SEMICONTINUITY OF ATTRACTORS FOR IMPULSIVE DYNAMICAL SYSTEMS

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Abstract. In this paper we introduce the concept of collective tube conditions which assures a suitable behaviour for a family of dynamical systems close to impulsive sets. Using the collective tube conditions, we develop the theory of upper and lower semicontinuity of global attractors for a family of impulsive dynamical systems.

1. Introduction

Perturbations are present in every aspect of the modelling of real world phenomena. Approximate measurements, data collecting, empirical laws and simplifications, for instance, are procedures that introduce small changes in the modelled problem. Such small errors are expected, but they need to be carefully treated. Otherwise, how can we assure that the properties obtained for the model also hold true for the real problem?

To answer this question, we need to study the *continuity* of such problems under small perturbations. We will focus on the following question: what can be said about the asymptotic behaviour of a problem (that is, the behaviour of solutions for large times t) if we make a small perturbation of it?

Even in the case of *continuous dynamical systems*, this question has a very non-trivial answer and the study of the perturbations is divided in the literature, in general, in four steps: the *upper semicontinuity*, the *lower semicontinuity*, the *topological stability* and, lastly, the *geometric stability* (see for instance [1, 2, 8, 9, 10, 11, 17, 18]). In this paper, we will deal mainly with the upper semicontinuity of impulsive dynamical systems and, also, we shall give some preliminary results on the lower semicontinuity.

We say that a family $\{A_\eta\}_{\eta\in[0,1]}$ of non-empty sets in a metric space (X, d) is **upper semicontinuous** at $\eta = 0$ if

$$\lim_{\eta\to 0} \mathrm{d}_H(A_\eta, A_0) = 0$$

and it is **lower semicontinuous** at $\eta = 0$ if

$$\lim_{\eta\to 0} \mathrm{d}_H(A_0, A_\eta) = 0,$$

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where

$$\mathrm{d}_H(A,B) = \sup_{a\in A} \inf_{b\in B} \mathrm{d}(a,b)$$

is the *Hausdorff semidistance* between two non-empty subsets A, B of X.

Roughly speaking, the upper semicontinuity property ensures that the solutions of the perturbed system do not "explode" and follow some solution of the limiting problem. The lower semicontinuity ensures that the solutions of the perturbed system do not "implode" and the perturbed system has, at least, the same degree of complexity that the limiting system.

If one is familiar with the theory of impulsive dynamical systems and their global attractors (see detailed results in [3] and additional results in [4, 5, 6, 7, 12, 13, 14, 15, 16, 19, 20, 22]), there is a natural question to ask: how can we talk about continuity under perturbations of systems which have precisely the *discontinuity* as its main feature?

To answer this question we must remind that, basically, an impulsive dynamical system is formed by a continuous dynamical system and a continuous *impulsive function* (or *jump function*), which gives rise to a discontinuous semiflow, that is, for each initial state, the solution has "jumps" and it is clearly discontinuous. But when we look at *the whole impulsive semiflow*, if the continuous semiflow and the jump function behave continuously under perturbations, there is no reason why the impulsive semiflow would not behave the same. Realizing this, one can see that the study of continuity of impulsive dynamical systems is not a contradictory statement by itself, and involves the study of perturbations of continuous semiflows as well as the study of perturbations of the impulsive functions. This will be the main goal of this work, that is, to study in details the upper semicontinuity of global attractors for impulsive dynamical systems and give a first step towards the study of their lower semicontinuity.

This work begins with some basic concepts and preliminary results on impulsive dynamical systems, presented in Section 2. This section is divided in three subsections, for an easier reading. The first subsection is devoted to the definitions of an *impulsive dynamical system* and *impulsive positive trajectories*. Next, we present the so-called *tube conditions*, which are crucial for this theory, and finally we present the definitions and recent results on *global attractors* for impulsive systems.

As we said before, the *tube conditions* are crucial for the development of the theory of impulsive systems, and we must be able to reproduce these conditions when we work with perturbations. This is the main goal in Section 3, where we introduce the *collective tube conditions*. Furthermore, we study the continuity of the *impact time maps* for a family of impulsive dynamical systems.

The main result of this work, namely Theorem 4.2, is presented in Section 4, where we provide conditions that ensure the upper semicontinuity for a family of global attractors of impulsive dynamical systems.

In Section 5, we show an application of the previous theorem in a coupled system of ODEs with impulses. Finally, in Section 6, we give probably the first step towards the understanding of the lower semicontinuity for a family of global attractors of impulsive systems. We present a result of lower semicontinuity (Theorem 6.3) in a simple case, where the critical elements of the continuous semigroups are a finite number of equilibria.

2. Preliminaries

In this section, we present the basic definitions and results of the theory of impulsive dynamical systems.

2.1. Impulsive dynamical systems.

Let (X, d) be a metric space, $\mathbb{R}_+$ be the set of non-negative real numbers, $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 2.1. A **semigroup** (or **semiflow**) in X is a family of continuous maps $\{\pi(t)\colon t \geqslant 0\}$ from X to itself, indexed on $\mathbb{R}_+$, satisfying the following conditions:

- **(i)** $\pi(0)x = x$ for all $x \in X$;
- **(ii)** $\pi(t+s) = \pi(t)\pi(s)$ for all $t, s \in \mathbb{R}_+$;[1]
- **(iii)** the map $\mathbb{R}_+ \times X \ni (t, x) \mapsto \pi(t)x$ is continuous.

If $\mathbb{R}_+$ is replaced by $\mathbb{R}$ in this definition, the family $\{\pi(t)\colon t \in \mathbb{R}\}$ is called a **group** (or **flow**) in X.

Let $\{\pi(t)\colon t \geqslant 0\}$ be a semigroup in X. For each $D \subset X$ and $J \subset \mathbb{R}_+$ we define

$$F(D, J) = \bigcup_{t \in J} \pi(t)^{-1} D.$$

According to [21], a point $x \in X$ is called a **start event** if $F(x, t) = \varnothing$ for all $t > 0$. A start event is also known as an initial point, see [3, 4, 5, 6, 7, 12, 13, 15].

Definition 2.2. An **impulsive dynamical system** (IDS, for short) (X, π, M, I) consists of a semigroup $\{\pi(t)\colon t \geqslant 0\}$ on a metric space (X, d), a non-empty closed subset $M \subset X$ such that for every $x \in M$ there exists $\epsilon_x > 0$ such that

$$F(x, (0, \epsilon_x)) \cap M = \varnothing \quad \text{and} \quad \bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \varnothing, \tag{2.1}$$

and a continuous function $I\colon M \to X$ (its role will be specified below).

The set M is called the **impulsive set** and the function I is called the **impulsive function**.

Remark 2.3. Condition (2.1) means that the flow of the semigroup $\{\pi(t)\colon t \geqslant 0\}$ is, in some sense, transversal to M at any point of M.

[1]In this paper $\pi(t)\pi(s)$ denotes the composition $\pi(t) \circ \pi(s)$ and the composition sign "$\circ$" is omitted.

We also define

$$M^+(x) = \left(\bigcup_{t>0} \pi(t)x\right) \cap M.$$

It follows immediately from the definition of $M^+(x)$ and (2.1) that if $M^+(x) \neq \varnothing$, then there exists $s > 0$ such that $\pi(s)x \in M$ and $\pi(t)x \notin M$ for $0 < t < s$. Thus, we are able to define the function $\phi\colon X \to (0, +\infty]$ by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(s)x \in M \text{ and } \pi(t)x \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \varnothing. \end{cases} \tag{2.2}$$

If $M^+(x) \neq \varnothing$, the value $\phi(x)$ represents the smallest positive time such that the positive trajectory of x meets M, which we will call the **impact time map**. In this case, we say that the point $\pi(\phi(x))x$ is the **impulsive point** of x.

The **impulsive positive trajectory** of $x \in X$ by the IDS (X, π, M, I) is a map $\tilde{\pi}(\cdot)x$ defined in an interval $J_x \subset \mathbb{R}_+$, $0 \in J_x$, with values in X given inductively by the following rule: if $M^+(x) = \varnothing$, then $\tilde{\pi}(t)x = \pi(t)x$ for all $t \in \mathbb{R}_+$. However, if $M^+(x) \neq \varnothing$ then we denote $x = x_0^+$ and we define $\tilde{\pi}(\cdot)x$ on $[0, \phi(x_0^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t)x_0^+, & \text{if } \quad 0 \leqslant t < \phi(x_0^+), \\ I(\pi(\phi(x_0^+))x_0^+), & \text{if } \quad t = \phi(x_0^+). \end{cases}$$

Now let $s_0 = \phi(x_0^+)$, $x_1 = \pi(s_0)x_0^+$ and $x_1^+ = I(\pi(s_0)x_0^+)$. In this case $s_0 < +\infty$ and the process can go on, but now starting at x_1^+. If $M^+(x_1^+) = \varnothing$ then we define $\tilde{\pi}(t)x = \pi(t - s_0)x_1^+$ for $s_0 \leqslant t < +\infty$ and we have $\phi(x_1^+) = +\infty$. However, if $M^+(x_1^+) \neq \varnothing$ we define $\tilde{\pi}(\cdot)x$ on $[s_0, s_0 + \phi(x_1^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - s_0)x_1^+, & \text{if } \quad s_0 \leqslant t < s_0 + \phi(x_1^+), \\ I(\pi(\phi(x_1^+))x_1^+), & \text{if } \quad t = s_0 + \phi(x_1^+). \end{cases}$$

Define $s_1 = \phi(x_1^+)$, $x_2 = \pi(s_1)x_1^+$ and $x_2^+ = I(\pi(s_1)x_1^+)$. Assume now that $\tilde{\pi}(\cdot)x$ is defined on the interval $[t_{n-1}, t_n]$ and that $\tilde{\pi}(t_n)x = x_n^+$, where $t_0 = 0$ and $t_n = \sum_{i=0}^{n-1} s_i$ for $n \in \mathbb{N}$. If $M^+(x_n^+) = \varnothing$, then $\tilde{\pi}(t)x = \pi(t - t_n)x_n^+$ for $t_n \leqslant t < +\infty$ and $\phi(x_n^+) = +\infty$. However, if $M^+(x_n^+) \neq \varnothing$, then we define $\tilde{\pi}(\cdot)x$ on $[t_n, t_n + \phi(x_n^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - t_n)x_n^+, & \text{if } \quad t_n \leqslant t < t_n + \phi(x_n^+), \\ I(\pi(\phi(x_n^+))x_n^+), & \text{if } \quad t = t_n + \phi(x_n^+), \end{cases}$$

and we set, inductively, $s_n = \phi(x_n^+)$, $x_{n+1} = \pi(s_n)x_n^+$ and $x_{n+1}^+ = I(\pi(s_n)x_n^+)$. This process ends after a finite number of steps if $M^+(x_n^+) = \varnothing$ for some $n \in \mathbb{N}_0$, or it may proceed indefinitely if $M^+(x_n^+) \neq \varnothing$ for all $n \in \mathbb{N}$ and in this case $\tilde{\pi}(\cdot)x$ is defined in the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{+\infty} s_i$.

We shall assume throughout this paper the following global existence condition:

$$T(x) = +\infty \text{ for all } x \in X. \qquad (\mathbf{G})$$

Note that $\tilde{\pi}(0)x = x$ for all $x \in X$. It is simple to see that if hypothesis ($\mathbf{G}$) holds then the family $\{\tilde{\pi}(t) \colon t \geqslant 0\}$ of maps satisfies an analogous property to (ii) of Definition 2.1, that is:

$$\tilde{\pi}(t+s)x = \tilde{\pi}(t)\tilde{\pi}(s)x \text{ for all } t, s \in \mathbb{R}_+ \text{ and } x \in X.$$

Remark 2.4.

1. If there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$, then the condition ($\mathbf{G}$) is verified. This assumption says that there is a positive lower bound for time which the semigroup π takes to reach M, when leaving from $I(M)$, and it is satisfied in several examples, for instance, when $I(M)$ is compact and $I(M) \cap M = \varnothing$ (see [3]).
2. Some important and interesting cases are the impulsive dynamical systems in which the impulsive trajectory is defined for all $t \in \mathbb{R}$. In many cases we may restrict ourselves to such systems, due to the existence of suitable isomorphisms (the reader may see [14]).

2.2. **Tube conditions for impulsive dynamical systems.**

In order to obtain important topological properties for impulsive systems which have counterparts in continuous systems, we must ensure that the original semiflow $\{\pi(t) \colon t \geqslant 0\}$ behaves nicely near to the impulsive set M. Therefore, we define the so called "tube conditions" (see [12] for more details).

Definition 2.5. Let $\{\pi(t) \colon t \geqslant 0\}$ be a semigroup on X. A closed set S containing $x \in X$ is called a **section** through x if there exist $\lambda > 0$ and a closed subset L of X such that:

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ contains a neighbourhood of x;
(c) $F(L, \nu) \cap F(L, \zeta) = \varnothing$ if $0 \leqslant \nu < \zeta \leqslant 2\lambda$.

We say that the set $F(L, [0, 2\lambda])$ is a $\lambda-$**tube** (or simply **tube**) and the set L is a **bar**.

Lemma 2.6. *If S is a section and $\lambda > 0$ is given as in the previous definition, then any $0 < \mu \leqslant \lambda$ satisfies conditions (a), (b) and (c) above with L replaced by $L_\mu = F(L, \lambda - \mu)$ and λ replaced by μ.*

Proof. See [15, Lemma 1.9]. □

Definition 2.7. Let (X, π, M, I) be an IDS. We say that a point $x \in M$ satisfies the **strong tube condition (STC)** if there exists a section S through x such that $S = F(L, [0, 2\lambda]) \cap M$. Also, we say that a point $x \in M$ satisfies the **special strong tube condition (SSTC)** if it satisfies STC and the λ-tube $F(L, [0, 2\lambda])$ is such that $F(L, [0, \lambda]) \cap I(M) = \varnothing$.

The strong tube conditions are the key notions of the theory developed in [3, 4, 5] and also give us both of the following results.

Theorem 2.8. *Let (X, π, M, I) be an IDS such that each point of M satisfies* STC. *Then ϕ is upper semicontinuous in X and it is continuous in $X \setminus M$. Moreover, if there are no start events in M and ϕ is continuous at x then $x \notin M$.*

Proof. See [12, Theorems 3.4, 3.5, 3.8]. ☐

Proposition 2.9. *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$ and let $y \in M$ satisfy* SSTC *with λ-tube $F(L, [0, 2\lambda])$. Then $\tilde{\pi}(t)X \cap F(L, [0, \lambda]) = \varnothing$ for all $t > \lambda$.*

Proof. See [3, Proposition 2.6]. ☐

2.3. **Global attractors for impulsive dynamical systems.**

In this subsection we present the definition of a global attractor for an impulsive dynamical system and an existence result, which can be found in details in [3].

We say that a subset A of X is **$\tilde{\pi}$-invariant** if $\tilde{\pi}(t)A = A$ for all $t \geqslant 0$. Also, we say that A **$\tilde{\pi}$-attracts** $B \subset X$ if

$$\lim_{t \to +\infty} \mathrm{d}_H(\tilde{\pi}(t)B, A) = 0.$$

With these concepts, we can present the definition of a *global attractor* for the IDS (X, π, M, I), which was first introduced in [3].

Definition 2.10. A subset $\mathcal{A} \subset X$ is called a **global attractor** for the IDS (X, π, M, I) if it satisfies the following conditions:

- **(i)** $\mathcal{A}$ is precompact and $\mathcal{A} = \overline{\mathcal{A}} \setminus M$;
- **(ii)** $\mathcal{A}$ is $\tilde{\pi}$-invariant;
- **(iii)** $\mathcal{A}$ $\tilde{\pi}$-attracts all bounded subsets of X.

To prove the existence of global attractors, we formulate the following:

Definition 2.11. An impulsive dynamical system (X, π, M, I) is called **strongly bounded dissipative** if there exists a non-empty precompact set K in X such that $K \cap M = \varnothing$ and $\tilde{\pi}$-absorbs all bounded subsets of X, i.e., for any bounded subset B of X there exists $t_B \geqslant 0$ such that $\tilde{\pi}(t)B \subset K$ for all $t \geqslant t_B$.

With this definition, we are able to present a result on the existence of global attractors for impulsive dynamical systems.

Theorem 2.12. *Let (X, π, M, I) be a strongly bounded dissipative IDS with $\tilde{\pi}$-absorbing set K such that $I(M) \cap M = \varnothing$, every point in M satisfies* SSTC *and there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. Then (X, π, M, I) has a global attractor $\mathcal{A}$ and we have $\mathcal{A} = \tilde{\omega}(K) \setminus M$, where $\tilde{\omega}(K)$ is the impulsive ω-limit of K, i.e.,*

$$\tilde{\omega}(K) = \bigcap_{t \geqslant 0} \overline{\bigcup_{s \geqslant t} \tilde{\pi}(s)K}.$$

To see the relation of this global attractor with its counterpart in the continuous case, we will use the characterization via *global solutions*. We say that a function $\psi\colon \mathbb{R} \to X$ is a **global solution** of $\tilde{\pi}$ (or a $\tilde{\pi}$**-global solution**) if

$$\tilde{\pi}(t)\psi(s) = \psi(t+s) \text{ for all } t \geqslant 0 \text{ and } s \in \mathbb{R}. \tag{2.3}$$

Moreover, if $\psi(0) = x$ we say that ψ is a $\tilde{\pi}$**-global solution through** x. We say that ψ is **bounded** if $\psi(\mathbb{R})$ is a bounded subset of X.

Thus, we have the same characterization of the global attractor for the impulsive case as in the continuous case, which is the content of the next result.

Proposition 2.13. *If the IDS (X, π, M, I) has a global attractor $\mathcal{A}$ and $I(M) \cap M = \varnothing$ then*

$$\mathcal{A} = \{x \in X\colon \textit{there exists a bounded global solution of } \tilde{\pi} \textit{ through } x\}.$$

Proof. See [3, Proposition 4.3]. ☐

Remark 2.14. Using the proof of Proposition 2.13 (see [3, Proposition 4.3]), the bounded global solution through $x \in \mathcal{A}$ is given by

$$\psi(t) = \begin{cases} \tilde{\pi}(t+n)x_{-n}, & \text{if } t \in [-n, -n+1],\ n \in \mathbb{N}, \\ \tilde{\pi}(t)x_0, & \text{if } t \geqslant 0, \end{cases}$$

where $x_0 = x$ and $\tilde{\pi}(1)x_{-n-1} = x_{-n}$ for all $n \in \mathbb{N}_0$.

For more properties of global attractors for impulsive dynamical systems we refer the reader to [3].

3. Collective tube conditions and impact time maps

We now focus on the problem of defining suitable tube conditions for a family of impulsive dynamical systems $\{(X, \pi_\eta, M_\eta, I_\eta)\}_{\eta \in [0,1]}$ in such a way that the property of upper semicontinuity will hold. Also, using these tube conditions, we will deal with the family of impact time maps - recall (2.2) - generated by this family of systems.

3.1. **Collective tube conditions.**

In this subsection we establish some *collective tube conditions* for a family of impulsive dynamical systems $\{(X, \pi_\eta, M_\eta, I_\eta)\}_{\eta \in [0,1]}$ so that the semiflows π_η and $\tilde{\pi}_\eta$ have suitable behaviours in their evolutions. From now on, we shall assume the following general conditions[2]: first, the continuity with respect to the parameter η of the continuous semigroups π_η given by

$$\pi_\eta(t)x \xrightarrow{\eta \to 0} \pi_0(t)x \ \text{ uniformly for } (t, x) \text{ in compact subsets of } \mathbb{R}_+ \times X. \tag{\textbf{C1}}$$

Also, we assume the continuity of the impulsive sets M_η, which is given by

$$\mathrm{d}_H(M_\eta, M_0) + \mathrm{d}_H(M_0, M_\eta) \xrightarrow{\eta \to 0} 0 \tag{\textbf{C2}}$$

[2]Along with condition (**G**) for each $\eta \in [0, 1]$.

and a *collective continuity* of the impulsive functions I_η:

$$\begin{aligned} &\text{given } \varepsilon > 0 \text{ and } w_0 \in M_0 \text{ there exists } \delta > 0 \text{ such that if } \eta \in [0,\delta), \\ &\qquad w \in M_\eta \text{ and } \mathrm{d}(w, w_0) < \delta \text{ then } \mathrm{d}(I_\eta(w), I_0(w_0)) < \varepsilon. \end{aligned} \tag{\textbf{C3}}$$

Finally, we assume that

$$\text{there exists } \overline{\eta} \in (0,1] \text{ such that } I_\eta(M_\eta) \cap M_\eta = \varnothing \quad \text{for all } \eta \in [0, \overline{\eta}). \tag{\textbf{C4}}$$

Condition (**C2**) provides us a simple and useful result.

Lemma 3.1. *If* $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ *and* $\{w_k\}_{k\in\mathbb{N}} \subset X$ *are sequences such that* $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$, $w_k \in M_{\eta_k}$ *for* $k \in \mathbb{N}$ *and* $w_k \stackrel{k\to+\infty}{\longrightarrow} w_0$, *then* $w_0 \in M_0$.

Proof. From (**C2**) we have

$$\mathrm{d}_H(w_0, M_0) = \inf_{z\in M_0} \mathrm{d}(w_0, z) \leqslant \inf_{z\in M_0} (\mathrm{d}(w_0, w_k) + \mathrm{d}(w_k, z)) \leqslant \mathrm{d}(w_0, w_k) + \mathrm{d}_H(M_{\eta_k}, M_0) \stackrel{k\to+\infty}{\longrightarrow} 0,$$

that is, $w_0 \in M_0$, since M_0 is closed. □

Remark 3.2. If M_0 is compact then we may replace condition (**C4**) by the following condition:

$$I_0(M_0) \cap M_0 = \varnothing. \tag{\textbf{C4'}}$$

It can be done since the relaxed condition (**C4'**) implies condition (**C4**). In fact, if there exists a sequence $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0^+$ with $w_k \in I_{\eta_k}(M_{\eta_k}) \cap M_{\eta_k}$, we have $w_k = I_{\eta_k}(z_k)$ for some $z_k \in M_{\eta_k}$. Now, using (**C2**) and the compactness of M_0 we may assume (taking subsequences if necessary) that $w_k \stackrel{k\to+\infty}{\longrightarrow} w_0 \in M_0$ and $z_k \stackrel{k\to+\infty}{\longrightarrow} z_0 \in M_0$. Hence, (**C3**) implies that $w_0 = I_0(z_0) \in I_0(M_0) \cap M_0$, which contradicts (**C4'**).

In the sequel, we introduce a more specific collective tube condition to assure that the semigroup $\{\pi_\eta(t)\colon t \geqslant 0\}$ behaves nicely near to its associated impulsive set M_η when $\eta \to 0$.

Definition 3.3. Let $\{(X, \pi_\eta, M_\eta, I_\eta)\}_{\eta\in[0,1]}$ be a family of impulsive dynamical systems. We say that a point $w_0 \in M_0$ satisfies the **collective strong tube condition** (**C-STC**) if given a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ such that $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$ and a sequence of points $w_k \in M_{\eta_k}$, $k \in \mathbb{N}$, with $w_k \stackrel{k\to+\infty}{\longrightarrow} w_0$, there exists $\lambda_0 > 0$ such that for each $0 < \lambda \leqslant \lambda_0$ one can find $\delta = \delta(\lambda) > 0$ such that $F_0(L_0, [0, 2\lambda])$ is a λ-tube through w_0 with section $S_0 = F_0(L_0, [0, 2\lambda]) \cap M_0$ such that $B(w_0, \delta) \subset F_0(L_0, [0, 2\lambda])$ and there exists $k_0 \in \mathbb{N}$ such that $\eta_k < \overline{\eta}$ for $k \geqslant k_0$ ($\overline{\eta}$ comes from (**C4**)) and we have a λ-tube $F_{\eta_k}(L_k, [0, 2\lambda])$ through w_k with section $S_k = F_{\eta_k}(L_k, [0, 2\lambda]) \cap M_{\eta_k}$ satisfying $B(w_k, \delta) \subset F_{\eta_k}(L_k, [0, 2\lambda])$ for $k \geqslant k_0$.

Definition 3.4. If, additionally, $F_0(L_0, [0, \lambda]) \cap I_0(M_0) = \varnothing$ and $F_{\eta_k}(L_k, [0, \lambda]) \cap I_{\eta_k}(M_{\eta_k}) = \varnothing$ for all $k \geqslant k_0$ in Definition 3.3, then we say that $w_0 \in M_0$ satisfies the **collective special strong tube condition** (**C-SSTC**).

To illustrate the previous concepts, we present a simple example.

Example 3.5. Consider the family of impulsive differential equations

$$\begin{cases} \dot{x} = -(1+\eta)x, & \eta \in [0,1], \\ I_\eta \colon M_\eta \to \mathbb{R}, \end{cases}$$

where $M_\eta = \{z+\eta \colon z \in \mathbb{N}\}$ and $I_\eta(z) = z + \eta - \frac{1}{2}$ for all $z \in M_\eta$ and $\eta \in [0,1]$.

Note that $\pi_\eta(t)x = xe^{-(1+\eta)t}$ for all $x \in \mathbb{R}$, $t \geqslant 0$ and $\eta \in [0,1]$, and conditions (**C1**)-(**C4**) are satisfied. Moreover, each point in M_0 satisfies C-SSTC. In fact, let $w_0 \in M_0$, $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$ and $\{w_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be such that $w_k \in M_{\eta_k}$ and $w_k \stackrel{k\to+\infty}{\longrightarrow} w_0$. Let $\eta_0 = 0$, $0 < \lambda_0 < \ln(\frac{2w_0}{2w_0-1})$, $0 < \lambda \leqslant \lambda_0$ and set $L_k = \{w_k e^{-(1+\eta_k)\lambda}\}$ for $k \in \mathbb{N}_0$. Note that there exists $k_0 \in \mathbb{N}$ such that $\eta_k < \frac{1}{2}$ for $k \geqslant k_0$, $F_{\eta_k}(L_k,[0,2\lambda]) = [w_k e^{-(1+\eta_k)\lambda}, w_k e^{(1+\eta_k)\lambda}]$ is a λ-tube through w_k, $F_{\eta_k}(L_k,[0,2\lambda]) \cap M_{\eta_k} = \{w_k\}$ and $F_{\eta_k}(L_k,[0,\lambda]) \cap I_{\eta_k}(M_{\eta_k}) = \varnothing$ for all $k \geqslant k_0$ and $k=0$.

Furthermore, for $0 < \delta < w_0(1-e^{-\lambda})$ we have $B(w_k,\delta) \subset F_{\eta_k}(L_k,[0,2\lambda])$ for all $k \geqslant k_0$ and $k=0$, which proves the claim.

From Definition 3.3 we have the following straightforward result.

Lemma 3.6. *Assume that $w_0 \in M_0$ satisfies* C-STC. *If a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ is such that $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$ and $w_k \in M_{\eta_k}$, $k \in \mathbb{N}$, is a sequence of points with $w_k \stackrel{k\to+\infty}{\longrightarrow} w_0$, then there exists an integer $k_1 \geqslant k_0$ such that $B(w_0,\frac{\delta}{2}) \subset B(w_k,\delta) \subset F_{\eta_k}(L_k,[0,2\lambda])$ for all $k \geqslant k_1$.*

Lemma 3.7. *Let (X,π_0,M_0,I_0) be an IDS such that X is locally compact and $\{\pi_0(t)\colon t \in \mathbb{R}\}$ is a group. Assume that $w_0 \in M_0$ satisfies* STC *with a λ-tube. Then it also satisfies* STC *with a compact λ-tube.*

Proof. Since w_0 satisfies STC there exist a tube $F_0(L_0,[0,2\lambda])$ through w_0 with section $S_0 = F_0(L_0,[0,2\lambda]) \cap M_0$ and $\delta > 0$ such that $B(w_0,\delta) \subset F_0(L_0,[0,2\lambda])$.

By the local compactness of X one can obtain $\epsilon > 0$ such that $\overline{B(w_0,\epsilon)}$ is compact. Now, let us define

$$\mathcal{S}_0 = S_0 \cap \overline{B(w_0,\epsilon)} \quad \text{and} \quad \mathcal{L}_0 = \pi_0(\lambda)\mathcal{S}_0.$$

Note that $\mathcal{S}_0$ and $\mathcal{L}_0$ are compact sets and $F_0(\mathcal{L}_0,\lambda) = \mathcal{S}_0$.

We claim that there is $\gamma > 0$ such that $B(w_0,\gamma) \subset F_0(\mathcal{L}_0,[0,2\lambda])$. Suppose to the contrary that there is a sequence $\{z_k\}_{k\in\mathbb{N}} \subset X$ such that $z_k \stackrel{k\to+\infty}{\longrightarrow} w_0$ and $z_k \notin F_0(\mathcal{L}_0,[0,2\lambda])$ for all $k \in \mathbb{N}$. Since $z_k \stackrel{k\to+\infty}{\longrightarrow} w_0$ there is an integer $\overline{k}_0 > 0$ such that $z_k \in B(w_0,\delta)$ for all $k \geqslant \overline{k}_0$. On the other hand, we have $B(w_0,\delta) \subset F_0(L_0,[0,2\lambda])$, which implies that there are $v_k \in S_0$ and $s_k \in [-\lambda,\lambda]$ such that

$$\pi_0(s_k)v_k = z_k \quad \text{for all} \quad k \geqslant \overline{k}_0.$$

We may assume that $s_k \stackrel{k\to+\infty}{\longrightarrow} s_0 \in [-\lambda,\lambda]$. Then

$$v_k = \pi_0(-s_k)z_k \stackrel{k\to+\infty}{\longrightarrow} \pi_0(-s_0)w_0$$

with $\pi_0(-s_0)w_0 \in S_0$, because S_0 is closed. But, by the tube condition, $\pi_0([-\lambda,\lambda])w_0 \cap M_0 = \{w_0\}$, which shows that $s_0 = 0$. Hence,

$$v_k \overset{k\to+\infty}{\longrightarrow} w_0.$$

Thus there is $\overline{k}_1 > \overline{k}_0$ such that $v_k \in S_0 \cap \overline{B(w_0,\epsilon)} = \mathcal{S}_0$ for all $k \geqslant \overline{k}_1$. Consequently,

$$z_k = \pi_0(s_k)v_k \in F_0(\mathcal{L}_0,[0,2\lambda]) \quad \text{for all} \quad k \geqslant \overline{k}_1,$$

which is a contradiction. This shows the claim.

It is not difficult to see that $F_0(\mathcal{L}_0,\mu) \cap F_0(\mathcal{L}_0,\nu) = \varnothing$ for $0 \leqslant \mu < \nu \leqslant 2\lambda$ and $\mathcal{S}_0 = F_0(\mathcal{L}_0,[0,2\lambda]) \cap M_0$. Hence, $w_0 \in M_0$ satisfies STC with the tube $F_0(\mathcal{L}_0,[0,2\lambda])$ through w_0 with $\mathcal{S}_0$ and $\mathcal{L}_0$ compact sets.

In order to see the compactness of the tube, let $\{w_k\}_{k\in\mathbb{N}}$ be a sequence in $F_0(\mathcal{L}_0,[0,2\lambda])$. Then there are $\alpha_k \in [-\lambda,\lambda]$ and $b_k \in \mathcal{S}_0$ such that $\pi_0(\alpha_k)b_k = w_k$ for all $k \in \mathbb{N}$. We may assume that $\alpha_k \overset{k\to+\infty}{\longrightarrow} \alpha_0 \in [-\lambda,\lambda]$ and $b_k \overset{k\to+\infty}{\longrightarrow} b_0 \in \mathcal{S}_0$. Then we have

$$w_k \overset{k\to+\infty}{\longrightarrow} \pi_0(\alpha_0)b_0 \in F_0(\mathcal{L}_0,[0,2\lambda])$$

which concludes the proof. □

In the next result, we present sufficient conditions to obtain C-STC in locally compact spaces.

Theorem 3.8. *Let $\{(X,\pi_\eta,M_\eta,I_\eta)\}_{\eta\in[0,1]}$ be a family of impulsive dynamical systems such that X is locally compact and $\{\pi_\eta(t)\colon t\in\mathbb{R}\}$ is a group for each $\eta\in[0,1]$. Assume that condition* (**C1**) *holds uniformly for (t,x) in compact subsets of $\mathbb{R}\times X$. Also, assume that the following conditions hold:*

- **(i)** *$w_0 \in M_0$ satisfies* STC *with respect to the group π_0;*
- **(ii)** *there are $\beta > 0$, $\delta_0 > 0$ and $\eta_0 > 0$ such that for $0 \leqslant \eta \leqslant \eta_0$ we have $B_\eta = B(w_0,\delta_0) \cap M_\eta \neq \varnothing$,*

$$\pi_\eta\left((-\beta,0)\cup(0,\beta)\right)B_\eta \cap M_\eta = \varnothing$$

and

$$\pi_\eta([-\beta,\beta])z \cap M_\eta \neq \varnothing \quad \textit{for all} \quad z \in B(w_0,\delta_0).$$

Then w_0 satisfies C-STC.

Proof. Let $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ be such that $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$ and $\{w_k\}_{k\in\mathbb{N}} \subset X$ be such that $w_k \in M_{\eta_k}$ and $w_k \overset{k\to+\infty}{\longrightarrow} w_0$. By assumption there is a λ_0-tube through w_0 with $0 < \lambda_0 < \beta$. Let $0 < \lambda \leqslant \lambda_0$. By Lemma 2.6 let $F_0(L_0,[0,2\lambda])$ be a λ-tube through w_0 with section $S_0 = F_0(L_0,[0,2\lambda]) \cap M_0$. We may assume that $F_0(L_0,[0,2\lambda])$ is compact taking in account Lemma 3.7. Moreover, there exists $\delta_1 \in (0,\delta_0)$ such that $B(w_0,\delta_1) \subset F_0(L_0,[0,2\lambda])$, where δ_0 comes from condition (ii). Let $\overline{k}_1 \in \mathbb{N}$ be such that $w_k \in B(w_0,\delta_1)$ for all $k \geqslant \overline{k}_1$.

Define $S_k = M_{\eta_k} \cap \overline{B(w_0, \delta_1)} \subset B_{\eta_k}$ for all $k \geqslant \overline{k}_1$. Note that S_k is compact and $w_k \in S_k$ for all $k \geqslant \overline{k}_1$. Using Lemma 3.1 and the compactness of $F_0(L_0, [0, 2\lambda])$ we conclude that

$$d_H(S_k, S_0) \overset{k \to +\infty}{\longrightarrow} 0. \tag{3.1}$$

Now, we define $L_k = \pi_{\eta_k}(\lambda) S_k$ for all $k \geqslant \overline{k}_1$. In the sequel, we shall show that $F_{\eta_k}(L_k, [0, 2\lambda])$ is a λ-tube through w_k for k sufficiently large.

Note that L_k is compact and

$$F_{\eta_k}(L_k, \lambda) = \pi_{\eta_k}(-\lambda) L_k = S_k \text{ for all } k \geqslant \overline{k}_1.$$

Moreover, $F_{\eta_k}(L_k, [0, 2\lambda]) \cap M_{\eta_k} = S_k$ for all $k \geqslant \overline{k}_1$. In fact, fix $k \geqslant \overline{k}_1$. If $z \in S_k$ then $z \in M_{\eta_k}$ and $\pi_{\eta_k}(\lambda) z \in L_k$, that is, $z \in F_{\eta_k}(L_k, [0, 2\lambda]) \cap M_{\eta_k}$. On the other hand, if $z \in F_{\eta_k}(L_k, [0, 2\lambda]) \cap M_{\eta_k}$ then there is $s_k \in [0, 2\lambda]$ such that $\pi_{\eta_k}(s_k) z \in L_k = \pi_{\eta_k}(\lambda) S_k$, that is, $\pi_{\eta_k}(s_k - \lambda) z \in S_k$. We claim that $s_k = \lambda$. Indeed, if $s_k \neq \lambda$ then by the definition of S_k, condition (ii) and by the fact that $\pi_{\eta_k}(\lambda - s_k) \pi_{\eta_k}(s_k - \lambda) z = z \in M_{\eta_k}$ we have

$$|\lambda - s_k| \geqslant \beta,$$

which is a contradiction, since $\lambda < \beta$ and $s_k \in [0, 2\lambda]$. Consequently, $s_k = \lambda$ and $z \in S_k$.

We still have to show items (b) and (c) from Definition 2.5. To this end, we present some assertions.

Assertion 1: There are $\delta \in (0, \delta_1)$ and $k_2 \geqslant \overline{k}_1$ such that $B(w_k, \delta) \subset F_{\eta_k}(L_k, [0, 2\lambda])$ for all $k \geqslant k_2$.

Indeed, suppose to the contrary that there are $k_m \overset{m \to +\infty}{\longrightarrow} +\infty$, $\delta_m \overset{m \to +\infty}{\longrightarrow} 0^+$, $z_m \in B(w_{k_m}, \delta_m)$ and $z_m \notin F_{\eta_{k_m}}(L_{k_m}, [0, 2\lambda])$ for all $m \in \mathbb{N}$. We may assume that $k_m \geqslant \overline{k}_1$ and $\delta_m \in (0, \delta_1)$ for all $m \in \mathbb{N}$. As $w_{k_m} \overset{m \to +\infty}{\longrightarrow} w_0$ and $\delta_m \overset{m \to +\infty}{\longrightarrow} 0^+$, there is $\overline{m}_0 \in \mathbb{N}$ such that $z_m \in B(w_0, \delta_1)$ for all $m \geqslant \overline{m}_0$. Condition (ii) ensures the existence of $\alpha_m \in [-\beta, \beta]$ such that

$$\pi_{\eta_{k_m}}(\alpha_m) z_m \in M_{\eta_{k_m}} \text{ for all } m \geqslant \overline{m}_0. \tag{3.2}$$

We may assume that $\alpha_m \overset{m \to +\infty}{\longrightarrow} \alpha \in [-\beta, \beta]$. Then as $m \to +\infty$ in (3.2) we obtain

$$\pi_0(\alpha) w_0 \in M_0,$$

which shows that $\alpha = 0$, since $\pi_0((0, \lambda]) w_0 \cap M_0 = \varnothing$ and $F_0(w_0, (0, \lambda]) \cap M_0 = \varnothing$. Consequently, $\lambda + \alpha_m \in [0, 2\lambda]$ and $\pi_{\eta_{k_m}}(\alpha_m) z_m \in S_{k_m}$ for m sufficiently large. This shows that

$$\pi_{\eta_{k_m}}(\lambda + \alpha_m) z_m \in L_{k_m},$$

hence $z_m \in F_{\eta_{k_m}}(L_{k_m}, [0, 2\lambda])$ for m sufficiently large, which is a contradiction and proves Assertion 1.

Assertion 2: There exists $k_0 \geqslant k_2$ such that $F_{\eta_k}(L_k, \nu) \cap F_{\eta_k}(L_k, \mu) = \varnothing$ for all $0 \leqslant \nu < \mu \leqslant 2\lambda$ and $k \geqslant k_0$.

Again, we suppose to the contrary that there exist $k_m \overset{m\to+\infty}{\longrightarrow} +\infty$, $0 \leqslant \nu_m < \mu_m \leqslant 2\lambda$ and $z_m \in F_{\eta_{k_m}}(L_{k_m}, \nu_m) \cap F_{\eta_{k_m}}(L_{k_m}, \mu_m)$ for all $m \in \mathbb{N}$. Then we have

$$\pi_{\eta_{k_m}}(\nu_m)z_m \in L_{k_m} \quad \text{and} \quad \pi_{\eta_{k_m}}(\mu_m)z_m \in L_{k_m} \text{ for all } m \in \mathbb{N},$$

which implies that

$$\pi_{\eta_{k_m}}(\nu_m - \lambda)z_m \in S_{k_m} \quad \text{and} \quad \pi_{\eta_{k_m}}(\mu_m - \lambda)z_m \in S_{k_m} \text{ for all } m \in \mathbb{N}.$$

Since $\pi_{\eta_{k_m}}(\nu_m - \lambda)z_m \in B_{\eta_{k_m}}$ and $\pi_{\eta_{k_m}}(\mu_m - \nu_m)\pi_{\eta_{k_m}}(\nu_m - \lambda)z_m \in M_{\eta_{k_m}}$, it follows by condition (ii) that

$$|\mu_m - \nu_m| \geqslant \beta \quad \text{for all} \quad m \in \mathbb{N}. \tag{3.3}$$

By (3.1) and (3.3), we may assume that $\pi_{\eta_{k_m}}(\nu_m - \lambda)z_m \overset{m\to+\infty}{\longrightarrow} a \in S_0$, $\pi_{\eta_{k_m}}(\mu_m - \lambda)z_m \overset{m\to+\infty}{\longrightarrow} b \in S_0$, $\nu_m \overset{m\to+\infty}{\longrightarrow} \nu \in [0, 2\lambda]$ and $\mu_m \overset{m\to+\infty}{\longrightarrow} \mu \in [0, 2\lambda]$ with $\nu \neq \mu$. Then we get

$$\pi_0(\lambda - \nu)a = \lim_{m\to+\infty} z_m = \pi_0(\lambda - \mu)b,$$

hence $F_0(L_0, \nu) \cap F_0(L_0, \mu) \neq \varnothing$, which is a contradiction.

In conclusion, w_0 satisfies C-STC and it proves the theorem. □

Corollary 3.9. *Under the assumptions of Theorem 3.8 assume additionally that $I_0(M_0)$ is closed, $\mathrm{d}_H(I_\eta(M_\eta), I_0(M_0)) \overset{\eta\to 0}{\longrightarrow} 0$ and $w_0 \in M_0$ satisfies* SSTC. *Then w_0 satisfies* C-SSTC.

Proof. Since, in particular, $w_0 \in M_0$ satisfies STC, it follows from Theorem 3.8 that w_0 satisfies C-STC. Moreover, since $w_0 \in M_0$ satisfies SSTC, Lemmas 2.6 and 3.7 used in the proof of Theorem 3.8 allow us to consider $F_0(L_0, [0, 2\lambda])$ compact with $F_0(L_0, [0, \lambda]) \cap I_0(M_0) = \varnothing$. Continuing the argument of the proof of Theorem 3.8 we are left to show that there exists $\overline{k}_0 \geqslant k_0$ such that

$$F_{\eta_k}(L_k, [0, \lambda]) \cap I_{\eta_k}(M_{\eta_k}) = \varnothing \text{ for } k \geqslant \overline{k}_0.$$

Suppose to the contrary that there exists $z_n \in F_{\eta_{k_n}}(L_{k_n}, [0, \lambda]) \cap I_{\eta_{k_n}}(M_{k_n})$, $n \in \mathbb{N}$. Then there exists $s_n \in [0, \lambda]$, which we may assume to converge to $s_0 \in [0, \lambda]$, such that $\pi_{\eta_{k_n}}(s_n - \lambda)z_n \in F_{\eta_{k_n}}(L_{k_n}, \lambda) = S_{k_n}$. Using (3.1) and compactness of S_0, by taking subsequences if necessary, we may assume that

$$\pi_{\eta_{k_n}}(s_n - \lambda)z_n \overset{n\to+\infty}{\longrightarrow} y_0 \in S_0.$$

Therefore, by **(C1)** we have

$$z_n = \pi_{\eta_{k_n}}(\lambda - s_n)\pi_{\eta_{k_n}}(s_n - \lambda)z_n \overset{n\to+\infty}{\longrightarrow} \pi_0(\lambda - s_0)y_0 = z_0.$$

Thus, we obtain $z_0 \in F_0(L_0, [0, \lambda])$. On the other hand, since $z_n \in I_{\eta_{k_n}}(M_{\eta_{k_n}})$, $n \in \mathbb{N}$, and $\mathrm{d}_H(I_\eta(M_\eta), I_0(M_0)) \overset{\eta\to 0}{\longrightarrow} 0$, we find a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $I_0(M_0)$, which converges to z_0. By the closedness of $I_0(M_0)$ we get $z_0 \in I_0(M_0)$. This contradiction ends the proof. □

3.2. Collective continuity of impact time maps.

As defined previously in (2.2), we consider the impact time map $\phi_\eta\colon (0,+\infty] \to X$, for each $\eta \in [0,1]$, given by

$$\phi_\eta(x) = \begin{cases} s, & \text{if } \pi_\eta(s)x \in M_\eta \text{ and } \pi_\eta(t)x \notin M_\eta \text{ for } 0 < t < s, \\ +\infty, & \text{if } M_\eta^+(x) = \varnothing, \end{cases}$$

where

$$M_\eta^+(x) = \left(\bigcup_{t>0} \pi_\eta(t)x \right) \cap M_\eta.$$

In the next lines, we discuss the behaviour of the family $\{\phi_\eta\}_{\eta\in[0,1]}$.

Lemma 3.10. *Let $x_0 \in X \setminus M_0$ and $\{x_k\}_{k\in\mathbb{N}} \subset X$ be a sequence such that $x_k \overset{k\to+\infty}{\longrightarrow} x_0$. Let $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ be a sequence such that $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$, then $\liminf\limits_{k\to+\infty} \phi_{\eta_k}(x_k) \geqslant \phi_0(x_0)$.*

Proof. Suppose, contrary to the claim, that there exist subsequences $\{\eta_{k_j}\}_{j\in\mathbb{N}}$ and $\{x_{k_j}\}_{j\in\mathbb{N}}$ of $\{\eta_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k\in\mathbb{N}}$, respectively, such that $\phi_{\eta_{k_j}}(x_{k_j}) \overset{j\to+\infty}{\longrightarrow} t < \phi_0(x_0)$. Thus we know that $\pi_{\eta_{k_j}}(\phi_{\eta_{k_j}}(x_{k_j}))x_{k_j} \in M_{\eta_{k_j}}$, $j \in \mathbb{N}$, and by Lemma 3.1 and (**C1**) we have

$$\pi_{\eta_{k_j}}(\phi_{\eta_{k_j}}(x_{k_j}))x_{k_j} \overset{j\to+\infty}{\longrightarrow} \pi_0(t)x_0 \in M_0,$$

that is, $\phi_0(x_0) \leqslant t$, which gives a contradiction. □

Lemma 3.11. *Let $x_0 \in X$ and $\{x_k\}_{k\in\mathbb{N}} \subset X$ be a sequence such that $x_k \overset{k\to+\infty}{\longrightarrow} x_0$. Assume that every point from M_0 satisfies* C-STC. *If $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ is a sequence such that $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$, then $\limsup\limits_{k\to+\infty} \phi_{\eta_k}(x_k) \leqslant \phi_0(x_0)$.*

Proof. It is enough to consider $\phi_0(x_0) < +\infty$. Since $\pi_0(\phi_0(x_0))x_0 \in M_0$, condition (**C2**) implies that there is a subsequence of $\{\eta_k\}_{k\in\mathbb{N}}$, which we denote the same, and a sequence $\{w_k\}_{k\in\mathbb{N}} \subset X$, with $w_k \in M_{\eta_k}$, such that $w_k \overset{k\to+\infty}{\longrightarrow} \pi_0(\phi_0(x_0))x_0$. By C-STC, there exist $\lambda < \phi_0(x_0)$, $\delta = \delta(\lambda) > 0$ and $k_0 \in \mathbb{N}$ such that

$$B(\pi_0(\phi_0(x_0))x_0, \delta) \subset F_0(L_0, [0, 2\lambda]) \quad \text{and} \quad B(w_k, \delta) \subset F_{\eta_k}(L_k, [0, 2\lambda]),\ k \geqslant k_0,$$

where $F_{\eta_k}(L_k, [0, 2\lambda])$ is a λ-tube through w_k with section $S_k = F_{\eta_k}(L_k, [0, 2\lambda]) \cap M_{\eta_k}$ and $F_0(L_0, [0, 2\lambda])$ is a λ-tube through $\pi_0(\phi_0(x_0))x_0$ with section $S_0 = F_0(L_0, [0, 2\lambda]) \cap M_0$.

By Lemma 3.6 there exists $k_1 \geqslant k_0$ such that

$$B\left(\pi_0(\phi_0(x_0))x_0, \tfrac{\delta}{2}\right) \subset B(w_k, \delta) \subset F_{\eta_k}(L_k, [0, 2\lambda]),\ k \geqslant k_1,$$

and condition (**C1**) implies that

$$\pi_{\eta_k}(\phi_0(x_0))x_k \overset{k\to+\infty}{\longrightarrow} \pi_0(\phi_0(x_0))x_0.$$

Consequently, there exists an integer $k_2 \geqslant k_1$ such that

$$\pi_{\eta_k}(\phi_0(x_0))x_k \in B(w_k, \delta) \subset F_{\eta_k}(L_k, [0, 2\lambda]),\ k \geqslant k_2.$$

Without loss of generality we can distinguish two cases.

Case 1: $\pi_{\eta_k}(\phi_0(x_0))x_k \in F_{\eta_k}(L_k, (\lambda, 2\lambda])$ for all $k \geqslant k_2$.

In this case, there is $\alpha_k \in (\lambda, 2\lambda]$ such that

$$\pi_{\eta_k}(\alpha_k)\pi_{\eta_k}(\phi_0(x_0))x_k = \pi_{\eta_k}(\alpha_k + \phi_0(x_0))x_k \in L_k.$$

Thus,

$$\pi_{\eta_k}(\alpha_k + \phi_0(x_0) - \lambda)x_k \in F_{\eta_k}(L_k, \lambda) = S_k \subset M_{\eta_k},\ k \geqslant k_2, \tag{3.4}$$

and hence $\phi_{\eta_k}(x_k) \leqslant \alpha_k + \phi_0(x_0) - \lambda$ for all $k \geqslant k_2$.

We may assume without loss of generality that $\alpha_k \stackrel{k\to+\infty}{\longrightarrow} \alpha_0 \in [\lambda, 2\lambda]$. We assert that $\alpha_0 = \lambda$. In fact, since

$$\pi_{\eta_k}(\alpha_k + \phi_0(x_0) - \lambda)x_k \stackrel{k\to+\infty}{\longrightarrow} \pi_0(\alpha_0 + \phi_0(x_0) - \lambda)x_0,$$

it follows by (3.4) and Lemma 3.1 that $\pi_0(\alpha_0 + \phi_0(x_0) - \lambda)x_0 = \pi_0(\alpha_0 - \lambda)\pi_0(\phi_0(x_0))x_0 \in M_0$. Since $\pi_0((0, \lambda])\pi_0(\phi_0(x_0))x_0 \cap M_0 = \varnothing$, we see that $\alpha_0 = \lambda$. In conclusion, we get

$$\limsup_{k\to+\infty} \phi_{\eta_k}(x_k) \leqslant \limsup_{k\to+\infty}(\alpha_k + \phi_0(x_0) - \lambda) = \phi_0(x_0).$$

Case 2: $\pi_{\eta_k}(\phi_0(x_0))x_k \in F_{\eta_k}(L_k, [0, \lambda])$ for all $k \geqslant k_2$.

In this case, there is $\beta_k \in [0, \lambda]$ such that

$$\pi_{\eta_k}(\phi_0(x_0) + \beta_k)x_k \in L_k,\ k \geqslant k_2.$$

Thus,

$$\pi_{\eta_k}(\phi_0(x_0) + \beta_k - \lambda)x_k \in F_{\eta_k}(L_k, \lambda) = S_k \subset M_{\eta_k},\ k \geqslant k_2,$$

which implies that $\phi_{\eta_k}(x_k) \leqslant \phi_0(x_0) + \beta_k - \lambda$ for all $k \geqslant k_2$. Assuming that $\beta_k \stackrel{k\to+\infty}{\longrightarrow} \beta_0 \in [0, \lambda]$ we obtain by (**C1**) and Lemma 3.1 that

$$\pi_0(\phi_0(x_0) + \beta_0 - \lambda)x_0 \in M_0,$$

that is, $\phi_0(x_0) \leqslant \phi_0(x_0) + \beta_0 - \lambda$. If $\beta_0 \neq \lambda$ we get a contradiction. Hence, $\beta_0 = \lambda$ and

$$\limsup_{k\to+\infty} \phi_{\eta_k}(x_k) \leqslant \limsup_{k\to+\infty}(\phi_0(x_0) + \beta_k - \lambda) = \phi_0(x_0),$$

which ends the proof. □

In conclusion, by Lemmas 3.10 and 3.11, we have the following theorem.

Theorem 3.12. *Assume that every point of M_0 satisfies* C-STC. *Let $x_0 \in X \backslash M_0$ and $\{x_k\}_{k\in\mathbb{N}} \subset X$ be a sequence such that $x_k \stackrel{k\to+\infty}{\longrightarrow} x_0$. If $\{\eta_k\}_{k\in\mathbb{N}} \subset [0, 1]$ is a sequence such that $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$, then $\lim_{k\to+\infty} \phi_{\eta_k}(x_k) = \phi_0(x_0)$, i.e., the function*

$$[0, 1] \times X \ni (\eta, x) \mapsto \phi_\eta(x) \in (0, +\infty]$$

is continuous in $\{0\} \times X \setminus M_0$.

In the proof of Lemma 3.11 we used the lower semicontinuity at zero of the family $\{M_\eta\}_{\eta\in[0,1]}$ of impulsive sets assumed in (**C2**). A simple example shows that without this assumption, the conclusion of the above lemma may not be true.

Example 3.13. Consider the semigroup

$$\pi_\eta(t)x = -t + x,\ t \geqslant 0,\ x \in X = \mathbb{R},\ \eta \in [0,1],$$

the impulsive sets

$$M_0 = \{0, 2\},\ M_\eta = \{\eta\},\ \eta \in (0,1],$$

and the impulsive function $I_\eta(z) = -1$ for $z \in M_\eta$, $\eta \in [0,1]$.

Then, conditions (**C1**), (**C3**) and (**C4**) are satisfied. Furthermore, C-STC holds for each point in M_0. Note that we have

$$\mathrm{d}_H(M_\eta, M_0) \overset{\eta\to 0}{\longrightarrow} 0 \ \text{ and } \ \mathrm{d}_H(M_0, M_\eta) \overset{\eta\to 0}{\longrightarrow} 2.$$

Setting $x_k = x_0 = 3$, $k \in \mathbb{N}$, for any sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset (0,1]$ with $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$ we get

$$\limsup_{k\to+\infty} \phi_{\eta_k}(x_k) = 3 > 1 = \phi_0(x_0).$$

3.3. **Continuity of the impulsive semiflows $\tilde{\pi}_\eta$ at $\eta = 0$.**

With Theorem 3.12 in hand, we are able to obtain a convergence result for the family $\{(X, \pi_\eta, M_\eta, I_\eta)\}_{\eta\in[0,1]}$. Its proof follows the lines of the proofs of [3, Lemma 3.6] and [20, Lemma 2.3], but we include it for the sake of completeness.

Proposition 3.14. *Let $x_0 \in X \setminus M_0$, $\{x_k\}_{k\in\mathbb{N}} \subset X$ and $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ be sequences such that $x_k \overset{k\to+\infty}{\longrightarrow} x_0$ and $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$. Assume that each point of M_0 satisfies* C-STC*. Given $t \geqslant 0$ there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\varepsilon_k \overset{k\to+\infty}{\longrightarrow} 0$ and*

$$\tilde{\pi}_{\eta_k}(t + \varepsilon_k)x_k \overset{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(t)x_0.$$

Proof. If $\phi_0(x_0) = +\infty$, it follows from Theorem 3.12 that for a given $t \in [0, +\infty)$ there exists $\overline{k} \in \mathbb{N}$ such that $\phi_{\eta_k}(x_k) > t$ for all $k \geqslant \overline{k}$. Consequently, for $k \geqslant \overline{k}$, $\tilde{\pi}_{\eta_k}(t)x_k = \pi_{\eta_k}(t)x_k$, and the result follows by (**C1**) setting $\varepsilon_k = 0$ for $k \in \mathbb{N}$.

Now, let us assume that $\phi_0(x_0) < +\infty$. By Theorem 3.12 we may assume that $\phi_{\eta_k}(x_k) < +\infty$ for all $k \in \mathbb{N}$.

Case 1: $0 \leqslant t < \phi_0(x_0)$.

By Theorem 3.12 there exists $\overline{k}_1 \in \mathbb{N}$ such that $t < \phi_{\eta_k}(x_k)$ for all $k \geqslant \overline{k}_1$. Then $\tilde{\pi}_{\eta_k}(t)x_k = \pi_{\eta_k}(t)x_k$ for all $k \geqslant \overline{k}_1$ and taking $\varepsilon_k = 0$, $k \in \mathbb{N}$, we have by (**C1**)

$$\tilde{\pi}_{\eta_k}(t + \varepsilon_k)x_k = \pi_{\eta_k}(t)x_k \overset{k\to+\infty}{\longrightarrow} \pi_0(t)x_0 = \tilde{\pi}_0(t)x_0.$$

Case 2: $t = \phi_0(x_0)$.

Note that $\tilde{\pi}_0(t)x_0 = \tilde{\pi}_0(\phi_0(x_0))x_0 = (x_0)_1^+$. Thus by (**C1**)

$$(x_k)_1 = \pi_{\eta_k}(\phi_{\eta_k}(x_k))x_k \overset{k\to+\infty}{\longrightarrow} \pi_0(\phi_0(x_0))x_0 = (x_0)_1.$$

Using (**C3**) we have

$$(x_k)_1^+ = I_{\eta_k}((x_k)_1) \overset{k\to+\infty}{\longrightarrow} I_0((x_0)_1) = (x_0)_1^+.$$

By Theorem 3.12, if we define $\varepsilon_k = \phi_{\eta_k}(x_k) - \phi_0(x_0) = \phi_{\eta_k}(x_k) - t$, then $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers such that $\varepsilon_k \overset{k\to+\infty}{\longrightarrow} 0$. Hence, we get

$$\tilde{\pi}_{\eta_k}(t+\varepsilon_k)x_k = \tilde{\pi}_{\eta_k}(\phi_{\eta_k}(x_k))x_k = (x_k)_1^+ \overset{k\to+\infty}{\longrightarrow} (x_0)_1^+ = \tilde{\pi}_0(t)x_0.$$

Case 3: $t > \phi_0(x_0)$.

In this case there exists $m \in \mathbb{N}$ such that $t = \sum_{i=0}^{m-1} \phi_0((x_0)_i^+) + t'$ with $0 \leqslant t' < \phi_0((x_0)_m^+)$. Since $\phi_{\eta_k}(x_k) \overset{k\to+\infty}{\longrightarrow} \phi_0(x_0)$, we have by (**C1**) the following convergence

$$(x_k)_1 = \pi_{\eta_k}(\phi_{\eta_k}(x_k))x_k \overset{k\to+\infty}{\longrightarrow} \pi_0(\phi_0(x_0))x_0 = (x_0)_1.$$

Using (**C3**) and (**C4**) we have

$$(x_k)_1^+ = I_{\eta_k}((x_k)_1) \overset{k\to+\infty}{\longrightarrow} I_0((x_0)_1) = (x_0)_1^+ \notin M_0.$$

Since $\phi_{\eta_k}((x_k)_1^+) \overset{k\to+\infty}{\longrightarrow} \phi_0((x_0)_1^+)$ by Theorem 3.12, we get again by (**C1**)

$$(x_k)_2 = \pi_{\eta_k}(\phi_{\eta_k}((x_k)_1^+))(x_k)_1^+ \overset{k\to+\infty}{\longrightarrow} \pi_0(\phi_0((x_0)_1^+))(x_0)_1^+ = (x_0)_2.$$

Continuing with this process, we obtain

$$(x_k)_i = \pi_{\eta_k}(\phi_{\eta_k}((x_k)_{i-1}^+))(x_k)_{i-1}^+ \overset{k\to+\infty}{\longrightarrow} (x_0)_i \text{ and } (x_k)_i^+ = I_{\eta_k}((x_k)_i) \overset{k\to+\infty}{\longrightarrow} (x_0)_i^+, \; i = 1, \dots, m.$$

Thus we get $\sum_{i=0}^{m-1} \phi_{\eta_k}((x_k)_i^+) \overset{k\to+\infty}{\longrightarrow} \sum_{i=0}^{m-1} \phi_0((x_0)_i^+)$. Set $t_k = \sum_{i=0}^{m-1} \phi_{\eta_k}((x_k)_i^+)$ and define the sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ by $\varepsilon_k = t_k + t' - t$, $k \in \mathbb{N}$. Note that $\varepsilon_k \overset{k\to+\infty}{\longrightarrow} 0$ and $t + \varepsilon_k = t_k + t' \geqslant 0$. Then, since $t' < \phi_{\eta_k}((x_k)_m^+)$ for large k, we get by (**C1**)

$$\tilde{\pi}_{\eta_k}(t+\varepsilon_k)x_k = \pi_{\eta_k}(t')(x_k)_m^+ \overset{k\to+\infty}{\longrightarrow} \pi_0(t')(x_0)_m^+ = \tilde{\pi}_0(t)x_0,$$

which proves the result. □

Remark 3.15. If $t \neq \sum_{i=0}^{m-1} \phi_0((x_0)_i^+)$ for every $m \in \mathbb{N}$, then we can take $\varepsilon_k = 0$, $k \in \mathbb{N}$, in the above lemma.

Theorem 3.12 also allows us to obtain the following result.

Proposition 3.16. *Let $x_0 \in X \setminus M_0$, $\{x_k\}_{k\in\mathbb{N}} \subset X$ and $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ be sequences such that $x_k \overset{k\to+\infty}{\longrightarrow} x_0$ and $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$. Assume that each point of M_0 satisfies* C-STC. *Then, given $\alpha_k \overset{k\to+\infty}{\longrightarrow} 0$ with $\alpha_k \geqslant 0$ for all $k \in \mathbb{N}$, we have $\tilde{\pi}_{\eta_k}(\alpha_k)x_k \overset{k\to+\infty}{\longrightarrow} x_0$.*

Proof. Since $x_0 \notin M_0$, it follows by Theorem 3.12 that $\phi_{\eta_k}(x_k) \overset{k\to+\infty}{\longrightarrow} \phi_0(x_0)$. Then there exists $\overline{k} \in \mathbb{N}$ such that $\alpha_k < \phi_{\eta_k}(x_k)$ for all $k \geqslant \overline{k}$, and by **(C1)** we have

$$\tilde{\pi}_{\eta_k}(\alpha_k)x_k = \pi_{\eta_k}(\alpha_k)x_k \overset{k\to+\infty}{\longrightarrow} \pi_0(0)x_0 = x_0,$$

which concludes the proof. □

Using Propositions 3.14 and 3.16, we can state the next result.

Corollary 3.17. *Under the assumptions of Proposition 3.14, there exists a sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset [0, +\infty)$ *such that* $\varepsilon_k \overset{k\to+\infty}{\longrightarrow} 0$ *and* $\tilde{\pi}_{\eta_k}(t + \varepsilon_k)x_k \overset{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(t)x_0$.

Proof. By Proposition 3.14 there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\xi_k \overset{k\to+\infty}{\longrightarrow} 0$ and $\tilde{\pi}_{\eta_k}(t + \xi_k)x_k \overset{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(t)x_0 \notin M_0$. Thus from Proposition 3.16 we have

$$\tilde{\pi}_{\eta_k}(t + \xi_k + |\xi_k|)x_k = \tilde{\pi}_{\eta_k}(|\xi_k|)\tilde{\pi}_{\eta_k}(t + \xi_k)x_k \overset{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(t)x_0$$

and the claim follows by setting $\varepsilon_k = \xi_k + |\xi_k|$, $k \in \mathbb{N}$. □

4. Upper semicontinuity of global attractors

In the sequel, we deal with the upper semicontinuity at zero of a family $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ of global attractors of a family of impulsive dynamical systems $\{(X, \pi_\eta, M_\eta, I_\eta)\}_{\eta\in[0,1]}$. Our goal is to establish sufficient conditions to show the upper semicontinuity at zero of $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$.

Lemma 4.1. *Let* $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ *be a family of non-empty subsets of* X *such that* $\mathcal{A}_0$ *is precompact. Then* $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ *is upper semicontinuous at* $\eta = 0$ *if and only if given a sequence* $\{\eta_k\}_{k\in\mathbb{N}} \subset [0, 1]$ *such that* $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$ *and a sequence* $\{x_k\}_{k\in\mathbb{N}} \subset X$ *with* $x_k \in \mathcal{A}_{\eta_k}$ *for all* $k \in \mathbb{N}$*, there exists a convergent subsequence of* $\{x_k\}_{k\in\mathbb{N}}$ *with limit in* $\overline{\mathcal{A}_0}$.

Proof. Suppose first that $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is not upper semicontinuous at $\eta = 0$. Hence, there exist sequences $\{\eta_k\}_{k\in\mathbb{N}} \subset [0, 1]$, $\{x_k\}_{k\in\mathbb{N}} \subset X$ and $\epsilon > 0$ such that $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$, $x_k \in \mathcal{A}_{\eta_k}$ and

$$\mathrm{d}_H(x_k, \mathcal{A}_0) \geqslant \epsilon \text{ for all } k \in \mathbb{N}.$$

Therefore, $\{x_k\}_{k\in\mathbb{N}}$ has no convergent subsequence with limit in $\overline{\mathcal{A}_0}$, which is a contradiction. Conversely, if $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta = 0$, $\{\eta_k\}_{k\in\mathbb{N}} \subset [0, 1]$ and $\{x_k\}_{k\in\mathbb{N}} \subset X$ are sequences with $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$ and $x_k \in \mathcal{A}_{\eta_k}$, then

$$\mathrm{d}_H(x_k, \mathcal{A}_0) \leqslant \mathrm{d}_H(\mathcal{A}_{\eta_k}, \mathcal{A}_0) \overset{k\to+\infty}{\longrightarrow} 0,$$

and thus $\{x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence with limit in $\overline{\mathcal{A}_0}$, by the precompactness of $\mathcal{A}_0$. □

Now we present the main result of this work.

Theorem 4.2. *Let $(X, \pi_\eta, M_\eta, I_\eta)$ be an impulsive dynamical system with a global attractor $\mathcal{A}_\eta$, for each $\eta \in [0,1]$, and assume that each point of M_0 satisfies* C-SSTC. *Assume also that $\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta$ is precompact in X. Then the family of global attractors $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta = 0$.*

Proof. Let $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ be a sequence such that $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$ and $\{x_k\}_{k\in\mathbb{N}} \subset X$ be a sequence with $x_k \in \mathcal{A}_{\eta_k}$ for all $k \in \mathbb{N}$. Since $\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta$ is precompact in X, then there are $x_0 \in X$ and a subsequence of $\{x_k\}_{k\in\mathbb{N}}$, which we continue denoting by the same notation, such that $x_k \stackrel{k\to+\infty}{\longrightarrow} x_0$ and $\eta_k < \overline{\eta}$ for all $k \in \mathbb{N}$, where $\overline{\eta}$ comes from condition (**C4**). We need to prove that $x_0 \in \overline{\mathcal{A}_0}$.

Let $\psi_k \colon \mathbb{R} \to X$, $k \in \mathbb{N}$, be a bounded global solution through $x_k \in \mathcal{A}_{\eta_k}$ given by

$$\psi_k(t) = \begin{cases} \tilde{\pi}_{\eta_k}(t+m)(x_k)_{-m}, & \text{if } t \in [-m, -m+1],\ m \in \mathbb{N}, \\ \tilde{\pi}_{\eta_k}(t)x_k, & \text{if } t \geqslant 0, \end{cases}$$

where $\{(x_k)_{-m}\}_{m\in\mathbb{N}} \subset \mathcal{A}_{\eta_k}$ is a sequence such that $\tilde{\pi}_{\eta_k}(1)(x_k)_{-m} = (x_k)_{-m+1}$ for all $m \in \mathbb{N}$, with $(x_k)_0 = x_k$.

By the compactness of $\overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$, we may assume that for each $m \in \mathbb{N}_0$ there is $(x_0)_{-m} \in \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$ such that

$$(x_k)_{-m} \stackrel{k\to+\infty}{\longrightarrow} (x_0)_{-m},\ m \in \mathbb{N}_0.$$

Moreover, there exists $\xi > 0$ such that $\phi_0(z) \geqslant \xi$ for all $z \in I_0\left(M_0 \cap \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}\right)$, see Remark 2.4.

Case 1: $x_0 \notin M_0$.

Subcase 1.1: Suppose there exists a subsequence $\{m_j\}_{j\in\mathbb{N}} \subset \mathbb{N}_0$ such that $m_{j+1} > m_j$ and $(x_0)_{-m_j} \notin M_0$ for all $j \in \mathbb{N}$, where $m_1 = 0$.

By Corollary 3.17, for each $j \in \mathbb{N}$, there is $\{\beta_k^j\}_{k\in\mathbb{N}} \subset [0, +\infty)$ such that $\beta_k^j \stackrel{k\to+\infty}{\longrightarrow} 0$ and

$$\tilde{\pi}_{\eta_k}(m_{j+1} - m_j + \beta_k^j)(x_k)_{-m_{j+1}} \stackrel{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(m_{j+1} - m_j)(x_0)_{-m_{j+1}}.$$

Since $\tilde{\pi}_{\eta_k}(m_{j+1} - m_j)(x_k)_{-m_{j+1}} = (x_k)_{-m_j}$, $k, j \in \mathbb{N}$, using Proposition 3.16 we get for each $j \in \mathbb{N}$

$$\tilde{\pi}_{\eta_k}(m_{j+1} - m_j + \beta_k^j)(x_k)_{-m_{j+1}} = \tilde{\pi}_{\eta_k}(\beta_k^j)(x_k)_{-m_j} \stackrel{k\to+\infty}{\longrightarrow} (x_0)_{-m_j}.$$

Thus, for each $j \in \mathbb{N}$ we obtain

$$\tilde{\pi}_0(m_{j+1} - m_j)(x_0)_{-m_{j+1}} = (x_0)_{-m_j}.$$

We define

$$\psi_0(t) = \begin{cases} \tilde{\pi}_0(t + m_{j+1})(x_0)_{-m_{j+1}}, & \text{if } t \in [-m_{j+1}, -m_j],\ j \in \mathbb{N}, \\ \tilde{\pi}_0(t)x_0, & \text{if } t \geqslant 0. \end{cases}$$

Note that ψ_0 is a global solution of $\tilde{\pi}_0$ through x_0. Now, let us show that $\psi_0(\mathbb{R}) \subset \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$. In fact, let $s \in \mathbb{R}$ and observe that if $s \geqslant 0$, then $\psi_0(s) = \tilde{\pi}_0(s)x_0$. By Corollary 3.17, there is a sequence $\{\gamma_k\}_{k\in\mathbb{N}} \subset [0, +\infty)$ such that $\gamma_k \overset{k\to+\infty}{\longrightarrow} 0$ and

$$\tilde{\pi}_{\eta_k}(s + \gamma_k)x_k \overset{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(s)x_0.$$

Since $\tilde{\pi}_{\eta_k}(s + \gamma_k)x_k \in \mathcal{A}_{\eta_k}$ for all $k \in \mathbb{N}$, we have $\psi_0(s) \in \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$. On the other hand, suppose that $s \in [-m_{j+1}, -m_j]$ for some $j \in \mathbb{N}$. Thus, we have $\psi_0(s) = \tilde{\pi}_0(s + m_{j+1})(x_0)_{-m_{j+1}}$. By Corollary 3.17, there is a sequence $\{\theta_k^j\}_{k\in\mathbb{N}} \subset [0, +\infty)$ such that $\theta_k^j \overset{k\to+\infty}{\longrightarrow} 0$ and

$$\tilde{\pi}_{\eta_k}(s + m_{j+1} + \theta_k^j)(x_k)_{-m_{j+1}} \overset{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(s + m_{j+1})(x_0)_{-m_{j+1}},$$

which implies that $\psi_0(s) \in \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$. Hence, ψ_0 is a bounded global solution of $\tilde{\pi}_0$ through x_0, that is, $x_0 \in \mathcal{A}_0$ by Proposition 2.13.

Subcase 1.2: Suppose there exists $m_0 \in \mathbb{N}$ such that $(x_0)_{-m} \in M_0$ for all $m \geqslant m_0$.

Since $(x_k)_{-m} \overset{k\to+\infty}{\longrightarrow} (x_0)_{-m} \in M_0$ for all $m \geqslant m_0$, taking a subsequence if necessary, we may assume that

$$\phi_{\eta_k}((x_k)_{-m}) \overset{k\to+\infty}{\longrightarrow} 0 \text{ for all } m \geqslant m_0.$$

Indeed, fixing $m \geqslant m_0$ it follows by (**C2**) that there exists a subsequence of $\{\eta_k\}_{k\in\mathbb{N}}$, which we denote the same, and a sequence $\{w_k\}_{k\in\mathbb{N}} \subset X$, $w_k \in M_{\eta_k}$, such that $w_k \overset{k\to+\infty}{\longrightarrow} (x_0)_{-m}$. By the C-SSTC and Lemma 3.6, there exist $\lambda, \delta > 0$ and $k_1 \in \mathbb{N}$ such that

$$B(w_k, \delta) \subset F_{\eta_k}(L_k, [0, 2\lambda]) \text{ for all } k \geqslant k_1,$$

where $F_{\eta_k}(L_k, [0, 2\lambda])$ is a λ-tube through w_k with section $S_k = F_{\eta_k}(L_k, [0, 2\lambda]) \cap M_{\eta_k}$ and $F_0(L_0, [0, 2\lambda])$ is a λ-tube through $(x_0)_{-m}$ with section $S_0 = F_0(L_0, [0, 2\lambda]) \cap M_0$. Moreover, we have

$$F_{\eta_k}(L_k, [0, \lambda]) \cap I(M_{\eta_k}) = \varnothing \text{ and } B\left((x_0)_{-m}, \tfrac{\delta}{2}\right) \subset F_0(L_0, [0, 2\lambda]) \cap B(w_k, \delta), \quad k \geqslant k_1.$$

Hence, there is $k_2 \geqslant k_1$ such that $(x_k)_{-m} \in B(w_k, \delta)$ for all $k \geqslant k_2$. Since through $(x_k)_{-m}$ passes a bounded global solution of $\tilde{\pi}_{\eta_k}$, Proposition 2.9 implies that $(x_k)_{-m} \in F_{\eta_k}(L_k, (\lambda, 2\lambda])$ for $k \geqslant k_2$ and, consequently, there is $\alpha_k \in (\lambda, 2\lambda]$, $k \geqslant k_2$, such that $\pi_{\eta_k}(\alpha_k)(x_k)_{-m} \in L_k$. Then we have

$$\pi_{\eta_k}(\alpha_k - \lambda)(x_k)_{-m} \in F_{\eta_k}(L_k, \lambda) = S_k \subset M_{\eta_k} \text{ for all } k \geqslant k_2.$$

We may assume that $\alpha_k \overset{k\to+\infty}{\longrightarrow} \alpha_0 \in [\lambda, 2\lambda]$ and obtain from (**C1**) and Lemma 3.1

$$\pi_{\eta_k}(\alpha_k - \lambda)(x_k)_{-m} \overset{k\to+\infty}{\longrightarrow} \pi_0(\alpha_0 - \lambda)(x_0)_{-m} \in M_0.$$

Since $\pi_0((0, \lambda])(x_0)_{-m} \cap M_0 = \varnothing$, it follows that $\alpha_0 = \lambda$. Hence, since $0 < \phi_{\eta_k}((x_k)_{-m}) \leqslant \alpha_k - \lambda$ for all $k \geqslant k_2$, we get $\phi_{\eta_k}((x_k)_{-m}) \overset{k\to+\infty}{\longrightarrow} 0$ which concludes the assertion.

Fix $0 < \beta < \min\{\xi, 1\}$ and define $(y_k)_{-m} = \tilde{\pi}_{\eta_k}(\beta)(x_k)_{-m} \in \mathcal{A}_{\eta_k}$ for all $k \in \mathbb{N}$ and $m \geqslant m_0$. Fix $m \geqslant m_0$ and note that by (**C1**) we have

$$w_k := \pi_{\eta_k}(\phi_{\eta_k}((x_k)_{-m}))(x_k)_{-m} \overset{k\to+\infty}{\longrightarrow} \pi_0(0)(x_0)_{-m} = (x_0)_{-m} := w_0 \in M_0 \cap \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}.$$

Since $w_k \in M_{\eta_k}$, $k \in \mathbb{N}$, we obtain from (**C3**)

$$I_{\eta_k}(w_k) \overset{k\to+\infty}{\longrightarrow} I_0(w_0) \notin M_0.$$

Hence, by Theorem 3.12 we have

$$\phi_{\eta_k}(I_{\eta_k}(w_k)) \overset{k\to+\infty}{\longrightarrow} \phi_0(I_0(w_0)).$$

Since $\phi_0(I_0(w_0)) \geqslant \xi > \beta$, we get $\phi_{\eta_k}(I_{\eta_k}(w_k)) > \beta$ for k sufficiently large. Thus, for k sufficiently large we have

$$\begin{aligned}(y_k)_{-m} &= \tilde{\pi}_{\eta_k}(\beta)(x_k)_{-m} = \tilde{\pi}_{\eta_k}(\beta - \phi_{\eta_k}((x_k)_{-m}))\tilde{\pi}_{\eta_k}(\phi_{\eta_k}((x_k)_{-m}))(x_k)_{-m} \\ &= \tilde{\pi}_{\eta_k}(\beta - \phi_{\eta_k}((x_k)_{-m}))I_{\eta_k}(w_k) = \pi_{\eta_k}(\beta - \phi_{\eta_k}((x_k)_{-m}))I_{\eta_k}(w_k).\end{aligned}$$

Using (**C1**) we get

$$(y_k)_{-m} \overset{k\to+\infty}{\longrightarrow} \pi_0(\beta)I_0((x_0)_{-m}) = \tilde{\pi}_0(\beta)I_0((x_0)_{-m}) := (y_0)_{-m} \notin M_0.$$

Note that $(y_k)_{-m} \in \mathcal{A}_{\eta_k}$ and $\tilde{\pi}_{\eta_k}(1)(y_k)_{-m} = (y_k)_{-m+1}$ for all $k \in \mathbb{N}$ and $m \geqslant m_0 + 1$. Since the points $(y_k)_{-m}$ belong to the bounded global solution ψ_k through $x_k \in \mathcal{A}_{\eta_k}$ as $(y_k)_{-m} = \psi_k(-m + \beta)$ for $m \geqslant m_0$, we may repeat the same construction carried out in Case 1 using the sequence $\{(y_0)_{-m_j}\}$ with $m_1 = 0$ and $m_j = m_0 + j - 2$, $j \geqslant 2$, where $(y_0)_0 = x_0$, to obtain a bounded global solution of $\tilde{\pi}_0$ through x_0. Hence, we see that $x_0 \in \mathcal{A}_0$.

Case 2: $x_0 \in M_0$.

Since $x_k \overset{k\to+\infty}{\longrightarrow} x_0 \in M_0$, repeating the argument of Subcase 1.2 we may assume that $\phi_{\eta_k}(x_k) \overset{k\to+\infty}{\longrightarrow} 0$. We may also suppose that $0 < \phi_{\eta_k}(x_k) < \frac{\xi}{4}$ for all $k \in \mathbb{N}$. Since there is $\varepsilon_{x_0} > 0$ such that $F_0(x_0, (0, \varepsilon_{x_0})) \cap M_0 = \varnothing$, we take $\overline{m} \in \mathbb{N}$ such that $\frac{1}{\overline{m}} < \min\{\varepsilon_{x_0}, \frac{\xi}{4}\}$. We fix $m \geqslant \overline{m}$. To simplify the notation, we set $w_k := \psi_k(-\frac{1}{m}) \in \mathcal{A}_{\eta_k}$ for all $k \in \mathbb{N}$. Taking a subsequence, if necessary, let $y_m \in \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$ be the limit of $\{w_k\}_{k\in\mathbb{N}}$. Below we will show that

$$\text{there exists } k_3 \in \mathbb{N} \text{ such that } \phi_{\eta_k}(w_k) > \tfrac{1}{m} \text{ for } k \geqslant k_3. \tag{4.1}$$

Indeed, suppose to the contrary that, up to a choice of a subsequence, we have

$$\phi_{\eta_k}(w_k) \leqslant \tfrac{1}{m} \text{ for all } k \in \mathbb{N}. \tag{4.2}$$

Since $\{\phi_{\eta_k}(w_k)\}_{k\in\mathbb{N}}$ is bounded by (4.2) and $w_k \overset{k\to+\infty}{\longrightarrow} y_m$, we may assume using (**C1**) that

$$z_k := \pi_{\eta_k}(\phi_{\eta_k}(w_k))w_k \overset{k\to+\infty}{\longrightarrow} z.$$

Since $z_k \in M_{\eta_k}$, we see from Lemma 3.1 that $z \in M_0$. Then by (**C3**) we have

$$v_k := \tilde{\pi}_{\eta_k}(\phi_{\eta_k}(w_k))w_k = I_{\eta_k}(z_k) \stackrel{k\to+\infty}{\longrightarrow} I_0(z) \in I_0\left(M_0 \cap \overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}\right).$$

Since $\phi_0(I_0(z)) \geqslant \xi$, by Theorem 3.12 we know that $\phi_{\eta_k}(v_k) > \frac{\xi}{2} > \frac{1}{m}$ for k sufficiently large. Hence, for k sufficiently large, we have

$$\begin{aligned}\pi_{\eta_k}\left(\phi_{\eta_k}(x_k) + \tfrac{1}{m} - \phi_{\eta_k}(w_k)\right) v_k &= \pi_{\eta_k}(\phi_{\eta_k}(x_k))\pi_{\eta_k}\left(\tfrac{1}{m} - \phi_{\eta_k}(w_k)\right) v_k \\ &= \pi_{\eta_k}(\phi_{\eta_k}(x_k))\tilde{\pi}_{\eta_k}(\tfrac{1}{m})w_k = \pi_{\eta_k}(\phi_{\eta_k}(x_k))x_k \in M_{\eta_k},\end{aligned}$$

which is a contradiction, since

$$\phi_{\eta_k}(x_k) + \tfrac{1}{m} - \phi_{\eta_k}(w_k) < \phi_{\eta_k}(x_k) + \tfrac{1}{m} < \tfrac{\xi}{4} + \tfrac{1}{\overline{m}} < \tfrac{\xi}{2}.$$

Hence, (4.1) holds and

$$\pi_{\eta_k}\left(\tfrac{1}{m}\right) w_k = \tilde{\pi}_{\eta_k}\left(\tfrac{1}{m}\right) w_k = \tilde{\pi}_{\eta_k}\left(\tfrac{1}{m}\right) \psi_k(-\tfrac{1}{m}) = x_k \notin M_{\eta_k},\ k \geqslant k_3.$$

As $k \to +\infty$, we get by (**C1**)

$$\pi_0\left(\tfrac{1}{m}\right) y_m = x_0 \in M_0,\ m \geqslant \overline{m}. \tag{4.3}$$

Thus, we have $y_m \notin M_0$ for $m \geqslant \overline{m}$, since $\frac{1}{m} < \varepsilon_{x_0}$. Since $w_k \in \mathcal{A}_{\eta_k}$ and $w_k \stackrel{k\to+\infty}{\longrightarrow} y_m \notin M_0$, by the proof of Case 1 we can construct a bounded global solution of $\tilde{\pi}_0$ through y_m, hence $y_m \in \mathcal{A}_0$ for all $m \geqslant \overline{m}$. Thus $\{y_m\}_{m\in\mathbb{N}}$ has a convergent subsequence, which converges to x_0 by (4.3). In particular $x_0 \in \overline{\mathcal{A}_0}$, which ends the proof. □

Remark 4.3. In Theorem 4.2, we assumed that the global attractors exist for each $\eta \in [0, 1]$. It is clear that this assumption is not required to prove upper semicontinuity at zero. We just need that the global attractors exist for all $\eta \leqslant \eta_0$, for some $0 < \eta_0 \leqslant 1$. However, we can always rescale the parameter η to obtain the required assumption for all $\eta \in [0, 1]$.

5. Application

Consider the following system of impulsive ODEs:

$$\begin{cases}\dot{x} = -x,\ x(0) = x_0,\\ \dot{y} = -y,\ y(0) = y_0,\\ I_0\colon M_0 \to I_0(M_0),\end{cases} \tag{5.1}$$

where $M_0 = \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 = 1\}$ and the function $I_0\colon M_0 \to I_0(M_0)$ is given as follows: given $(x, y) \in M_0$ we consider the line segment $\Gamma_{(x,y)}$ that connects the points (x, y) and $(3, y)$. The point $I_0(x, y)$ is the point in the intersection $\Gamma_{(x,y)} \cap \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 = 9\}$. Note that $I_0(M_0) \subset \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 = 9\}$. In [3, Example 4.8] the authors show that system (5.1) possesses a global attractor given by $\mathcal{A}_0 = \{(0, 0)\} \cup \{(x, 0)\colon x \in (1, 3]\}$.

Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ be C^1 maps, and assume that there exist $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \in \mathbb{R}$ such that

$$xf(x,y) \leqslant \alpha_1 x^2 + \alpha_2 \quad \text{and} \quad yg(x,y) \leqslant \beta_1 y^2 + \beta_2 \text{ for all } (x,y) \in \mathbb{R}^2. \tag{5.2}$$

Now, consider the following autonomous perturbed system of coupled ODEs:

$$\begin{cases} \dot{x} = -x + \eta f(x,y), \quad \eta \in [0,1], \\ \dot{y} = -y + \eta g(x,y), \\ (x(0), y(0)) = (x_0, y_0). \end{cases} \tag{5.3}$$

For each $\eta \in [0,1]$, let $\{\pi_\eta(t)\colon t \in \mathbb{R}\}$ be the flow associated to the system (5.3). By the continuous dependence on initial data and parameters, we have

$$\pi_\eta(t)(x,y) \xrightarrow{\eta \to 0} \pi_0(t)(x,y)$$

uniformly for $(t,(x,y))$ in compact subsets of $\mathbb{R} \times \mathbb{R}^2$.

Now, we define $M_\eta = \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = (1+\eta)^2\}$, $\eta \in [0,1]$. Note that $M_\eta = H^{-1}((1+\eta)^2)$, where $H\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $H(x,y) = x^2 + y^2$. Let

$$G_\eta(x,y) = (-x + \eta f(x,y), -y + \eta g(x,y)),\ (x,y) \in \mathbb{R}^2 \text{ and } \eta \in [0,1].$$

Observe that there exists $\overline{\eta} \in (0,1]$ such that

$$\nabla H(x,y) \cdot G_\eta(x,y) < 0 \quad \text{for all } (x,y) \in M_\eta \text{ and } \eta \in [0, \overline{\eta}]. \tag{5.4}$$

Thus, given $p = (x,y) \in M_\eta$, with $\eta \in [0,\overline{\eta}]$, there is $\epsilon_p^\eta > 0$ such that

$$F_\eta(p, (0, \epsilon_p^\eta)) \cap M_\eta = \varnothing \quad \text{and} \quad \pi_\eta((0,\epsilon_p^\eta))p \cap M_\eta = \varnothing.$$

Hence, rescaling the parameter η if necessary, we consider M_η as an impulsive set for each $\eta \in [0,1]$.

Let $I_\eta\colon M_\eta \to \mathbb{R}^2$ be an impulsive function defined as follows: given $(x,y) \in M_\eta$ we consider the line segment $\Gamma^\eta_{(x,y)}$ that connects the points (x,y) and $(3+\eta, y)$. The point $I_\eta(x,y)$ is the point in the intersection $\Gamma^\eta_{(x,y)} \cap \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = (3+\eta)^2\}$. Note that $I_\eta(M_\eta) \subset \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = (3+\eta)^2\}$.

Thus, we have the following properties:

$$\mathrm{d}_H(M_\eta, M_0) + \mathrm{d}_H(M_0, M_\eta) \xrightarrow{\eta \to 0} 0,$$

given $\epsilon > 0$ and $(x_0,y_0) \in M_0$ there exists $\delta > 0$ such that

$$\mathrm{d}(I_\eta(x,y), I_0(x_0,y_0)) < \epsilon \text{ if } \eta \in [0,\delta),\ (x,y) \in M_\eta \text{ and } \mathrm{d}((x,y),(x_0,y_0)) < \delta,$$

and lastly

$$I_\eta(M_\eta) \cap M_\eta = \varnothing \text{ for all } \eta \in [0,1].$$

Therefore, we obtain the following system

$$\begin{cases} \dot{x} = -x + \eta f(x,y), \quad \eta \in [0,1], \\ \dot{y} = -y + \eta g(x,y), \\ (x(0), y(0)) = (x_0, y_0), \\ I_\eta \colon M_\eta \to I_\eta(M_\eta), \end{cases} \tag{5.5}$$

which defines a family of impulsive dynamical systems $\{(\mathbb{R}^2, \pi_\eta, M_\eta, I_\eta)\}_{\eta\in[0,1]}$ that satisfies conditions (**G**), (**C1**)-(**C4**), where (**C1**) is satisfied uniformly for (t, x) in compact subsets of $\mathbb{R} \times \mathbb{R}^2$. Also, each point of M_0 satisfies C-SSTC as we can see in the next lemma.

Lemma 5.1. *Each point of M_0 satisfies* C-SSTC*.*

Proof. It is easy to see that each point of M_0 satisfies SSTC. Fix $(x_0, y_0) \in M_0$. Let us show that there are $\gamma > 0$, $\delta_0 > 0$ and $\eta_0 > 0$ such that for $0 \leqslant \eta \leqslant \eta_0$ we have:

(*a*) $B((x_0, y_0), \delta_0) \cap M_\eta \neq \varnothing$;

(*b*) $\pi_\eta((-\gamma, 0) \cup (0, \gamma))(x, y) \cap M_\eta = \varnothing$ for all $(x, y) \in B((x_0, y_0), \delta_0) \cap M_\eta$;

(*c*) if $(x, y) \in B((x_0, y_0), \delta_0)$ then $\pi_\eta([-\gamma, \gamma])(x, y) \cap M_\eta \neq \varnothing$.

In fact, it is clear that given $\overline{\delta}_1 > 0$ there is $\overline{\eta}_1 > 0$ such that $B((x_0, y_0), \overline{\delta}_1) \cap M_\eta \neq \varnothing$ for all $0 \leqslant \eta \leqslant \overline{\eta}_1$.

In order to show (*b*), suppose to the contrary that there are sequences $\{\eta_k\}_{k\in\mathbb{N}} \subset (0, \overline{\eta}_1)$, $\{s_k\}_{k\in\mathbb{N}} \subset \mathbb{R} \setminus \{0\}$, $\{\delta_k\}_{k\in\mathbb{N}} \subset (0, +\infty)$ and $(x_k, y_k) \in B((x_0, y_0), \delta_k) \cap M_{\eta_k}$ such that $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$, $s_k \overset{k\to+\infty}{\longrightarrow} 0$, $\delta_k \overset{k\to+\infty}{\longrightarrow} 0$ and $\pi_{\eta_k}(s_k)(x_k, y_k) \in M_{\eta_k}$ for all $k \in \mathbb{N}$. Note that

$$(x_k, y_k) \overset{k\to+\infty}{\longrightarrow} (x_0, y_0) \in M_0.$$

For each $k \in \mathbb{N}$, define the mapping $\Phi_k \colon \mathbb{R} \to \mathbb{R}$ by $\Phi_k(t) = H(\pi_{\eta_k}(t)(x_k, y_k))$, $t \in \mathbb{R}$. Note that $\Phi_k(0) = \Phi_k(s_k) = (1 + \eta_k)^2, k \in \mathbb{N}$. We may assume without loss of generality that $s_k > 0$ for all $k \in \mathbb{N}$. Using the Mean Value Theorem we conclude that there is $\overline{s}_k \in (0, s_k)$ such that

$$\nabla H(\pi_{\eta_k}(\overline{s}_k)(x_k, y_k)) \cdot G_{\eta_k}(\pi_{\eta_k}(\overline{s}_k)(x_k, y_k)) = 0 \text{ for all } k \in \mathbb{N}.$$

When $k \to +\infty$ we get

$$\nabla H(x_0, y_0) \cdot G_0(x_0, y_0) = 0,$$

which contradicts (5.4). Hence, there are $\gamma_1 > 0$, $\overline{\delta}_2 \in (0, \overline{\delta}_1)$ and $\overline{\eta}_2 \in (0, \overline{\eta}_1)$ satisfying conditions (*a*) and (*b*).

Now, let us suppose to the contrary that there exist sequences $\{\eta_k\}_{k\in\mathbb{N}} \subset (0, \overline{\eta}_2)$ and $(a_k, b_k) \in B((x_0, y_0), \overline{\delta}_2))$ such that $\eta_k \overset{k\to+\infty}{\longrightarrow} 0$, $(a_k, b_k) \overset{k\to+\infty}{\longrightarrow} (x_0, y_0)$ and

$$\pi_{\eta_k}([-\gamma_1, \gamma_1])(a_k, b_k) \cap M_{\eta_k} = \varnothing \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.6}$$

We choose $\vartheta \in (0, 1)$ such that

$$H(\pi_0(\gamma_1)(x_0, y_0)) < (1 - \vartheta)^2 \quad \text{and} \quad H(\pi_0(-\gamma_1)(x_0, y_0)) > (1 + \vartheta)^2.$$

Then there is $\overline{k}_0 > 0$ such that

$$H(\pi_{\eta_k}(\gamma_1)(a_k, b_k)) < (1-\vartheta)^2, \quad H(\pi_{\eta_k}(-\gamma_1)(a_k, b_k)) > (1+\vartheta)^2, \quad \eta_k < \vartheta \quad \text{for all } k \geqslant \overline{k}_0.$$

By continuity there is $s_k \in [-\gamma_1, \gamma_1]$ such that $H(\pi_{\eta_k}(s_k)(a_k, b_k)) = (1+\eta_k)^2$ for all $k \geqslant \overline{k}_0$, which contradicts (5.6). Hence, one may choose $\gamma = \gamma_1$, $\delta_0 \in (0, \overline{\delta}_2)$ and $\eta_0 \in (0, \overline{\eta}_2)$ satisfying conditions (a), (b) and (c).

It is not difficult to see that $\mathrm{d}_H(I_\eta(M_\eta), I_0(M_0)) \stackrel{\eta\to 0}{\longrightarrow} 0$. Therefore, the result follows by Corollary 3.9. ☐

The next result concerns the existence of global attractors for the system (5.5).

Lemma 5.2. *There is $\eta_0 \in (0,1)$ such that system (5.5) has a global attractor $\mathcal{A}_\eta$ for each $\eta \in [0, \eta_0]$.*

Proof. Indeed, let $\alpha_0 = \max\{\alpha_1, \beta_1\}$ and $0 < \overline{\eta}_0 < \min\{\alpha_0^{-1}, 1\}$ be such that $0 \leqslant \frac{\eta(|\alpha_2|+|\beta_2|)}{1-\eta\alpha_0} \leqslant 1$ for all $\eta \in [0, \overline{\eta}_0]$. Let B be a bounded set in $\mathbb{R}^2$. For $(x_0, y_0) \in B$, let us denote $\pi_\eta(t)(x_0, y_0) = (x_\eta(t), y_\eta(t))$, $t \in \mathbb{R}$. Then, using (5.2), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|(x_\eta(t), y_\eta(t))|^2 &= x_\eta'(t)x_\eta(t) + y_\eta'(t)y_\eta(t) \\ &= x_\eta(t)(-x_\eta(t) + \eta f(x_\eta(t), y_\eta(t))) + y_\eta(t)(-y_\eta(t) + \eta g(x_\eta(t), y_\eta(t))) \\ &\leqslant -(1-\alpha_0\eta)|(x_\eta(t), y_\eta(t))|^2 + \eta(|\alpha_2| + |\beta_2|). \end{aligned} \tag{5.7}$$

Hence, we obtain

$$\begin{aligned} |\pi_\eta(t)(x_0, y_0)|^2 &\leqslant |(x_0, y_0)|^2 e^{-2(1-\eta\alpha_0)t} + \frac{\eta(|\alpha_2| + |\beta_2|)}{1-\eta\alpha_0}\left(1 - e^{-2(1-\eta\alpha_0)t}\right) \\ &\leqslant |(x_0, y_0)|^2 e^{-2(1-\eta\alpha_0)t} + 1, \ t \geqslant 0, \ \eta \in [0, \overline{\eta}_0]. \end{aligned} \tag{5.8}$$

Since B is bounded, we find $T_B > 0$ such that

$$|\pi_\eta(t)(x, y)| \leqslant 5 \quad \text{for all} \quad (x, y) \in B \quad \text{and} \quad t \geqslant T_B.$$

Note that if $(x, y) \in I_\eta(M_\eta)$ and $\eta \in [0, \overline{\eta}_0]$, then

$$|\pi_\eta(t)(x, y)|^2 \leqslant |(x, y)|^2 e^{-2(1-\eta\alpha_0)t} + 1 \leqslant (3+\eta)^2 + 1 \leqslant 25 \quad \text{for all } t \geqslant 0.$$

Thus, we conclude that

$$|\tilde{\pi}_\eta(t)(x, y)| \leqslant 5 \quad \text{for all} \quad (x, y) \in B, \quad t \geqslant T_B \quad \text{and} \quad \eta \in [0, \overline{\eta}_0].$$

The set $K_\eta = \overline{B((0,0), 5)} \setminus M_\eta$ $\tilde{\pi}_\eta$-absorbs all bounded subsets of $\mathbb{R}^2$ for each $\eta \in [0, \overline{\eta}_0]$.

Note that there exists $0 < \eta_0 \leqslant \overline{\eta}_0$ such that every point of M_η satisfies SSTC for $\eta \in [0, \eta_0]$. Indeed, suppose to the contrary that there are sequences $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$ and $w_k \in M_{\eta_k}$ such that w_k does not satisfy SSTC. By (**C2**) and the compactness of M_0, we may assume that $w_k \stackrel{k\to+\infty}{\longrightarrow} w_0$ with $w_0 \in M_0$. However, since w_0 satisfies C-SSTC by Lemma 5.1, we see that $w_k \in M_{\eta_k}$ satisfies SSTC for large k, a contradiction.

Hence, by Theorem 2.12 the system $(X, \pi_\eta, M_\eta, I_\eta)$ admits a global attractor $\mathcal{A}_\eta$ such that $\mathcal{A}_\eta \subset K_\eta$, $\eta \in [0, \eta_0]$. □

Note in the proof of Lemma 5.2 that $\bigcup_{\eta\in[0,\eta_0]} \mathcal{A}_\eta \subset \overline{B((0,0),5)}$, that is, $\bigcup_{\eta\in[0,\eta_0]} \mathcal{A}_\eta$ is precompact in $\mathbb{R}^2$. Thus, by the previous results and Theorem 4.2 we have the following conclusion.

Theorem 5.3. *The family of global attractors* $\{\mathcal{A}_\eta\}_{\eta\in[0,\eta_0]}$ *of* (5.5) *is upper semicontinuous at* $\eta = 0$.

6. Lower semicontinuity

In this section, we provide a first step in the study of lower semicontinuity for global attractors of impulsive dynamical systems. Such topic is far too extensive to be discussed here in details, with examples and applications. We will describe the theoretic foundation to the study and present a result of lower semicontinuity in a particular case. To this end, let (X, π, M, I) be an impulsive dynamical system.

If we recall the concept of a $\tilde{\pi}$-global solution (see (2.3)), we will define a **$\tilde{\pi}$-backwards solution through** $x \in X$ as a function $\psi\colon (-\infty, 0] \to X$ satisfying $\psi(0) = x$ and

$$\tilde{\pi}(t)\psi(s) = \psi(t+s) \text{ for all } t \geqslant 0 \text{ and } s \leqslant 0 \text{ such that } t+s \leqslant 0.$$

Given an $\tilde{\pi}$-invariant subset Ξ of X, we define the **unstable set** of Ξ by

$$W^u(\Xi) = \{y \in X\colon \text{ there is a } \tilde{\pi}\text{-backwards solution } \psi \text{ through } y \\ \text{such that } \mathrm{d}_H(\psi(t), \Xi) \overset{t\to-\infty}{\longrightarrow} 0\}.$$

We also define the **δ-unstable set** of Ξ by

$$W^u_\delta(\Xi) = \{y \in \mathcal{O}_\delta(\Xi)\colon \text{ there exists a } \tilde{\pi}\text{-backward solution } \psi \text{ through } y \\ \text{such that } \psi(t) \in \mathcal{O}_\delta(\Xi) \text{ for all } t \leqslant 0 \text{ and } \mathrm{d}_H(\psi(t), \Xi) \overset{t\to-\infty}{\longrightarrow} 0\},$$

for $\delta > 0$ sufficiently small, where $\mathcal{O}_\delta(\Xi) = \{z \in X\colon \inf_{x\in\Xi} \mathrm{d}(z, x) < \delta\}$.

A point $y^* \in X$ is called an **equilibrium point** for $\tilde{\pi}$ if $\tilde{\pi}(t)y^* = y^*$ for all $t \geqslant 0$. With an equilibrium point y^* for $\tilde{\pi}$, we can construct a $\tilde{\pi}$-global bounded solution ψ^* by setting

$$\psi^*(t) = y^* \text{ for all } t \in \mathbb{R}.$$

Such a solution is called a **stationary solution of** $\tilde{\pi}$ and we often say that the point y^* is a stationary solution of $\tilde{\pi}$.

Remark 6.1. It is clear that, with assumptions (2.1), if y^* is a stationary point of $\tilde{\pi}$, then $y^* \notin M$. Thus $\tilde{\pi}(t)y^* = \pi(t)y^*$, $t \geqslant 0$, and hence y^* is an equilibrium point for the continuous semigroup $\{\pi(t)\colon t \geqslant 0\}$. Also, since M is closed, the behaviour of $\tilde{\pi}$ in a suitable small neighbourhood of y^* is qualitatively no different from the behaviour of π.

First we give an equivalent condition for the lower semicontinuity at $\eta = 0$ of a family $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$.

Lemma 6.2. *Let $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ be a family of non-empty subsets of X such that $\mathcal{A}_0$ is precompact. Then $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$ if and only if given $x_0 \in \overline{\mathcal{A}_0}$ and $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ with $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$, there exist a subsequence $\{\eta_{k_j}\}_{j\in\mathbb{N}}$ and $\{x_j\}_{j\in\mathbb{N}} \subset X$ with $x_j \in \mathcal{A}_{\eta_{k_j}}$ for all $j \in \mathbb{N}$ such that $x_j \stackrel{j\to+\infty}{\longrightarrow} x_0$.*

Proof. Suppose that $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is not lower semicontinuous at $\eta = 0$. Hence, there exists a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ with $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$, $\epsilon > 0$ and a sequence $z_k \in \mathcal{A}_0$, $k \in \mathbb{N}$, such that

$$\mathrm{d}_H(z_k, \mathcal{A}_{\eta_k}) \geqslant \epsilon, \quad k \in \mathbb{N}.$$

Since $\mathcal{A}_0$ is precompact, we may assume that $z_k \stackrel{k\to+\infty}{\longrightarrow} z_0 \in \overline{\mathcal{A}_0}$. Hence, there cannot exist a subsequence $\{\eta_{k_j}\}_{j\in\mathbb{N}}$ and $\{x_j\}_{j\in\mathbb{N}}$ with $x_j \in \mathcal{A}_{\eta_{k_j}}$ for all $j \in \mathbb{N}$ such that $x_j \stackrel{j\to+\infty}{\longrightarrow} z_0$, which is a contradiction. Conversely, if $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$, $x_0 \in \overline{\mathcal{A}_0}$ and $\{\eta_k\}_{k\in\mathbb{N}} \subset [0,1]$ with $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$, then there exists a subsequence $\{\eta_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\mathrm{d}_H(\mathcal{A}_0, \mathcal{A}_{\eta_{k_j}}) < \tfrac{1}{j}, \ j \in \mathbb{N}.$$

Thus, there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ with $x_j \in \mathcal{A}_{\eta_{k_j}}$ for all $j \in \mathbb{N}$ such that $x_j \stackrel{j\to+\infty}{\longrightarrow} x_0$. □

Now, we consider a family of impulsive dynamical systems $\{(X, \pi_\eta, M_\eta, I_\eta)\}_{\eta\in[0,1]}$ satisfying **(G)**, **(C1)**-**(C4)** and assume that each point of M_0 satisfies C-STC. We set $\mathcal{E}_\eta$ as the set of all stationary solutions of $\{\pi_\eta(t) \colon t \geqslant 0\}$. Hence, it coincides with the set of stationary solution of $\tilde{\pi}_\eta$.

The following result provides sufficient conditions for the lower semicontinuity at $\eta = 0$ of a family of global attractors.

Theorem 6.3. *With the above conditions, assume that $(X, \pi_\eta, M_\eta, I_\eta)$ has a global attractor $\mathcal{A}_\eta$, for each $\eta \in [0,1]$. Additionally, assume that:*

- **(a)** *there exists $p \in \mathbb{N}$ such that $\{y_1^{*,\eta}, \ldots, y_p^{*,\eta}\} \subset \mathcal{E}_\eta$, for each $\eta \in [0,1]$;*
- **(b)** *there exists $\delta > 0$ such that the family $\{W^u_\delta(y_j^{*,\eta})\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$, for all $j = 1, \ldots, p$;*
- **(c)** *$\mathcal{A}_0 = \bigcup_{j=1}^p W^u(y_j^{*,0})$.*

Then the family of global attractors $\{\mathcal{A}_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$.

Proof. Let $u_0 \in \overline{\mathcal{A}_0}$ and $\{\eta_k\}_{k\in\mathbb{N}} \subset (0,1]$ be a sequence such that $\eta_k \stackrel{k\to+\infty}{\longrightarrow} 0$. First we suppose that $u_0 \in \mathcal{A}_0$. Then $u_0 \in W^u(y_r^{*,0})$ for some $r \in \{1, \ldots, p\}$. Thus, there exists a $\tilde{\pi}_0$-global solution ψ_{u_0} through u_0 such that $\psi_{u_0}(t) \stackrel{t\to-\infty}{\longrightarrow} y_r^{*,0}$, and let $\tau \geqslant 0$ be such that $\psi_{u_0}(-\tau) \in W^u_\delta(y_r^{*,0})$.

By item **(b)**, taking a subsequence if necessary, there exist $u_{\eta_k}^{-\tau} \in W^u_\delta(y_r^{*,\eta_k})$ such that $u_{\eta_k}^{-\tau} \stackrel{k\to+\infty}{\longrightarrow} \psi_{u_0}(-\tau)$ and a $\tilde{\pi}_{\eta_k}$-global solution $\psi^{(\eta_k)}$ through $u_{\eta_k}^{-\tau}$ such that

$$\psi^{(\eta_k)}(0) \stackrel{k\to+\infty}{\longrightarrow} \psi_{u_0}(-\tau).$$

Since $\psi_{u_0}(-\tau) \notin M_0$ it follows by Corollary 3.17 that there is a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\alpha_k \stackrel{k\to+\infty}{\longrightarrow} 0$ and

$$\tilde{\pi}_{\eta_k}(\tau+\alpha_k)\psi^{(\eta_k)}(0) \stackrel{k\to+\infty}{\longrightarrow} \tilde{\pi}_0(\tau)\psi_{u_0}(-\tau) = u_0,$$

with $\tilde{\pi}_{\eta_k}(\tau+\alpha_k)\psi^{(\eta_k)}(0) = \tilde{\pi}_{\eta_k}(\tau+\alpha_k)u^{-\tau}_{\eta_k} \in \mathcal{A}_{\eta_k}$.

It remains the case where $u_0 \in \overline{\mathcal{A}_0} \setminus \mathcal{A}_0$. Given $\epsilon > 0$ there is $u' \in \mathcal{A}_0$ such that $\mathrm{d}(u', u_0) < \frac{\epsilon}{2}$, and since $u' \in \mathcal{A}_0$, it follows by the previous case that there are $x_k \in \mathcal{A}_{\eta_k}$ and $\overline{k}_0 \in \mathbb{N}$ such that

$$\mathrm{d}(x_k, u') < \tfrac{\epsilon}{2} \text{ for all } k \geqslant \overline{k}_0.$$

Hence, for $k \geqslant \overline{k}_0$ we have $\mathrm{d}(x_k, u_0) < \epsilon$ and the proof is complete. □

Using Remark 6.1, this result might be applied, for instance, when the continuous semigroups $\{\pi_\eta(t)\colon t \geqslant 0\}$ are differentiable in X (or at least in a small neighbourhood of $\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta$) and the equilibrium points of $\{\pi_0(t)\colon t \geqslant 0\}$ are all hyperbolic, that is, the spectrum $\sigma(D_x\pi_0(1))$ of the derivative $D_x\pi_0(1)$ is disjoint from the unit circle S^1 in $\mathbb{C}$.

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