of (γ, δ) parameters

$$0 \leqslant \gamma \leqslant \sqrt{\nu_C \delta}, \tag{1.11}$$

where $\nu_C > 0$ depends exclusively on the properties of $C(x)$ from the structure condition (1.2) (cp. (3.7)). Observe that (1.11) is similar in vein to the restriction (1.10) for the bounded domain case.

In our considerations, the right-hand side of (1.6) takes the form

$$f(x,s) = g(x) + m(x)s + f_0(x,s),\ x \in \mathbb{R}^N,\ s \in \mathbb{R}, \tag{1.12}$$

with mildly integrable potential $m\colon \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\|m\|_{L^r_U(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \|m\|_{L^r(B(y,1))} < \infty \quad \text{for some } r \text{ as in (1.3)}, \tag{1.13}$$

$$g \in L^2(\mathbb{R}^N), \tag{1.14}$$

and $f_0\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that

$$f_0(x,0) = 0,\ x \in \mathbb{R}^N, \tag{1.15}$$

$$|f_0(x,s_1) - f_0(x,s_2)| \leqslant c_0|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}),\ x \in \mathbb{R}^N,\ s_1, s_2 \in \mathbb{R}, \tag{1.16}$$

where c_0 is a certain positive constant and the exponent

$$\rho \geqslant 1 \text{ is arbitrarily large for } N \leqslant 4 \text{ and } 1 \leqslant \rho \leqslant \tfrac{N}{N-4} \quad \text{for} \quad N \geqslant 5. \tag{1.17}$$

In fact, except for the result of Section 5, we can strengthen (1.17) to

$$\rho \geqslant 1 \text{ is arbitrarily large for } N \leqslant 4 \text{ and } 1 \leqslant \rho < \tfrac{N+4}{N-4} \quad \text{for} \quad N \geqslant 5. \tag{1.18}$$

If $\rho \in (\frac{N}{N-4}, \frac{N+4}{N-4})$ then we limit ourselves only to $\alpha = \frac{1}{2}$ below and instead of standard $X^{\frac{1}{2}} = H^2(\mathbb{R}^N)$ solutions we consider $\epsilon-$regular mild solutions in the sense of [1] (see Remark 2.4).

The main result of the paper is the following.

Theorem 1.1 *Let f be in the class of functions satisfying (1.12)–(1.16) and (1.17) or even (1.18) for $\alpha = \frac{1}{2}$ only and assume the dissipativity mechanism (1.2), (1.3), (1.4) with the equation (1.5) having exponentially decaying solutions. If the condition (1.11) holds, then the Cauchy problem (1.6), (1.7) with u_0 from the fractional power space X^α, $\alpha \in [\frac{1}{2}, 1)$, corresponding to the operator $\mathcal{A}$ given in (2.5), defines a C^0 semigroup $\{S(t) \colon t \geqslant 0\}$ of global solutions possessing a compact global attractor $\mathbf{A}$ in X^α, which coincides with $H^2(\mathbb{R}^N)$ for $\alpha = \frac{1}{2}$. More precisely, we have*

$$\mathbf{A} = W^u(\mathcal{E}),$$

where $W^u(\mathcal{E})$ denotes the unstable manifold of the set of stationary solutions of (1.6). Moreover, if $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and ρ satisfies (1.17), then $\mathbf{A}$ is bounded in $L^\infty(\mathbb{R}^N)$ and is contained in a positively invariant bounded absorbing set from $L^\infty(\mathbb{R}^N)$.

The article is organized as follows. In Section 2, we give a short summary of properties of the operator $\Delta^2 - V(x)I$ with a mildly integrable potential V and justify existence of unique local X^α, $\alpha \in [\frac{1}{2}, 1)$, solutions of the abstract Cauchy problem corresponding to (1.6) within the considered class of nonlinearities. In Section 3, we use the above-described dissipative mechanism to show boundedness of X^α solutions, which, therefore, generate a C^0 semigroup on X^α. Moreover, we also provide examples of nonlinear terms to which the mechanism applies. Next, in Section 4, we show that the semigroup is asymptotically compact and prove the existence of the global attractor from Theorem 1.1 using the existing Lyapunov function. Finally, we discuss in Section 5 regularity properties of the constructed global attractor and showing its boundedness in $L^\infty(\mathbb{R}^N)$, we complete the proof of Theorem 1.1.

2. Local solvability of the problem

We consider the Cauchy problem (1.6), (1.7) for a modified Swift–Hohenberg equation in $\mathbb{R}^N$, that is

$$\begin{cases} u_t + \Delta^2 u + \gamma \Delta u + \delta u = f(x, u), \; x \in \mathbb{R}^N, \; t > 0, \\ u(0) = u_0, \end{cases}$$

where $\gamma, \delta \geqslant 0$ and

$$f(x, s) = g(x) + m(x)s + f_0(x, s), \; x \in \mathbb{R}^N, \; s \in \mathbb{R},$$

belongs to the class of functions satisfying (1.13), (1.14), (1.15) and the growth condition (1.16) with (1.17), formulated in the Introduction.

Remark 2.1 *A direct consequence of (1.15) and (1.16) is the following growth estimate*

$$|f_0(x, s)| \leqslant c_0(|s| + |s|^\rho), \; x \in \mathbb{R}^N, \; s \in \mathbb{R} \;\; \textit{with } c_0, \rho \textit{ as in } (1.16). \tag{2.1}$$

The local existence of solutions, but most of all the dissipativity of the problem, will be based on the regularity of the potential m and the function C belonging to the common space $L^r_U(\mathbb{R}^N)$ (see (1.13) and (1.3)).

Properties of the operator $A_V = \Delta^2 - V(x)I$ with $V\colon \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\|V\|_{L^r_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|V\|_{L^r(B(y,1))} < \infty \text{ for some } r \text{ satisfying } \max\left\{\tfrac{N}{4}, 1\right\} < r \leqslant \infty, \tag{2.2}$$

were investigated in [6] from where we collect below the relevant facts. For simplicity, we denote the inner product in $X = L^2(\mathbb{R}^N)$ by $\langle \varphi, \psi\rangle$ and the corresponding norm by $\|\varphi\|$ for $\varphi, \psi \in L^2(\mathbb{R}^N)$.

Proposition 2.2 *(cf. [6]) Under the assumption (2.2),*

(i) the operator A_V is a densely defined self-adjoint operator in $L^2(\mathbb{R}^N)$ with its domain $D(A_V)$ dense in $H^2(\mathbb{R}^N)$; for $r \geqslant 2$ we have $D(A_V) = H^4(\mathbb{R}^N)$,

(ii) the operator A_V is bounded below and there exists $\omega_V \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \left(|\Delta\varphi|^2 - V(x)\varphi^2\right) \geqslant \omega_V \|\varphi\|^2, \ \varphi \in H^2(\mathbb{R}^N), \tag{2.3}$$

(iii) if E^β denotes the extrapolated fractional power scale associated to A_V in $L^2(\mathbb{R}^N)$, then

$$E^\beta = \begin{cases} H^{4\beta}(\mathbb{R}^N), & \beta \in [0, \beta^*], \\ (H^{4\beta}(\mathbb{R}^N))' = H^{-4\beta}(\mathbb{R}^N), & \beta \in [-\beta^*, 0), \end{cases}$$

where $\beta^* = 1 + \left(\frac{N}{8} - \frac{N}{4r}\right)_- \in (\frac{1}{2}, 1]$ and $x_- = \min\{x, 0\}$ for $x \in \mathbb{R}$,

(iv) the analytic semigroup $\{e^{-A_V t}\colon t \geqslant 0\}$ in $L^2(\mathbb{R}^N)$ decays exponentially if and only if (2.3) holds with a certain $\omega_V > 0$,

(v) (see [6, (2.24)]) for any $\varepsilon_0 > 0$ there exists $\xi_V(\varepsilon_0) > 0$ such that

$$\left|\int_{\mathbb{R}^N} V(x)\varphi^2\right| \leqslant \varepsilon_0 \|\varphi\|^2_{H^2(\mathbb{R}^N)} + \xi_V(\varepsilon_0) \|\varphi\|^2, \ \varphi \in H^2(\mathbb{R}^N). \tag{2.4}$$

If additionally ω_V in (2.3) is positive, then

(a) $A_V^{\frac{1}{2}}$ is a positive definite self-adjoint operator in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$,

(b) norms $\left\|A_V^{\frac{1}{2}}\varphi\right\|$ and $\|\varphi\|_{H^2(\mathbb{R}^N)}$ are equivalent and

$$\langle A_V^{\frac{1}{2}}\varphi, A_V^{\frac{1}{2}}\psi\rangle = \langle A_V\varphi, \psi\rangle, \ \varphi \in D(A_V), \ \psi \in D(A_V^{\frac{1}{2}}).$$

For the potential $m\colon \mathbb{R}^N \to \mathbb{R}$ satisfying (1.13) we choose a constant $\mu > 0$ large enough so that

$$\int_{\mathbb{R}^N} \left(|\Delta\varphi|^2 - m(x)\varphi^2 + \mu\varphi^2 \right) \geqslant \omega_{m-\mu} \|\varphi\|^2, \ \varphi \in H^2(\mathbb{R}^N),$$

holds with $\omega_{m-\mu} > 0$. Then the operator

$$\mathcal{A} = A_{m-\mu} = A_m + \mu I \tag{2.5}$$

is positive definite self-adjoint in $L^2(\mathbb{R}^N)$ with the domain $D(A_m)$ dense in $H^2(\mathbb{R}^N)$ and thus generates a strongly continuous analytic semigroup $\{e^{-\mathcal{A}t}\colon t \geqslant 0\}$ on X. Moreover, its fractional power space $X^{\frac{1}{2}} = D(\mathcal{A}^{\frac{1}{2}})$ coincides up to the equivalence of norms with $H^2(\mathbb{R}^N)$ endowed with the equivalent norm

$$\|\varphi\|^2_{H^2(\mathbb{R}^N)} = \|\Delta\varphi\|^2 + \|\varphi\|^2.$$

We consider the problem (1.6), (1.7) in the abstract form

$$\begin{cases} u_t + \mathcal{A}u = \mathcal{F}(u), \ t > 0, \\ u(0) = u_0 \end{cases} \tag{2.6}$$

with

$$\mathcal{F}(u)(x) = -\gamma\Delta u(x) + (\mu - \delta)u(x) + g(x) + \mathcal{F}_0(u)(x), \quad \mathcal{F}_0(u)(x) = f_0(x, u(x)).$$

We fix $\alpha \in [\frac{1}{2}, 1)$ and note that $\mathcal{F}\colon X^\alpha \to X$ is well defined, since (2.1) yields

$$\|\mathcal{F}_0(u)\|^2 \leqslant 2c_0^2(\|u\|^2 + \|u\|^{2\rho}_{L^{2\rho}(\mathbb{R}^N)}),$$

which combined with the Sobolev type embedding (see [29, 2.8.1/15]), due to (1.17),

$$H^2(\mathbb{R}^N) \hookrightarrow L^{2\rho}(\mathbb{R}^N), \tag{2.7}$$

implies

$$\|\mathcal{F}(u)\| \leqslant \|g\| + c(\|u\|_{H^2(\mathbb{R}^N)} + \|u\|^\rho_{H^2(\mathbb{R}^N)}), \ u \in X^\alpha \subset H^2(\mathbb{R}^N), \tag{2.8}$$

with some positive c.

In order to prove local solvability of (2.6) it suffices to show that $\mathcal{F}\colon X^\alpha \to X$ is Lipschitz continuous on bounded subsets of X^α and apply the standard theory of analytic semigroups (see [5, 14]).

Theorem 2.3 *Under the assumptions (1.12)–(1.17) for each $u_0 \in X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$, there exists a unique local X^α solution*

$$u \in C([0, \tau_{u_0}); X^\alpha) \cap C((0, \tau_{u_0}); D(A_m)) \cap C^1((0, \tau_{u_0}); X^{1-})$$

of the problem (2.6) in X satisfying the Duhamel formula

$$u(t) = e^{-\mathcal{A}t}u_0 + \int_0^t e^{-\mathcal{A}(t-s)}\mathcal{F}(u(s))ds, \ t \in [0, \tau_{u_0}), \tag{2.9}$$

and defined on its maximal interval of existence $[0, \tau_{u_0})$, i.e.,

$$\limsup_{t \to \tau_{u_0}^-} \|u(t)\|_{X^\alpha} = \infty \ \textit{unless} \ \tau_{u_0} = \infty.$$

Proof For a fixed bounded subset B of X^{α} and $u_1, u_2 \in B$ by (1.16) and the Hölder inequality we deduce that

$$\|\mathcal{F}_0(u_1)-\mathcal{F}_0(u_2)\| \leqslant c\left(\|u_1-u_2\|+\|u_1-u_2\|_{L^{2\rho}(\mathbb{R}^N)}(\|u_1\|^{\rho-1}_{L^{2\rho}(\mathbb{R}^N)}+\|u_2\|^{\rho-1}_{L^{2\rho}(\mathbb{R}^N)})\right).$$

Thus, by (2.7), we get

$$\|\mathcal{F}(u_1)-\mathcal{F}(u_2)\| \leqslant c\left(\|u_1-u_2\|_{H^2(\mathbb{R}^N)}(1+\gamma+\mu+\delta+\|u_1\|^{\rho-1}_{H^2(\mathbb{R}^N)}+\|u_2\|^{\rho-1}_{H^2(\mathbb{R}^N)})\right)$$

and hence $\mathcal{F}\colon X^{\alpha}\to X$ is Lipschitz continuous on bounded subsets of X^{α}. □

Remark 2.4 *Observe that a different approach of $\epsilon-$regular mild solutions from [1] for $N \geqslant 5$ allows to extend the admissible interval $[1, \frac{N}{N-4}]$ in (1.17) of the growth parameter ρ also to the interval $(\frac{N}{N-4}, \frac{N+4}{N-4}]$ if $u_0 \in H^2(\mathbb{R}^N) = X^{\frac{1}{2}}$. Indeed, we decompose $\mathcal{F} = \widetilde{\mathcal{F}}_0 + \widetilde{\mathcal{F}}_1$ with*

$$\widetilde{\mathcal{F}}_0(u)(x) = f_0(x,u(x)) + g(x), \quad \widetilde{\mathcal{F}}_1(u)(x) = -\gamma\Delta u(x) + (\mu-\delta)u(x).$$

It has already been shown in [6, Section 3.2] that for $\rho \in (\frac{N}{N-4}, \frac{N+4}{N-4}]$ the map $\mathcal{F}_0$ is $\epsilon-$regular relative to $(X^{\frac{1}{2}}, X^{-\frac{1}{2}})$. More precisely, we have

$$\left\|\widetilde{\mathcal{F}}_0(v)-\widetilde{\mathcal{F}}_0(w)\right\|_{X^{\widetilde{\gamma}(\epsilon)-\frac{1}{2}}} \leqslant c\,\|v-w\|_{X^{\frac{1}{2}+\epsilon}}\left(1+\|v\|^{\rho-1}_{X^{\frac{1}{2}+\epsilon}}+\|w\|^{\rho-1}_{X^{\frac{1}{2}+\epsilon}}\right),\ v,w\in X^{\frac{1}{2}+\epsilon}$$

with some constants $c>0$, $\epsilon>0$ and $\widetilde{\gamma}(\epsilon)\in[\rho\epsilon,\frac{1}{2}]$. Since

$$\left\|\widetilde{\mathcal{F}}_1(v)-\widetilde{\mathcal{F}}_1(w)\right\|_{X^{\widetilde{\gamma}(\epsilon)-\frac{1}{2}}} \leqslant c\,\|v-w\|_{X^{\widetilde{\gamma}(\epsilon)}}$$

and $\widetilde{\gamma}(\epsilon)\leqslant\frac{1}{2}$, it follows that also $\mathcal{F}$ is an $\epsilon-$regular map relative to $(X^{\frac{1}{2}}, X^{-\frac{1}{2}})$. Consequently, there exists a unique local $\epsilon-$regular mild solution of (2.6) with $u_0\in H^2(\mathbb{R}^N)$ such that

$$u\in C([0,\tau_{u_0});H^2(\mathbb{R}^N))\cap C((0,\tau_{u_0});H^{4\beta^*}(\mathbb{R}^N))\cap C^1((0,\tau_{u_0});L^2(\mathbb{R}^N))$$

satisfies (2.9) and τ_0 is its maximal existence time for $\rho\in(\frac{N}{N-4},\frac{N+4}{N-4})$ in the sense that

$$\limsup_{t\to\tau_{u_0}^-}\|u(t)\|_{H^2(\mathbb{R}^N)}=\infty \text{ unless } \tau_{u_0}=\infty.$$

Therefore, if $\rho\in(\frac{N}{N-4},\frac{N+4}{N-4})$ in (1.16) then we limit ourselves only to $\alpha=\frac{1}{2}$ and instead of standard X^{α} solutions we consider $\epsilon-$regular mild solutions.

3. Semigroup of global solutions

In this section, we will take advantage of the interplay between the linear term and the nonlinearity in (1.6). To this end, we assume here the structure condition (1.2) along with (1.3), (1.4) and require that solutions of (1.5) decay exponentially, which by Proposition 2.2 (iv) means that

$$\int_{\mathbb{R}^N}\left(|\Delta\varphi|^2-C(x)\varphi^2\right)\geqslant\omega_C\,\|\varphi\|^2,\ \varphi\in H^2(\mathbb{R}^N)\text{ with some }\omega_C>0. \tag{3.1}$$

Multiplying (1.6) by $2u_t$ in $L^2(\mathbb{R}^N)$, we obtain

$$\frac{d}{dt}\left(\mathcal{L}(u)\right) = -2\left\|u_t\right\|^2 \leqslant 0, \tag{3.2}$$

where

$$\mathcal{L}(u) = \|\Delta u\|^2 + \delta\|u\|^2 - \gamma\|\nabla u\|^2 - 2\int_{\mathbb{R}^N} F(x,u) \tag{3.3}$$

with the antiderivative

$$F(x,s) = \int_0^s f(x,\tau)d\tau, \; x \in \mathbb{R}^N, \; s \in \mathbb{R}.$$

Remark 3.1 *Note that the structure condition (1.2) implies*

$$F(x,s) \leqslant \frac{1}{2}C(x)s^2 + D(x)\left|s\right|, \; x \in \mathbb{R}^N, \; s \in \mathbb{R}. \tag{3.4}$$

Moreover, Proposition 2.2 (v) for $C \in L^r_U(\mathbb{R}^N)$ yields in particular

$$\int_{\mathbb{R}^N} C(x)\varphi^2 \leqslant \frac{1}{4}\|\varphi\|^2_{H^2(\mathbb{R}^N)} + \zeta_C\|\varphi\|^2, \; \varphi \in H^2(\mathbb{R}^N), \tag{3.5}$$

with some positive constant ζ_C.

Proposition 3.2 *The functional $\mathcal{L}$ from (3.3) is well defined for $u \in H^2(\mathbb{R}^N)$ and for some $c_1 > 0$, we have*

$$\mathcal{L}(u) \leqslant c_1(1 + \|u\|^2_{H^2(\mathbb{R}^N)} + \|u\|^{\rho+1}_{H^2(\mathbb{R}^N)}), \; u \in H^2(\mathbb{R}^N). \tag{3.6}$$

Moreover, if

$$\delta \geqslant \frac{\gamma^2}{\nu_C} = 2\left(1 + \frac{\zeta_C + \frac{1}{2}}{\omega_C}\right)\gamma^2, \tag{3.7}$$

where ω_C comes from (3.1) and ζ_C from (3.5), then for some constants $c_2, c_3 > 0$ we have

$$\mathcal{L}(u) \geqslant c_2\|u\|^2_{H^2(\mathbb{R}^N)} - c_3\|D\|^2_{L^q(\mathbb{R}^N)}, \; u \in H^2(\mathbb{R}^N). \tag{3.8}$$

In consequence, for any bounded subset B of X^α, $\alpha \in [\frac{1}{2}, 1)$, there exists $R_B > 0$ such that

$$\sup_{t \geqslant 0}\sup_{u_0 \in B}\|u(t,u_0)\|_{X^\alpha} \leqslant R_B. \tag{3.9}$$

Proof Using the mean value theorem, (1.12) and (2.4) for $|m| \in L^r_U(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|F(x,u)| &= \int_{\mathbb{R}^N}|F(x,u) - F(x,0)| = \int_{\mathbb{R}^N}|u|\,|f(x,\theta u)| \\ &\leqslant \varepsilon_0\|u\|^2_{H^2(\mathbb{R}^N)} + \xi_{|m|}(\varepsilon_0)\|u\|^2 + \|g\|\|u\| + \int_{\mathbb{R}^N}|u|\,|f_0(x,\theta u)|\end{aligned}$$

where $\theta = \theta(x) \in (0,1)$. Moreover, due to the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{\rho+1}(\mathbb{R}^N)$ for $\rho \geqslant 1$ satisfying (1.18), we get from (2.1)

$$\int_{\mathbb{R}^N} |u|\,|f_0(x,\theta u)| \leqslant c(\|u\|^2_{H^2(\mathbb{R}^N)} + \|u\|^{\rho+1}_{H^2(\mathbb{R}^N)})$$

and in consequence (3.6).

For $u \in H^2(\mathbb{R}^N)$, we estimate $\mathcal{L}(u)$ from below. By (3.4), we get

$$\mathcal{L}(u) \geqslant \|\Delta u\|^2 + \delta\,\|u\|^2 - \gamma\,\|\nabla u\|^2 - \int_{\mathbb{R}^N} C(x)u^2 - 2\int_{\mathbb{R}^N} D(x)\,|u|\,. \tag{3.10}$$

We estimate the last three terms. If q' denotes the conjugate exponent to q from (1.4), we have $H^2(\mathbb{R}^N) \hookrightarrow L^{q'}(\mathbb{R}^N)$. Thus, by Hölder and Cauchy inequalities, we obtain for any $\varepsilon_1 > 0$

$$\int_{\mathbb{R}^N} D(x)\,|u| \leqslant \|D\|_{L^q(\mathbb{R}^N)}\,\|u\|_{L^{q'}(\mathbb{R}^N)} \leqslant \frac{\varepsilon_1}{2}\,\|u\|^2_{H^2(\mathbb{R}^N)} + \frac{c^2}{2\varepsilon_1}\,\|D\|^2_{L^q(\mathbb{R}^N)}\,. \tag{3.11}$$

Following [6, Corollary 2.11], by (3.5), we get for $0 < \nu < \frac{1}{2}$

$$-2\nu\int_{\mathbb{R}^N} C(x)u^2 \geqslant -\frac{\nu}{2}\,\|u\|^2_{H^2(\mathbb{R}^N)} - 2\nu\zeta_C\,\|u\|^2\,. \tag{3.12}$$

Moreover, for any $\varepsilon_2 > 0$, we have

$$\gamma\,\|\nabla u\|^2 \leqslant \gamma\,\|\Delta u\|\,\|u\| \leqslant \varepsilon_2\,\|\Delta u\|^2 + \frac{\gamma^2}{4\varepsilon_2}\,\|u\|^2\,. \tag{3.13}$$

Combining these estimates and using (3.1), we obtain for any $0 < \nu < \frac{1}{2}$

$$\mathcal{L}(u) \geqslant \left(\frac{3}{2}\nu - \varepsilon_2 - \varepsilon_1\right)\|\Delta u\|^2 + \left(\delta - \frac{\gamma^2}{4\varepsilon_2} - \varepsilon_1 - \frac{\nu}{2} + (1-2\nu)\omega_C - 2\nu\zeta_C\right)\|u\|^2 - \frac{c^2}{\varepsilon_1}\,\|D\|^2_{L^q(\mathbb{R}^N)}\,.$$

Setting

$$\nu_C = \frac{\omega_C}{2(\omega_C + \zeta_C + \frac{1}{2})} \in \left(0, \frac{1}{2}\right) \tag{3.14}$$

and $\varepsilon_1 = \varepsilon_2 = \frac{\nu_C}{4}$, we have $(1-2\nu_C)\omega_C - 2\nu_C\zeta_C = \nu_C$ and

$$\mathcal{L}(u) \geqslant \nu_C\,\|\Delta u\|^2 + \frac{\nu_C}{4}\,\|u\|^2 + \left(\delta - \frac{\gamma^2}{\nu_C}\right)\|u\|^2 - \frac{4c^2}{\nu_C}\,\|D\|^2_{L^q(\mathbb{R}^N)}\,.$$

Thus, we get (3.8) provided that (3.7) holds.

Applying (3.2), (3.6), and (3.8), we obtain

$$c_2\,\|u(t)\|^2_{H^2(\mathbb{R}^N)} - c_3\,\|D\|^2_{L^q(\mathbb{R}^N)} \leqslant \mathcal{L}(u(t)) \leqslant \mathcal{L}(u_0) \leqslant c_1(1 + \|u_0\|^2_{H^2(\mathbb{R}^N)} + \|u_0\|^{\rho+1}_{H^2(\mathbb{R}^N)}), \tag{3.15}$$

which yields the a priori estimate (3.9) with $\alpha = \frac{1}{2}$ for a bounded subset B of $X^{\frac{1}{2}} = H^2(\mathbb{R}^N)$. The a priori estimate (3.9) for $\alpha \in (\frac{1}{2}, 1)$ is then a consequence of (2.8) and (3.15), see [5, Theorem 3.1.1] for details. □

Summarizing the above considerations, we conclude that the modified Swift–Hohenberg equation (1.6) generates a C^0 semigroup $\{S(t)\colon t \geqslant 0\}$ on X^α for a given $\alpha \in [\frac{1}{2}, 1)$, that is, the mapping

$$[0, \infty) \times X^\alpha \ni (t, u_0) \mapsto S(t)u_0 \in X^\alpha$$

is continuous (cp. [5, Remark 3.1.1]).

Theorem 3.3 *Additionally to the conditions of Theorem 2.3 guaranteeing local solvability (with ρ satisfying the enhanced bound (1.18) as in Remark 2.4 for $\alpha = \frac{1}{2}$ only), assume the structure condition (1.2) together with (1.3), (1.4), and (3.1). If (3.7) holds, then the problem (2.6) with $u_0 \in X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$, defines a C^0 semigroup of global X^α solutions*

$$S(t)u_0 = u(t; u_0), \ t \geqslant 0, \ u_0 \in X^\alpha,$$

which has orbits of bounded sets bounded.

We emphasize that the restriction (3.7) is only related to properties of the function C, since ω_C and ζ_C come from (3.1) and (3.5), respectively. Note that a similar limitation was also required for the modified Swift–Hohenberg equation considered in a bounded domain of $\mathbb{R}^N$ to generate a semigroup of solutions (see [17] and (1.10)).

Remark 3.4 *Observe that in a simple situation when the structure condition (1.2) holds with $C(x) \equiv -d$, d being a positive constant, and $D \in L^2(\mathbb{R}^N)$, the estimate of $\mathcal{L}(u)$ from below given in (3.8) holds provided that*

$$\delta + d > \frac{\gamma^2}{4}, \tag{3.16}$$

which replaces the restriction (3.7) in this case. Indeed, we write (3.10), take $\varepsilon > 0$ so small that $\delta + d - \varepsilon > \frac{\gamma^2}{4}$ and apply the estimates

$$\int_{\mathbb{R}^N} D(x)\,|u| \leqslant \|D\|\,\|u\| \leqslant \frac{\varepsilon}{4}\,\|u\|^2 + \frac{1}{\varepsilon}\,\|D\|^2$$

and

$$\gamma\,\|\nabla u\|^2 \leqslant \gamma\,\|\Delta u\|\,\|u\| \leqslant \frac{\gamma^2}{\gamma^2 + 2\varepsilon}\,\|\Delta u\|^2 + \left(\frac{\gamma^2}{4} + \frac{\varepsilon}{2}\right)\|u\|^2$$

(compare (3.13) with $\varepsilon_2 = \frac{\gamma^2}{\gamma^2 + 2\varepsilon}$). Thus, we obtain in this case

$$\mathcal{L}(u) \geqslant \left(1 - \frac{\gamma^2}{\gamma^2 + 2\varepsilon}\right)\|\Delta u\|^2 + (\delta + d - \frac{\gamma^2}{4} - \varepsilon)\,\|u\|^2 - \frac{2}{\varepsilon}\,\|D\|^2, \tag{3.17}$$

which proves the claim.

Example 3.5 *A specific simple version of the dissipative mechanism was considered in [30] for the hyperbolic relaxation of the Swift–Hohenberg equation. Namely, there $\gamma = 2$, $m(x) \equiv \delta > 0$, $g \in L^2(\mathbb{R}^N)$ and $f_0 \in C^2(\mathbb{R})$ is an $x-$independent function satisfying*

$$|f_0''(s)| \leqslant C_0(1 + |s|^{\rho - 2}), \ s \in \mathbb{R}, \tag{3.18}$$

with $\rho > 2$, $(N-4)\rho \leqslant N$ and

$$f_0(s)s \leqslant -(1+\delta)s^2, \ s \in \mathbb{R}. \tag{3.19}$$

It is easy to see that (3.18) yields (1.16). Moreover, (3.19) implies that $f_0(0) = 0$ and the structure condition (1.2) holds with $C(x) \equiv -1$ and $D(x) = |g(x)|$, which is the case discussed in Remark 3.4 with $d = 1$. Since (3.16) is trivially satisfied (one can take e.g. $\varepsilon_2 = \frac{2}{2+\delta} \in (0,1)$ in (3.17)), it follows that (3.8) holds without any further restriction on $\delta > 0$. Thus, (3.7) is not needed in Theorem 3.3 in this example.

Now we give other examples of nonlinearities of the form (1.12) satisfying assumptions (1.13)–(1.16) and the dissipative mechanism (1.2)–(1.4) with (3.1).

Example 3.6 *Remaining in the simple setting of Remark 3.4 for $N \leqslant 4$, we can take the nonlinearity*

$$f(x,s) = -ds - cs^{2k-1} + g(x), \ x \in \mathbb{R}^N, \, s \in \mathbb{R},$$

where $k \in \mathbb{N}$, $k > 1$, c, d are positive constants and $g \in L^2(\mathbb{R}^N)$. The abovementioned assumptions are then satisfied with $f_0(x,s) = -cs^{2k-1}$, $C(x) = m(x) \equiv -d$, $D(x) = |g(x)|$, $q = 2$ and $\omega_C = d$. Moreover, (1.18) also holds for $N = 5$ if $k = 2,3,4$ and for $N = 6,7$ for $k = 2$.

Example 3.7 *Consider now the nonlinear term of the form*

$$f(x,s) = m(x)s + p^2(x)s - |p(x)|^{\rho+1}s|s|^{\rho-1}, \ x \in \mathbb{R}^N, \ s \in \mathbb{R},$$

with m satisfying (1.13) and (2.3) with $\omega_m > 0$, $p \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\rho > 1$ as in (1.18). Using Young's inequality $ab \leqslant \frac{a^{\tilde{p}}}{\tilde{p}} + \frac{b^{\tilde{q}}}{\tilde{q}}$ with

$$a = |p(x)s|^{\frac{\rho-1}{\rho}}, \quad b = |p(x)s|^{\frac{\rho+1}{\rho}}, \quad \tilde{p} = \frac{\rho}{\rho-1} \quad \text{and} \quad \tilde{q} = \rho,$$

we obtain

$$f(x,s)s = m(x)s^2 + |p(x)s|^{\frac{\rho-1}{\rho}}|p(x)s|^{\frac{\rho+1}{\rho}} - |p(x)s|^{\rho+1} \leqslant m(x)s^2 + \frac{\rho-1}{\rho}|p(x)s| + \left(\frac{1}{\rho} - 1\right)|p(x)s|^{\rho+1}.$$

Note that our assumptions are satisfied with

$$f_0(x,s) = p^2(x)s - |p(x)|^{\rho+1}s|s|^{\rho-1}$$

and $C(x) = m(x)$, $D(x) = \frac{\rho-1}{\rho}|p(x)|$ and $\omega_C = \omega_m$.

For more examples of suitable right-hand sides of (1.6), we refer the reader to [8, Section 7].

4. Asymptotic compactness and global attractor

In this section, we consider the very same assumptions as in Theorem 3.3 so that (1.6), (1.7) defines a semigroup on X^α with a given $\alpha \in [\frac{1}{2}, 1)$. Our aim is to show that this semigroup is asymptotically compact and in consequence has a global attractor. To this end, we will estimate the tails of orbits of bounded subsets of X^α in the sense of the following lemma.

Lemma 4.1 *For each bounded subset B of X^α and each $\varepsilon > 0$, there exist $T_{\varepsilon,B} > 0$ and $k_{\varepsilon,B} \in \mathbb{N}$ such that for any $k \geqslant k_{\varepsilon,B}$ we have*

$$\sup_{u_0\in B}\sup_{t\geqslant T_{\varepsilon,B}} \|S(t)u_0\|_{L^2(|x|\geqslant 2k)} < \varepsilon.$$

Proof Let $\theta\colon [0,\infty) \to [0,1]$ be a C^∞ function such that $\theta(s) = 0$ for $0 \leqslant s \leqslant 1$ and $\theta(s) = 1$ for $s \geqslant 2$ and let $\theta_0 > 0$ be such that $|\theta'(s)| \leqslant \theta_0$ and $|\theta''(s)| \leqslant \theta_0$ for $s \in \mathbb{R}$. We define

$$\eta_k(x) = \theta^2\left(\frac{|x|}{k}\right) \text{ for } x \in \mathbb{R}^N \text{ and } k \in \mathbb{N}$$

and note that

$$|\nabla\eta_k|(x) \leqslant \frac{2\theta_0}{k},\ \left|\nabla\eta_k^{\frac{1}{2}}\right|(x) \leqslant \frac{\theta_0}{k},\ |\Delta\eta_k|(x) \leqslant \frac{\theta_1}{k^2},\ \left|\Delta\eta_k^{\frac{1}{2}}\right|(x) \leqslant \frac{\theta_1}{k^2},\ x \in \mathbb{R}^N, \tag{4.1}$$

with some $\theta_1 > 0$ depending on θ_0 and N.

We multiply (1.6) by $\eta_k u$ in $L^2(\mathbb{R}^N)$ and get

$$\int_{\mathbb{R}^N}\eta_k u_t u + \int_{\mathbb{R}^N}\eta_k(\Delta u)^2 + \gamma\int_{\mathbb{R}^N}\eta_k(\Delta u)u + \delta\int_{\mathbb{R}^N}\eta_k u^2 = \int_{\mathbb{R}^N}\eta_k f(x,u)u - \int_{\mathbb{R}^N}(\Delta\eta_k(\Delta u)u + 2(\Delta u)\nabla\eta_k\cdot\nabla u).$$

By (4.1) and (3.9), we obtain

$$-\int_{\mathbb{R}^N}(\Delta\eta_k(\Delta u)u + 2(\Delta u)\nabla\eta_k\cdot\nabla u) \leqslant \frac{c_B}{k},$$

where henceforth c_B denotes a positive constant depending on the set B, which may change from line to line. Applying (1.2), we get for $0 < \nu < 2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\eta_k^{\frac{1}{2}}u\right\|^2 &+ \gamma\int_{\mathbb{R}^N}\eta_k(\Delta u)u + \frac{\nu}{2}\left\|\eta_k^{\frac{1}{2}}\Delta u\right\|^2 + \left(1-\frac{\nu}{2}\right)\int_{\mathbb{R}^N}\eta_k(\Delta u)^2\\ &+ \delta\int_{\mathbb{R}^N}\eta_k u^2 - \int_{\mathbb{R}^N}C(x)(\eta_k^{\frac{1}{2}}u)^2 \leqslant \frac{c_B}{k} + \int_{\mathbb{R}^N}D(x)\,|u|\,\eta_k.\end{aligned}$$

Since we have

$$\Delta(\eta_k^{\frac{1}{2}}u) = \Delta(\eta_k^{\frac{1}{2}})u + 2\nabla(\eta_k^{\frac{1}{2}})\cdot\nabla u + \eta_k^{\frac{1}{2}}\Delta u \tag{4.2}$$

and as a result

$$\eta_k(\Delta u)^2 = \left(\Delta(\eta_k^{\frac{1}{2}}u) - 2\nabla(\eta_k^{\frac{1}{2}})\cdot\nabla u - \Delta(\eta_k^{\frac{1}{2}})u\right)^2,$$

it follows from (3.9), (4.1), and (4.2) that

$$\int_{\mathbb{R}^N}\left(4\Delta(\eta_k^{\frac{1}{2}}u)\nabla(\eta_k^{\frac{1}{2}})\cdot\nabla u + 2\Delta(\eta_k^{\frac{1}{2}}u)\Delta(\eta_k^{\frac{1}{2}})u - 4\left(\nabla(\eta_k^{\frac{1}{2}})\cdot\nabla u\right)\Delta(\eta_k^{\frac{1}{2}})u\right) \leqslant \frac{c_B}{k}$$

and in consequence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\eta_k^{\frac{1}{2}}u\right\|^2 &+ \gamma\int_{\mathbb{R}^N}\eta_k(\Delta u)u + \frac{\nu}{2}\left\|\eta_k^{\frac{1}{2}}\Delta u\right\|^2 + \left(1-\frac{\nu}{2}\right)\left\|\Delta(\eta_k^{\frac{1}{2}}u)\right\|^2\\ &+ \delta\int_{\mathbb{R}^N}\eta_k u^2 - \int_{\mathbb{R}^N}C(x)(\eta_k^{\frac{1}{2}}u)^2 \leqslant \frac{c_B}{k} + \int_{\mathbb{R}^N}D(x)\,|u|\,\eta_k.\end{aligned}$$

We estimate the last term as in (3.11) with arbitrary $\varepsilon_1 > 0$ by

$$\left\|\eta_k^{\frac{1}{2}} D\right\|_{L^q(\mathbb{R}^N)} \left\|\eta_k^{\frac{1}{2}} u\right\|_{L^{q'}(\mathbb{R}^N)} \leqslant \frac{\varepsilon_1}{2}\left(\left\|\Delta(\eta_k^{\frac{1}{2}} u)\right\|^2 + \left\|\eta_k^{\frac{1}{2}} u\right\|^2\right) + \frac{c^2}{2\varepsilon_1}\left\|\eta_k^{\frac{1}{2}} D\right\|^2_{L^q(\mathbb{R}^N)}.$$

Moreover, by (3.9) and (4.1), we get

$$-\gamma\int_{\mathbb{R}^N} \eta_k(\Delta u)u = \gamma\int_{\mathbb{R}^N} \nabla u\cdot\nabla(\eta_k)u + \gamma\int_{\mathbb{R}^N}\left|\eta_k^{\frac{1}{2}}\nabla u\right|^2 \leqslant \gamma\left\|\nabla(\eta_k^{\frac{1}{2}} u)\right\|^2 + \frac{c_B}{k},$$

which, as in (3.13), gives for arbitrary $\varepsilon_2 > 0$

$$-\gamma\int_{\mathbb{R}^N} \eta_k(\Delta u)u \leqslant \varepsilon_2\left\|\Delta(\eta_k^{\frac{1}{2}} u)\right\|^2 + \frac{\gamma^2}{4\varepsilon_2}\left\|\eta_k^{\frac{1}{2}} u\right\|^2 + \frac{c_B}{k}.$$

Thus, these inequalities, the estimate as in (3.12) with $\eta_k^{\frac{1}{2}} u$ in the role of u and (3.1) imply that for any $0 < \nu < \frac{1}{2}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\eta_k^{\frac{1}{2}} u\right\|^2 + \frac{\nu}{2}\left\|\eta_k^{\frac{1}{2}}\Delta u\right\|^2 + \left((1-2\nu)\omega_C - 2\nu\zeta_C + \delta - \frac{\gamma^2}{4\varepsilon_2} - \frac{\nu}{2} - \frac{\varepsilon_1}{2}\right)\left\|\eta_k^{\frac{1}{2}} u\right\|^2 \\ + \left(\nu - \varepsilon_2 - \frac{\varepsilon_1}{2}\right)\left\|\Delta(\eta_k^{\frac{1}{2}} u)\right\|^2 \leqslant \frac{c_B}{k} + \frac{c^2}{2\varepsilon_1}\left\|\eta_k^{\frac{1}{2}} D\right\|^2_{L^q(\mathbb{R}^N)}.\end{aligned}$$

Thus, taking $\nu = \nu_C$ from (3.14) and $\varepsilon_1 = \frac{\nu_C}{2}$, $\varepsilon_2 = \frac{\nu_C}{4}$, we obtain in particular

$$\frac{d}{dt}\left\|\eta_k^{\frac{1}{2}} u\right\|^2 + \frac{\nu_C}{2}\left\|\eta_k^{\frac{1}{2}} u\right\|^2 + 2\left(\delta - \frac{\gamma^2}{\nu_C}\right)\left\|\eta_k^{\frac{1}{2}} u\right\|^2 \leqslant \frac{c_B}{k} + \frac{2c^2}{\nu_C}\left\|\eta_k^{\frac{1}{2}} D\right\|^2_{L^q(\mathbb{R}^N)}.$$

Therefore, if (3.7) holds, by the Gronwall inequality, we get

$$\left\|\eta_k^{\frac{1}{2}} u(t)\right\|^2 \leqslant \left\|\eta_k^{\frac{1}{2}} u_0\right\|^2 e^{-\frac{\nu_C}{2}t} + \frac{2}{\nu_C}\left(\frac{c_B}{k} + \frac{2c^2}{\nu_C}\left\|\eta_k^{\frac{1}{2}} D\right\|^2_{L^q(\mathbb{R}^N)}\right),\ t \geqslant 0.$$

Note that $\|u\|^2_{L^2(|x|\geqslant 2k)} \leqslant \left\|\eta_k^{\frac{1}{2}} u\right\|^2 \leqslant \|u\|^2$ and

$$\left\|\eta_k^{\frac{1}{2}} D\right\|^2_{L^q(\mathbb{R}^N)} \leqslant \|D\|^2_{L^q(|x|>k)} \to 0 \ \text{ as } \ k\to\infty,$$

since $D \in L^q(\mathbb{R}^N)$. Thus, for each $\varepsilon > 0$, there exist $T_{\varepsilon,B} > 0$ and $k_{\varepsilon,B} \in \mathbb{N}$ such that for $t \geqslant T_{\varepsilon,B}$ and $k \geqslant k_{\varepsilon,B}$ we have $\|u(t)\|_{L^2(|x|\geqslant 2k)} < \varepsilon$, which ends the proof. □

Proposition 4.2 *The semigroup $\{S(t)\colon t \geqslant 0\}$ on X^α is asymptotically compact, that is, for any sequence $\{S(t_n)u_{0n}\}$, where $\{u_{0n}\}$ is bounded in X^α and $t_n \to \infty$, there exists a convergent subsequence in X^α.*

Proof By Lemma 4.1 for a given $\varepsilon > 0$, there exist $T > 0$, $r_0 > 0$ and $n_0 \in \mathbb{N}$ such that for $l, n \geqslant n_0$ we have $t_l, t_n \geqslant T + 1$ and

$$\|S(t_l - 1)u_{0l} - S(t_n - 1)u_{0n}\|_{L^2(|x|\geqslant r_0)} < \varepsilon. \tag{4.3}$$

Moreover, by (3.9), the sequence $\{S(t_n-1)u_{0n}\}$ is bounded in $X^\alpha \hookrightarrow H^2(\mathbb{R}^N)$, hence in $H^2(|x|<r_0)$, which is compactly embedded into $L^2(|x|<r_0)$. Thus, we can choose a convergent subsequence $\{S(t_{n_k}-1)u_{0n_k}\}$ in $L^2(|x|<r_0)$. Combining this with (4.3), we see that $\{S(t_{n_k}-1)u_{0n_k}\}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, hence convergent in $L^2(\mathbb{R}^N)$. Since this sequence was bounded in X^α, it follows from the Duhamel formula (2.9) and [5, Theorem 3.2.1] that the sequence

$$S(1)S(t_{n_k}-1)u_{0n_k} = S(t_{n_k})u_{0n_k}$$

contains a convergent subsequence in X^α, which proves the claim. □

Having proved the asymptotic compactness of the semigroup, our next goal is to show that it possesses a global attractor. For this purpose we make the following two observations.

Proposition 4.3 *The set $\mathcal{E}$ of stationary solutions of (1.6) is bounded in X^α.*

Proof Multiplying

$$\Delta^2 u + \gamma\Delta u + \delta u = f(x,u)$$

by u in $L^2(\mathbb{R}^N)$, we get from (1.2)

$$\|\Delta u\|^2 + \delta\|u\|^2 - \gamma\|\nabla u\|^2 = \int_{\mathbb{R}^N} f(x,u)u \leqslant \int_{\mathbb{R}^N} C(x)u^2 + \int_{\mathbb{R}^N} D(x)|u|.$$

Applying (3.1), (3.11), (3.12), and (3.13) with $\nu=\nu_C$ from (3.14) and $\varepsilon_1=\frac{\nu_C}{2}$, $\varepsilon_2=\frac{\nu_C}{4}$, we obtain

$$\nu_C\|\Delta u\|^2 + \frac{\nu_C}{4}\|u\|^2 + \left(\delta - \frac{\gamma^2}{\nu_C}\right)\|u\|^2 \leqslant \frac{c^2}{\nu_C}\|D\|^2_{L^q(\mathbb{R}^N)}.$$

Invoking (3.7), we conclude that

$$\|u\|^2_{H^2(\mathbb{R}^N)} \leqslant \frac{4c^2}{\nu_C^2}\|D\|^2_{L^q(\mathbb{R}^N)},$$

which shows boundedness of the set of equilibria of (1.6) in $H^2(\mathbb{R}^N)$. If ρ satisfies (1.17), then by (2.8) the set of stationary solutions is bounded in $X^1 = D(\mathcal{A})$. □

Proposition 4.4 *The functional $\mathcal{L}$ from (3.3) is continuous on X^α.*

Proof Since $X^\alpha \hookrightarrow H^2(\mathbb{R}^N)$, it suffices to show continuity of $\mathcal{L}$ on $H^2(\mathbb{R}^N)$. Let $u, u_n \in H^2(\mathbb{R}^N)$, $n\in\mathbb{N}$, be such that $\|u_n - u\|_{H^2(\mathbb{R}^N)} \to 0$ as $n\to\infty$. Since

$$\begin{aligned}|\mathcal{L}(u_n)-\mathcal{L}(u)| &\leqslant \|\Delta(u_n-u)\|(\|\Delta u_n\|+\|\Delta u\|) + 2\int_{\mathbb{R}^N}|F(x,u_n)-F(x,u)| \\ &\quad + \gamma\|\nabla(u_n-u)\|(\|\nabla u_n\|+\|\nabla u\|) + \delta\|u_n-u\|(\|u_n\|+\|u\|),\end{aligned}$$

it suffices to show that $\int_{\mathbb{R}^N}|F(x,u_n)-F(x,u)| \to 0$ as $n\to\infty$. Using the mean value theorem, (1.12) and (2.1), we have

$$\begin{aligned}\int_{\mathbb{R}^N}|F(x,u_n)-F(x,u)| &= \int_{\mathbb{R}^N}|f(x,u+\theta(u_n-u))||u_n-u| \leqslant \|g\|\|u_n-u\| \\ &+ 2\int_{\mathbb{R}^N}|m(x)|(|u_n|+|u|)|u_n-u| + c\int_{\mathbb{R}^N}|u_n-u|(|u_n|+|u|+|u_n|^\rho+|u|^\rho)\end{aligned} \tag{4.4}$$

with $\theta = \theta(x) \in (0,1)$. Considering a partition of $\mathbb{R}^N$ by disjoint unitary cubes Q_i centered at $i \in \mathbb{Z}^N$, by the Hölder inequality and Sobolev embeddings with r from (1.3) and its conjugate exponent r', we get

$$\int_{\mathbb{R}^N} |m(x)|(|u_n| + |u|)|u_n - u| \leqslant \sum_{i\in\mathbb{Z}^N} \|m\|_{L^r(Q_i)} \||u_n| + |u|\|_{L^{2r'}(Q_i)} \|u_n - u\|_{L^{2r'}(Q_i)}$$

$$\leqslant c\,\|m\|_{L^r_U(\mathbb{R}^N)} \big(\sum_{i\in\mathbb{Z}^N} \||u_n| + |u|\|^2_{H^2(Q_i)}\big)^{\frac{1}{2}} \big(\sum_{i\in\mathbb{Z}^N} \|u_n - u\|^2_{H^2(Q_i)}\big)^{\frac{1}{2}}$$

$$\leqslant c\,\|m\|_{L^r_U(\mathbb{R}^N)} (\|u_n\|_{H^2(\mathbb{R}^N)} + \|u\|_{H^2(\mathbb{R}^N)})\,\|u_n - u\|_{H^2(\mathbb{R}^N)}\,.$$

Applying the Hölder inequality to the last term on the right side of (4.4) and next using the embeddings (2.7) for ρ from (1.17) and $H^2(\mathbb{R}^N) \hookrightarrow L^{\frac{2N\rho}{N+4}}(\mathbb{R}^N)$ for $\rho \in \left(\frac{N}{N-4}, \frac{N+4}{N-4}\right)$, we consequently obtain

$$\int_{\mathbb{R}^N} |F(x,u_n) - F(x,u)| \leqslant c\|u_n - u\|_{H^2(\mathbb{R}^N)}$$
$$\times (\|g\| + (\|m\|_{L^r_U(\mathbb{R}^N)} + 1)(\|u_n\|_{H^2(\mathbb{R}^N)} + \|u\|_{H^2(\mathbb{R}^N)}) + \|u_n\|^\rho_{H^2(\mathbb{R}^N)} + \|u\|^\rho_{H^2(\mathbb{R}^N)}),$$

which implies the continuity of $\mathcal{L}$ on $H^2(\mathbb{R}^N)$. □

Theorem 4.5 *Under assumptions of Theorem 3.3 the semigroup $\{S(t)\colon t \geqslant 0\}$ on X^α with $\alpha \in [\frac{1}{2}, 1)$ has a global attractor $\mathbf{A}$ in X^α. More precisely, we have*

$$\mathbf{A} = W^u(\mathcal{E}), \tag{4.5}$$

where $W^u(\mathcal{E})$ denotes the unstable manifold of the set of stationary solutions of (1.6).

Proof The functional $\mathcal{L}\colon X^\alpha \to \mathbb{R}$ defined in (3.3) is a Lyapunov function for the semigroup $\{S(t)\colon t \geqslant 0\}$, since $\mathcal{L}$ is continuous on X^α by Proposition 4.4, bounded below by (3.8), nonincreasing along the solutions and constant only for stationary solutions due to (3.2). Since by (3.9) orbits of bounded subsets of X^α are bounded in X^α, Proposition 4.3 and [3, Theorems 2.41 and 2.43] imply the existence of a global attractor $\mathbf{A}$ for the semigroup in X^α with the structure given in (4.5). □

5. Regularity of global attractor

In this section, we strengthen the regularity of g assuming that

$$g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{5.1}$$

Therefore, for a given arbitrary $2 \leqslant p < \infty$, we have $g \in L^p(\mathbb{R}^N)$ and in the following paragraphs, we will rewrite the differential equation (1.6) as

$$u_t + \mathcal{A}u = \mathcal{F}_1(u) + \mathcal{F}_2(u) + \mathcal{F}_3(u), \tag{5.2}$$

where the operator $\mathcal{A} = \Delta^2 - m(x)I + \mu I$ is now considered over the base space $L^p(\mathbb{R}^N)$ and such that $-\mathcal{A}$ generates an exponentially decaying analytic semigroup $\{e^{-\mathcal{A}t}\colon t \geqslant 0\}$ on $L^p(\mathbb{R}^N)$. Moreover, the extrapolated scale of fractional power spaces corresponding to $\mathcal{A}$ is characterized by

$$E_p^\beta = \begin{cases} H_p^{4\beta}(\mathbb{R}^N), & \beta \in [0, \beta^*(p)], \\ (H_{p'}^{-4\beta}(\mathbb{R}^N))' = H_p^{4\beta}(\mathbb{R}^N), & \beta \in [-\beta_*(p), 0), \end{cases}$$

where $\beta^*(p) = 1 + \left(\frac{N}{4p} - \frac{N}{4r}\right)_- \in (0,1]$ and $\beta_*(p) = \beta^*(p')$ with r as in (1.3) and $x_- = \min\{x, 0\}$, and we have for some $\omega = \omega(p) > 0$ and $M = M(p) \geqslant 1$

$$\left\|e^{-\mathcal{A}t}\right\|_{\mathcal{L}(E_p^{\beta_1}, E_p^{\beta_2})} \leqslant M \frac{e^{-\omega t}}{t^{\beta_2 - \beta_1}},\ t > 0,\ -\beta_*(p) \leqslant \beta_1 \leqslant \beta_2 \leqslant \beta^*(p). \tag{5.3}$$

For the proofs of these statements, we refer the reader to [6, Theorem 1.1].

Furthermore, the right-hand side of (5.2) consists of

$$\mathcal{F}_1(u)(x) = g(x) + (\mu - \delta)u + f_{01}(x, u(x))$$

where $f_{01}(x, 0) = 0$ and f_{01} is globally Lipschitz w.r.t. the second variable,

$$\mathcal{F}_2(u)(x) = f_{02}(x, u(x))$$

with $f_{02}(x, 0) = 0$ and such that with ρ from (1.16)

$$|f_{02}(x, s_1) - f_{02}(x, s_2)| \leqslant c\,|s_1 - s_2|\,(|s_1|^{\rho-1} + |s_2|^{\rho-1}),\ s_1, s_2 \in \mathbb{R}, \tag{5.4}$$

and $f_0(x, s) = f_{01}(x, s) + f_{02}(x, s)$ (see [6, Lemma 3.1]), whereas $\mathcal{F}_3(u) = -\gamma\Delta u$. In order to deal with $\mathcal{F}_3$, it is useful to observe that

$$\beta^*(p) \geqslant \frac{1}{2} \text{ if and only if } r \geqslant \frac{Np}{N + 2p}.$$

Following the idea from [7, Proposition 3.3], below we make a key observation, setting

$$\beta(p) = \begin{cases} \beta^*(p), & 0 < \beta^*(p) < \frac{1}{2}, \\ \frac{1}{2}, & \beta^*(p) \geqslant \frac{1}{2}. \end{cases} \tag{5.5}$$

Lemma 5.1 *Assume that $u(\cdot, u_0)$ is an X^α solution of (2.6) for a given $\alpha \in [\frac{1}{2}, 1)$ and $N > 2$. If*

$$\sup_{t \geqslant 0} \|u(t, u_0)\|_{L^{p_0}(\mathbb{R}^N)} < \infty \text{ for some } p_0 \geqslant 2 \tag{5.6}$$

and for $p \geqslant p_0$ the following estimate

$$\|u\|_{C_b([2\varepsilon,\infty); E_p^{\beta(p)})} \leqslant K(\varepsilon, \|u\|_{C_b([\varepsilon,\infty); L^p(\mathbb{R}^N))}),\ u \in C_b([\varepsilon, \infty); L^p(\mathbb{R}^N)), \tag{5.7}$$

holds with some continuous function $K(\varepsilon, \cdot)$ and $\varepsilon > 0$ small enough, then

$$\|u\|_{C_b([\varepsilon,\infty); L^\infty(\mathbb{R}^N))} \leqslant K_0(\varepsilon, \|u\|_{C_b([0,\infty); L^{p_0}(\mathbb{R}^N))}) \tag{5.8}$$

for some continuous function $K_0(\varepsilon, \cdot)$ and any $\varepsilon > 0$ small enough.

Proof For the sake of completeness of the presentation, we include a sketch of the proof. We apply (5.7) with $p = p_0$. If $\beta^*(p) < \frac{1}{2}$ then

$$4\beta(p) - \frac{N}{p} = 4\beta^*(p) - \frac{N}{p} = 4 - \frac{N}{r} > 0,$$

whereas if $\beta^*(p) \geqslant \frac{1}{2}$ and $p > \frac{N}{2}$, then $4\beta(p) - \frac{N}{p} = 2 - \frac{N}{p} > 0$ and in both situations $E_p^{\beta(p)} \hookrightarrow L^\infty(\mathbb{R}^N)$ and (5.8) follows.

If $\beta^*(p) \geqslant \frac{1}{2}$ and $p \leqslant \frac{N}{2}$, then in particular we have $E_p^{\beta(p)} \hookrightarrow L^{p_1}(\mathbb{R}^N)$ with $p_1 = \frac{Np}{N-2} > p$ and apply (5.7) with $p = p_1$. In the worst case scenario, repeatedly using (5.7) with bigger and bigger p's, after a finite number of steps, we get some $p_n > \frac{N}{2}$ and (5.8) must hold regardless of the value of $\beta^*(p_n)$. □

We locally solve the Cauchy problem corresponding to (5.2) for initial data from $L^p(\mathbb{R}^N)$ with sufficiently large p.

Lemma 5.2 *For $N > 2$, we consider the problem*

$$\begin{cases} v_t + \mathcal{A}v = \mathcal{F}_1(v) + \mathcal{F}_2(v) + \mathcal{F}_3(v), \ t > t_0, \\ v(t_0) = v_0 \in L^p(\mathbb{R}^N). \end{cases} \tag{5.9}$$

If

$$p > \frac{N}{N-2} \quad \text{and} \quad p \geqslant \frac{N}{2}(\rho - 1), \quad \text{where } \rho \text{ comes from (1.16)}, \tag{5.10}$$

then there exists a unique maximally defined $L^p(\mathbb{R}^N)$ solution of (5.9) such that

$$v \in C([t_0, \tau_{v_0}); L^p(\mathbb{R}^N)) \cap C((t_0, \tau_{v_0}); E_p^{\frac{1}{2}}) \cap C^1((t_0, \tau_{v_0}); L^p(\mathbb{R}^N)).$$

Proof Since $p > \frac{N}{N-2}$, we have $\beta_*(p) > \frac{1}{2}$ and consider (5.9) in the base space $Y = E_p^{-\frac{1}{2}} = H_p^{-2}(\mathbb{R}^N)$. To guarantee its local solvability in $L^p(\mathbb{R}^N)$ via the semigroup approach of [5, 14], we need to check that the right-hand side of (5.9) is Lipschitz continuous on bounded sets as a map from $L^p(\mathbb{R}^N)$ into $H_p^{-2}(\mathbb{R}^N)$. Since $\mathcal{F}_1$ is a globally Lipschitz map from $L^p(\mathbb{R}^N)$ into itself and

$$\|\mathcal{F}_3(v_1) - \mathcal{F}_3(v_2)\|_{H_p^{-2}(\mathbb{R}^N)} \leqslant c \|v_1 - v_2\|_{L^p(\mathbb{R}^N)},$$

for a given bounded subset B of $L^p(\mathbb{R}^N)$, it is sufficient to show with some constant $L(B) > 0$ that

$$\|\mathcal{F}_2(v_1) - \mathcal{F}_2(v_2)\|_{H_p^{-2}(\mathbb{R}^N)} \leqslant L(B) \|v_1 - v_2\|_{L^p(\mathbb{R}^N)}, \ v_1, v_2 \in B. \tag{5.11}$$

From (5.4) and (5.10) using the Sobolev embedding $L^q(\mathbb{R}^N) \hookrightarrow H_p^{-2}(\mathbb{R}^N)$ with $q = \frac{p}{\rho} \geqslant \frac{Np}{N+2p}$ and the Hölder inequality, we obtain

$$\|\mathcal{F}_2(v_1) - \mathcal{F}_2(v_2)\|_{H_p^{-2}(\mathbb{R}^N)} \leqslant c\|v_1 - v_2\|_{L^p(\mathbb{R}^N)} \left(\|v_1\|^{\rho-1}_{L^p(\mathbb{R}^N)} + \|u_2\|^{\rho-1}_{L^p(\mathbb{R}^N)} \right)$$

and thus (5.11) in consequence. □

In the proposition below we assume that (1.16) holds with ρ satisfying (1.17).

Proposition 5.3 *Assume that B is a bounded subset of X^α for a given $\alpha \in [\frac{1}{2}, 1)$. Then, denoting by $u(\cdot, u_0)$ the X^α solution of (2.6), for any $\varepsilon > 0$ there exists $\widetilde{R}_{B,\varepsilon} > 0$ such that*

$$\sup_{t \geqslant \varepsilon} \sup_{u_0 \in B} \|u(t, u_0)\|_{L^\infty(\mathbb{R}^N)} \leqslant \widetilde{R}_{B,\varepsilon}. \tag{5.12}$$

Proof Observe that (3.9) implies an a priori bound of orbits in $H^2(\mathbb{R}^N)$. For $N \leqslant 3$, this automatically yields the $L^\infty(\mathbb{R}^N)$ bound. In view of Lemma 5.1 for $N \geqslant 4$ it suffices to show (5.6) uniform with respect to $u_0 \in B$ and (5.7). For $N = 4$, by (3.9), we get the a priori bound in $L^{p_0}(\mathbb{R}^N)$ with arbitrary $p_0 \geqslant 2$. Finally, for $N \geqslant 5$, it implies a bound in $L^{p_0}(\mathbb{R}^N)$ with $p_0 = \frac{2N}{N-4}$. Note that $p_0 > \frac{N}{N-2}$ and $p_0 \geqslant \frac{N}{2}(\rho - 1)$ due to (1.17). Therefore, for any $N \geqslant 4$, it follows that (5.6) is satisfied uniformly for $u_0 \in B$ with some $p_0 \geqslant 2$ such that

$$p_0 > \frac{N}{N-2} \quad \text{and} \quad p_0 \geqslant \frac{N}{2}(\rho - 1). \tag{5.13}$$

Therefore, we are left to show (5.7) for $N \geqslant 4$ and $p \geqslant p_0$ using (5.13). To this end, we suppose that $u \in C_b([\varepsilon, \infty); L^p(\mathbb{R}^N))$ with some positive ε small enough. In particular, we have $u(\varepsilon) \in L^p(\mathbb{R}^N)$ and consider the problem (5.9) with $t_0 = \varepsilon$, $v_0 = u(\varepsilon)$, which by Lemma 5.2 and (5.5) has a unique maximally defined $L^p(\mathbb{R}^N)$ solution such that

$$v \in C([\varepsilon, \tau_{v_0}); L^p(\mathbb{R}^N)) \cap C((\varepsilon, \tau_{v_0}); E_p^{\beta(p)}) \cap C^1((\varepsilon, \tau_{v_0}); L^p(\mathbb{R}^N)).$$

Since $u \in C([\varepsilon, \infty); L^p(\mathbb{R}^N))$ is also an $L^p(\mathbb{R}^N)$ solution of the same problem, these functions must coincide. In particular, we have $u \in C([2\varepsilon, \infty); H_p^{4\beta}(\mathbb{R}^N))$ with $\beta = \beta(p)$ as in (5.5).

We will follow some of the arguments of the proof of [7, Proposition 3.3], where (1.6) with $\gamma = \delta = 0$ was discussed. Firstly, we write the Duhamel's formula corresponding to (5.2) on a short interval

$$u(t) = e^{-\mathcal{A}(t-\varepsilon)}u(\varepsilon) + \int_\varepsilon^t e^{-\mathcal{A}(t-s)}[\mathcal{F}_1(u(s)) + \mathcal{F}_2(u(s)) + \mathcal{F}_3(u(s))]ds, \ \varepsilon < t \leqslant 1$$

with sufficiently small $0 < \varepsilon \leqslant \frac{1}{2}$ and estimate it in $H_p^{4\beta}(\mathbb{R}^N) = E_p^\beta$ with $\beta = \beta(p)$. Note that by (5.3), we get in particular

$$\left\| e^{-\mathcal{A}(t-\varepsilon)}u(\varepsilon) \right\|_{E_p^\beta} \leqslant \frac{c}{(t-\varepsilon)^\beta} \|u(\varepsilon)\|_{L^p(\mathbb{R}^N)} \leqslant \frac{c}{(t-\varepsilon)^\beta} \|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))},$$

$$\left\| \int_\varepsilon^t e^{-\mathcal{A}(t-s)}\mathcal{F}_1(u(s))ds \right\|_{E_p^\beta} \leqslant \frac{c}{(t-\varepsilon)^\beta} \left(\|g\|_{L^p(\mathbb{R}^N)} + \|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))} \right),$$

$$\left\| \int_\varepsilon^t e^{-\mathcal{A}(t-s)}\mathcal{F}_2(u(s))ds \right\|_{E_p^\beta} \leqslant \frac{c}{(t-\varepsilon)^\beta} \|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))}^{(1-\theta)\rho} \left(\sup_{s \in (\varepsilon,t]} \left\| (s-\varepsilon)^\beta u(s) \right\|_{E_p^\beta} \right)^{\theta\rho}$$

with some $\theta \in (0,1)$ such that $\theta\rho \in (0,1)$, since $p > \frac{N}{N-2}$ and $p \geqslant \frac{N}{2}(\rho - 1)$. Indeed, considering only $\rho > 1$ we have

$$\left\| \int_\varepsilon^t e^{-\mathcal{A}(t-s)}\mathcal{F}_2(u(s))ds \right\|_{E_p^\beta} \leqslant c \int_\varepsilon^t \frac{1}{(t-s)^{\beta-\zeta}} \|u(s)\|_{L^{q\rho}(\mathbb{R}^N)}^\rho ds$$

for some $-\beta_*(p) \leqslant \zeta \leqslant 0$ such that $\beta - 1 < \zeta$ provided that $4\zeta - \frac{N}{p} \leqslant -\frac{N}{q}$ and $p \geqslant q > 1$. By the interpolation inequality (see [29, 2.4.2/11, 1.9.3/3]), we further obtain

$$\left\|\int_\varepsilon^t e^{-\mathcal{A}(t-s)}\mathcal{F}_2(u(s))ds\right\|_{E_p^\beta} \leqslant c\int_\varepsilon^t \frac{1}{(t-s)^{\beta-\zeta}}\left\|u(s)\right\|_{H_p^{4\beta}(\mathbb{R}^N)}^{\theta\rho}\left\|u(s)\right\|_{L^p(\mathbb{R}^N)}^{(1-\theta)\rho} ds$$

if $4\beta\theta - \frac{N}{p} \geqslant -\frac{N}{q\rho}$ and $p \leqslant q\rho$ for some $\theta \in (0,1)$, which we additionally want such that $0 < \theta\rho < 1$.

Since $p > \frac{N}{N-2}$ and $p \geqslant \frac{N}{2}(\rho - 1)$, then $1 < \frac{Np}{N+2p} \leqslant \frac{p}{\rho}$ and all above restrictions will be fulfilled if we find

$$-\beta_*(p) \leqslant \zeta \leqslant 0, \quad \beta - 1 < \zeta, \tag{5.14}$$

and $q \in (\frac{p}{\rho}, p]$ such that

$$4\zeta - \frac{N}{p} \leqslant -\frac{N}{q} < 4\beta - \frac{N\rho}{p}. \tag{5.15}$$

Taking $q \in (\frac{p}{\rho}, p]$ such that $\frac{1}{q} > \frac{\rho}{p} - \frac{4\beta}{N}$, we get $-2 - \frac{N}{p} < -\frac{N}{q} < 4\beta - \frac{N\rho}{p}$. If $\beta = \frac{1}{2}$, we take $\zeta > \beta - 1 = -\frac{1}{2} > -\beta_*(p)$ so close to $-\frac{1}{2}$ that $-2 - \frac{N}{p} < 4\zeta - \frac{N}{p} < -\frac{N}{q}$, whereas if $0 < \beta < \frac{1}{2}$ we simply choose $\zeta = -\frac{1}{2} > \beta - 1$, hence fulfilling both (5.14) and (5.15). Then the claim follows noticing that

$$\int_\varepsilon^t \frac{(t-\varepsilon)^\beta}{(t-s)^{\beta-\zeta}(s-\varepsilon)^{\beta\theta\rho}}ds = (t-\varepsilon)^{1+\zeta-\beta\theta\rho}B(1-\beta\theta\rho, 1+\zeta-\beta) \leqslant B(1-\beta\theta\rho, 1+\zeta-\beta),$$

where we used Euler's Beta function.

For our problem, we also need to estimate the term with $\mathcal{F}_3$. Note that we have

$$\left\|\int_\varepsilon^t e^{-\mathcal{A}(t-s)}\mathcal{F}_3(u(s))ds\right\|_{E_p^\beta} \leqslant c\int_\varepsilon^t \frac{e^{-\omega(t-s)}}{(t-s)^{\beta-\xi}}\left\|u(s)\right\|_{H_p^{4\xi+2}(\mathbb{R}^N)} ds$$

for some $\xi = \xi(p) \in [-\beta_*(p), 0]$ such that $\beta - 1 < \xi$.

Observe further that

$$\|u\|_{H_p^{4\xi+2}(\mathbb{R}^N)} \leqslant c\,\|u\|_{H_p^{4\beta}(\mathbb{R}^N)}^{\theta_1}\|u\|_{L^p(\mathbb{R}^N)}^{1-\theta_1},\ u \in H_p^{4\beta}(\mathbb{R}^N), \tag{5.16}$$

provided that $4\theta_1\beta \geqslant 4\xi + 2$ for some $\theta_1 \in (0,1)$. Since $0 < \beta \leqslant \frac{1}{2}$ and $\beta_*(p) > \frac{1}{2}$, these restrictions are fulfilled with

$$-\frac{1}{2} < \xi < \beta - \frac{1}{2} \ \text{ and } \ \frac{1}{\beta}\big(\xi + \frac{1}{2}\big) < \theta_1 < 1. \tag{5.17}$$

Consequently, we obtain

$$\left\|\int_\varepsilon^t e^{-\mathcal{A}(t-s)}\mathcal{F}_3(u(s))ds\right\|_{E_p^\beta} \leqslant \frac{c}{(t-\varepsilon)^\beta}\,\|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))}^{1-\theta_1}\Big(\sup_{s\in(\varepsilon,t]}\left\|(s-\varepsilon)^\beta u(s)\right\|_{E_p^\beta}\Big)^{\theta_1},$$

since as before

$$\int_\varepsilon^t \frac{(t-\varepsilon)^\beta}{(t-s)^{\beta-\xi}(s-\varepsilon)^{\beta\theta_1}}ds \leqslant B(1-\beta\theta_1, 1+\xi-\beta),\ \varepsilon < t \leqslant 1.$$

Combining these estimates and denoting $z = \sup_{t\in(\varepsilon,1]} \left\|(t-\varepsilon)^\beta u(t)\right\|_{E^\beta_p}$, if $z \geqslant 1$, we get

$$z \leqslant c\left(\|g\|_{L^p(\mathbb{R}^N)} + \|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))}\right) + \widetilde{L}(\|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))})z^{\theta_2},$$

with $\theta_2 = \max\{\theta\rho, \theta_1\} \in (0,1)$ and a nonnegative continuous function $\widetilde{L}$. In particular, this gives

$$\varepsilon^\beta \|u(2\varepsilon)\|_{E^\beta_p} \leqslant z \leqslant \max\{1, z_0(\|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))})\}, \tag{5.18}$$

where $z_0(h)$ is the only nonnegative root of $z = c(\|g\|_{L^p(\mathbb{R}^N)} + h) + \widetilde{L}(h)z^{\theta_2}$.

Now we estimate in E^β_p the Duhamel's formula corresponding to (5.2) on an unbounded interval

$$u(t) = e^{-\mathcal{A}(t-2\varepsilon)}u(2\varepsilon) + \int_{2\varepsilon}^t e^{-\mathcal{A}(t-s)}[\mathcal{F}_1(u(s)) + \mathcal{F}_2(u(s)) + \mathcal{F}_3(u(s))]ds,\ t \geqslant 2\varepsilon.$$

This time, by (5.3) and (5.18), we have for $t \geqslant 2\varepsilon$

$$\left\|e^{-\mathcal{A}(t-2\varepsilon)}u(2\varepsilon)\right\|_{E^\beta_p} \leqslant M\|u(2\varepsilon)\|_{E^\beta_p} \leqslant M\varepsilon^{-\beta}\max\{1, z_0(\|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))})\}. \tag{5.19}$$

By (5.3), we also obtain

$$\left\|\int_{2\varepsilon}^t e^{-\mathcal{A}(t-s)}\mathcal{F}_1(u(s))ds\right\|_{E^\beta_p} \leqslant c\big(\|g\|_{L^p(\mathbb{R}^N)} + \|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))}\big), \tag{5.20}$$

and, using the fact that $p > \frac{N}{N-2}$ and $p \geqslant \frac{N}{2}(\rho - 1)$ and reasoning as before, we get

$$\left\|\int_{2\varepsilon}^t e^{-\mathcal{A}(t-s)}\mathcal{F}_2(u(s))ds\right\|_{E^\beta_p} \leqslant c\|u\|^{(1-\theta)\rho}_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))}\left(\sup_{s\in[2\varepsilon,t]}\|u(s)\|_{E^\beta_p}\right)^{\theta\rho}, \tag{5.21}$$

with some $\theta \in (0,1)$ such that $\theta\rho \in (0,1)$ and some $\zeta > \beta - 1$, since

$$\int_{2\varepsilon}^t \frac{e^{-\omega(t-s)}}{(t-s)^\beta}ds \leqslant \frac{\Gamma(1-\beta)}{\omega^{1-\beta}} \quad \text{and} \quad \int_{2\varepsilon}^t \frac{e^{-\omega(t-s)}}{(t-s)^{\beta-\zeta}}ds \leqslant \frac{\Gamma(1-\beta+\zeta)}{\omega^{1-\beta+\zeta}}.$$

Moreover, using (5.16) with ξ, θ_1 as in (5.17), we estimate the term with $\mathcal{F}_3$ by

$$\left\|\int_{2\varepsilon}^t e^{-\mathcal{A}(t-s)}\mathcal{F}_3(u(s))ds\right\|_{E^\beta_p} \leqslant c\|u\|^{1-\theta_1}_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))}\left(\sup_{s\in[2\varepsilon,t]}\|u(s)\|_{E^\beta_p}\right)^{\theta_1}. \tag{5.22}$$

Setting $z(t) = \sup_{s\in[2\varepsilon,t]}\|u(s)\|_{E^\beta_p}$, we join (5.19), (5.20), (5.21), (5.22) and if $z(t) \geqslant 1$, we get

$$z(t) \leqslant a(\varepsilon, \|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))}) + \overline{L}(\|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))})z(t)^{\theta_2}$$

with $\theta_2 = \max\{\theta\rho, \theta_1\} \in (0,1)$, a nonnegative continuous function $\overline{L}$ and

$$a(\varepsilon, h) = c(M\varepsilon^{-\beta}\max\{1, z_0(h)\} + \|g\|_{L^p(\mathbb{R}^N)} + h).$$

This gives

$$\|u(t)\|_{E_p^\beta} \leqslant z(t) \leqslant \max\{1, z_1(\varepsilon, \|u\|_{C_b([\varepsilon,\infty);L^p(\mathbb{R}^N))})\},\ t \geqslant 2\varepsilon, \tag{5.23}$$

where $z_1(\varepsilon, h)$ is the only nonnegative root of the equation $z = a(\varepsilon, h) + \overline{L}(h)z^{\theta_2}$. Thus, (5.23) yields (5.7), which ends the proof. □

Theorem 5.4 *Under the assumptions of Theorem 4.5 with ρ satisfying (1.17) and g as in (5.1), the global attractor $\mathbf{A}$ from Theorem 4.5 is bounded in $L^\infty(\mathbb{R}^N)$ and there exists a positively invariant bounded absorbing set $\mathbf{B}$ in $L^\infty(\mathbb{R}^N)$.*

Proof By (5.12), $\mathbf{A} = S(\varepsilon)\mathbf{A}$ is contained in a closed ball in $L^\infty(\mathbb{R}^N)$ centered at 0 and of some radius $\widetilde{R}_{\mathbf{A},\varepsilon}$. If $\mathbf{B}_0$ denotes an ε_0-neighborhood of $\mathbf{A}$ in X^α, then

$$\mathbf{B} = \bigcup_{t \geqslant T_{\mathbf{B}_0}} S(t)\mathbf{B}_0,$$

with $T_{\mathbf{B}_0} > 0$ being an absorption time of $\mathbf{B}_0$, is a positively invariant bounded absorbing set, which is bounded in $L^\infty(\mathbb{R}^N)$ again by (5.12). □

We now gather the results to complete the proof of Theorem 1.1 announced in Introduction.

Proof of Theorem 1.1 Under the assumptions of Theorem 1.1, the problem (1.6),(1.7) generates a C^0 semigroup of global X^α, $\alpha \in [\frac{1}{2}, 1)$, solutions with orbits of bounded sets bounded by Theorem 3.3 provided that γ and δ satisfy the condition (1.11). This semigroup is asymptotically compact by Proposition 4.2 and in consequence possesses the global attractor $\mathbf{A}$ coinciding with the unstable manifold $W^u(\mathcal{E})$ of the set of stationary solutions of (1.6) due to Theorem 4.5. Knowing that $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and ρ satisfies (1.17), it then follows from Theorem 5.4 that $\mathbf{A}$ is contained in a positively invariant bounded absorbing set from $L^\infty(\mathbb{R}^N)$, which completes the proof. □

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