GLOBAL ATTRACTORS FOR IMPULSIVE DYNAMICAL SYSTEMS - A PRECOMPACT APPROACH

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Abstract. In this work we give the definition of a global attractor of an impulsive dynamical system and obtain several important properties for this class of attractors. We prove the theorem on existence of such attractors and apply it to chosen ordinary and partial differential equations with impulsive functions.

1. Introduction

The theory of impulsive dynamical systems describes the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. This subject has been the research topic of many authors over the last four decades and first appeared in the 70's in the works of Rozko (see [31, 32]). In 1990, Kaul constructed the mathematical foundation of this theory with impulses at variable times in [25] and next, Kaul and Ciesielski published very important results in this area (see [16, 17, 18, 26, 27]). Thereon a vast literature on this topic has been developed and the reader can see for instance [7, 8, 9, 10, 11, 29] for details of this theory and [1, 5, 12, 13, 15, 20, 21, 22] for other results and applications. Many real world problems are defined in terms of impulsive systems; for instance, a simple medicine intake, which requires that a new dose must be taken in order to keep the disease under control.

The study of impulsive dynamical systems requires a previous knowledge of *continuous autonomous dynamical systems* (or simply, *semigroups*) and we now state very superficially this theory.

Let (X,d) be a metric space with metric d and $\mathbb{R}_+$ be the set of non-negative real numbers. A **semigroup** in X is a family of functions $\{\pi(t)\colon t \geqslant 0\}$, indexed on $\mathbb{R}_+$, satisfying

- **(i)** $\pi(0)x = x$, for all $x \in X$;
- **(ii)** $\pi(t+s) = \pi(t) \circ \pi(s)$, for all $t, s \geqslant 0$;
- **(iii)** the map $\mathbb{R}_+ \times X \ni (t,x) \mapsto \pi(t)x$ is continuous.

From now on we omit the composition sign '$\circ$' and will simply write property (ii) as $\pi(t+s) = \pi(t)\pi(s)$.

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A subset A of X is called $\pi-$**invariant** under $\{\pi(t)\colon t \geqslant 0\}$ if $\pi(t)A = A$, for all $t \geqslant 0$. Also A is $\pi-$**positively (negatively) invariant** if $\pi(t)A \subseteq A$ ($\pi(t)A \supseteq A$), for all $t \geqslant 0$.

Given two subsets $A, B \subseteq X$, we say that A $\pi-$**attracts** B if

$$\lim_{t\to\infty} \mathrm{d}_H(\pi(t)B, A) = 0,$$

where $\mathrm{d}_H(\cdot,\cdot)$ denotes the Hausdorff semidistance between two sets; i.e.,

$$\mathrm{d}_H(C, D) = \sup_{x\in C} \inf_{y\in D} d(x, y).$$

We can now define the notion of a global attractor for the semigroup $\{\pi(t)\colon t \geqslant 0\}$.

Definition 1.1. *A subset $\mathcal{A}$ of X is called a* **global attractor** *for the semigroup $\{\pi(t)\colon t \geqslant 0\}$ if it is compact, $\pi-$invariant and $\pi-$attracts all bounded subsets of X.*

We know that the global attractor $\mathcal{A}$ of the semigroup $\{\pi(t)\colon t \geqslant 0\}$ is unique and describes the long-time behavior dynamics of $\{\pi(t)\colon t \geqslant 0\}$; that is, to study the asymptotic dynamics of the semigroup $\{\pi(t)\colon t \geqslant 0\}$, one must understand completely the global attractor and its internal structures. This has been the main research topic for many authors (see for instance [4, 14, 23, 24, 28, 30, 33]).

Our goal is to develop an analogous theory for impulsive dynamical systems; more precisely, we want to define a useful notion of a global attractor for an impulsive dynamical system, in such a way that this object describes completely the long-time behavior of the system. To this end, we introduce some of the definitions and basic properties of impulsive dynamical systems and we attempt to find an appropriate notion of a global attractor.

Let $\{\pi(t)\colon t \geqslant 0\}$ be a semigroup in X. For each $D \subseteq X$ and $J \subseteq \mathbb{R}_+$ we define

$$F(D, J) = \bigcup_{t\in J} \pi(t)^{-1}(D).$$

A point $x \in X$ is called an **initial point** if $F(x, t) = \varnothing$ for all $t > 0$.

Definition 1.2. *An* **impulsive dynamical system** (*IDS, for short*) (X, π, M, I) *consists of a semigroup $\{\pi(t)\colon t \geqslant 0\}$ on a metric space (X, d), a nonempty closed subset $M \subseteq X$ such that for every $x \in M$ there exists $\epsilon_x > 0$ such that*

$$F(x, (0, \epsilon_x)) \cap M = \varnothing \quad \text{and} \quad \bigcup_{t\in(0,\epsilon_x)} \{\pi(t)x\} \cap M = \varnothing, \tag{1.1}$$

and a continuous function $I\colon M \to X$ (its role will be specified later).

The set M is called the **impulsive set** *and the function I is called* **impulsive function**. *We also define*

$$M^+(x) = \left(\bigcup_{t>0} \pi(t)x\right) \cap M.$$

Remark 1.3. *Condition* (1.1) *means that the flow of the semigroup $\{\pi(t)\colon t \geqslant 0\}$ is, in some sense, transversal to M at any point of M.*

Proposition 1.4. *Let (X, π, M, I) be an IDS and $x \in X$. If $M^+(x) \neq \varnothing$ then there exists $s > 0$ such that $\pi(s)x \in M$ and $\pi(t)x \notin M$ for $0 < t < s$.*

Proof: Since $M^+(x) \neq \varnothing$, there exists $s_0 > 0$ such that $\pi(s_0)x \in M$. Assume by contradiction that there exists a sequence $s_n \to 0$ such that $\pi(s_n)x \in M$. Since M is closed, the continuity of $\mathbb{R}_+ \ni t \mapsto \pi(t)x$ implies that $x = \pi(0)x \in M$. Thus we have two possibilities:

- if $x \notin M$ we have already reached a contradiction;
- if $x \in M$ then the condition $\pi((0, \epsilon_x))x \cap M = \varnothing$, for some $\epsilon_x > 0$, gives us a contradiction.

□

With this proposition at hand, we are able to define the function $\phi\colon X \to (0, \infty]$ by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(s)x \in M \text{ and } \pi(t)x \notin M \text{ for } 0 < t < s, \\ \infty, & \text{if } M^+(x) = \varnothing. \end{cases}$$

If $M^+(x) \neq \varnothing$, the value $\phi(x)$ represents the smallest positive time such that the trajectory of x meets M. In this case, we say that the point $\pi(\phi(x))x$ is the **impulsive point** of x.

Definition 1.5. *The* **impulsive trajectory** *of $x \in X$ by the IDS (X, π, M, I) is a map $\tilde{\pi}(\cdot)x$ defined in an interval $J_x \subseteq \mathbb{R}_+$ with values in X given inductively by the following rule: if $M^+(x) = \varnothing$, then $\tilde{\pi}(t)x = \pi(t)x$ for all $t \in \mathbb{R}_+$. However, if $M^+(x) \neq \varnothing$ then we denote $x = x_0^+$ and define $\tilde{\pi}(\cdot)x$ on $[0, \phi(x_0^+)]$ by*

$$\tilde{\pi}(t)x = \begin{cases} \pi(t)x_0^+, & \text{if} \quad 0 \leqslant t < \phi(x_0^+), \\ I(\pi(\phi(x_0^+))x_0^+), & \text{if} \quad t = \phi(x_0^+). \end{cases}$$

Now let $s_0 = \phi(x_0^+)$, $x_1 = \pi(s_0)x_0^+$, and $x_1^+ = I(\pi(s_0)x_0^+)$. In this case $s_0 < \infty$ and the process can go on, but now starting at x_1^+. If $M^+(x_1^+) = \varnothing$ then we define $\tilde{\pi}(t)x = \pi(t - s_0)x_1^+$ for $s_0 \leqslant t < \infty$ and in this case $\phi(x_1^+) = \infty$. However, if $M^+(x_1^+) \neq \varnothing$ we define $\tilde{\pi}(\cdot)x$ on $[s_0, s_0 + \phi(x_1^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - s_0)x_1^+, & \text{if} \quad s_0 \leqslant t < s_0 + \phi(x_1^+), \\ I(\pi(\phi(x_1^+))x_1^+), & \text{if} \quad t = s_0 + \phi(x_1^+). \end{cases}$$

Now let $s_1 = \phi(x_1^+)$, $x_2 = \pi(s_1)x_1^+$, and $x_2^+ = I(\pi(s_1)x_1^+)$. Assume now that $\tilde{\pi}(\cdot)x$ is defined on the interval $[t_{n-1}, t_n]$ and that $\tilde{\pi}(t_n)x = x_n^+$, where $t_0 = 0$ and $t_n = \sum_{i=0}^{n-1} s_i$ for $n \in \mathbb{N}$. If $M^+(x_n^+) = \varnothing$, then $\tilde{\pi}(t)x = \pi(t - t_n)x_n^+$ for $t_n \leqslant t < \infty$ and $\phi(x_n^+) = \infty$. However, if $M^+(x_n^+) \neq \varnothing$, then we define $\tilde{\pi}(\cdot)x$ on $[t_n, t_n + \phi(x_n^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - t_n)x_n^+, & \text{if} \quad t_n \leqslant t < t_n + \phi(x_n^+), \\ I(\pi(\phi(x_n^+))x_n^+), & \text{if} \quad t = t_n + \phi(x_n^+). \end{cases}$$

Now let $s_n = \phi(x_n^+)$*,* $x_{n+1} = \pi(s_n)x_n^+$*, and* $x_{n+1}^+ = I(\pi(s_n)x_n^+)$*. This process ends after a finite number of steps if* $M^+(x_n^+) = \varnothing$ *for some* $n \in \mathbb{N}$*, or it may proceed indefinitely, if* $M^+(x_n^+) \neq \varnothing$ *for all* $n \in \mathbb{N}$ *and in this case* $\tilde{\pi}(\cdot)x$ *is defined in the interval* $[0, T(x))$*, where* $T(x) = \sum_{i=0}^{\infty} s_i$*.*

With these basic definitions and results, assuming that all impulsive trajectories exist for all time $t \geqslant 0$, i.e., assuming that $T(x) = \infty$ for all $x \in X$, we are able to start our search for a suitable definition of a global attractor for the IDS (X, π, M, I) and from now on we will always assume this global existence condition:

(G) $T(x) = \infty$ for all $x \in X$, and thus $\{\tilde{\pi}(t) \colon t \geqslant 0\}$ satisfies the semigroup property:

$$\tilde{\pi}(t+s)x = \tilde{\pi}(t)\tilde{\pi}(s)x, \ t, s \geqslant 0, \ x \in X, \quad \tilde{\pi}(0)x = x, \ x \in X. \tag{1.2}$$

Remark 1.6.

(1) *Observe that if there exists* $\xi > 0$ *such that* $\phi(z) \geqslant \xi$ *for all* $z \in I(M)$*, then for every* $x \in X$ *we have* $T(x) = \infty$ *and* $\{\tilde{\pi}(t) \colon t \geqslant 0\}$ *satisfies* (1.2)*.*

(2) *The condition that there exists* $\xi > 0$ *such that* $\phi(z) \geqslant \xi$ *for all* $z \in I(M)$ *says that there is a minimum time for which the semigroup* π *takes to reach* M*, when leaving from* $I(M)$*. This condition is satisfied in several examples and, for instance, when* $I(M)$ *is compact and* $I(M) \cap M = \varnothing$*. Indeed, let* $\epsilon > 0$ *be such that* $\mathcal{O}_\epsilon(I(M)) \cap M = \varnothing$*. For each point* $z \in I(M)$ *either* $\pi(s)z \in \mathcal{O}_\epsilon(I(M))$ *for all* $s \geqslant 0$ *and we set* $s_z = \infty$ *or there exists a finite time* $s_z > 0$ *such that* $\pi(s_z)z \in \overline{\mathcal{O}_\epsilon(I(M))} \setminus \mathcal{O}_\epsilon(I(M))$ *and that* $\pi(t)z \in \mathcal{O}_\epsilon(I(M))$ *for* $0 \leqslant t < s_z$ *(here and from now on, for a set* $A \subseteq X$*,* $\overline{A}$ *denotes its closure in* X *with its metric* d*). Hence, by the joint continuity of the semigroup, the map* $I(M) \ni z \mapsto s_z$ *is lower semicontinuous, and since* $I(M)$ *is compact, it has a positive infimum* $\xi > 0$*.*

The definitions of $\tilde{\pi}$-invariance and $\tilde{\pi}-$attraction are analogous to the notions of $\pi-$invariance and $\pi-$attraction, respectively, simply replacing π by $\tilde{\pi}$. In Bonotto-Demuner [7], the authors present the following definition for a global attractor:

Definition 1.7. *A subset* $\mathcal{A}$ *of* X *is a* global attractor *for an IDS* (X, π, M, I) *if it is compact,* $\mathcal{A} \cap M = \varnothing$*,* $\tilde{\pi}-$*invariant and* $\tilde{\pi} -$ *attracts all bounded subsets of* X*.*

This definition is consistent with the notion of a global attractor for semigroups; that is, when $M = \varnothing$, both definitions coincide; and in fact, this notion of a global attractor is useful to describe the asymptotic dynamics of $\tilde{\pi}$ in many cases. However, this notion excludes large and very important classes of IDS, since with this definition, the asymptotic behavior of $\tilde{\pi}$ is qualitatively not different from the asymptotic behavior of π. Thus we must find a more suitable definition, that includes cases where the dynamics in long time of $\tilde{\pi}$ is different from the one of π. Let us present a simple example of a case in which the asymptotic dynamics of $\tilde{\pi}$ and π are different.

Example 1.8. *Consider the following continuous differential equation*

$$\dot{x} = \begin{cases} 1, & \text{if } x < 0, \\ 1 - x, & \text{if } x \geqslant 0 \end{cases} \tag{1.3}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}$ *and consider the action of the impulsive function* $I(0) = -1$. *The solutions of* (1.3) *without the action of* I *are given by*

$$\pi(t)x_0 = \begin{cases} t + x_0, \ x_0 < 0, \ t \in [0, -x_0), \\ -e^{-t-x_0} + 1, \ x_0 < 0, \ t \in [-x_0, \infty), \\ (x_0 - 1)e^{-t} + 1, \ x_0 \geqslant 0, \ t \in [0, \infty). \end{cases}$$

This problem has only one bounded invariant set; namely the asymptotically stable equilibrium solution $\{1\}$, *and it is also the global attractor for* (1.3). *Now, the solutions of* (1.3) *with the action of* I, *are given by*

$$\tilde{\pi}(t)x_0 = \begin{cases} t + x_0, \ x_0 < 0, \ t \in [0, -x_0), \\ t + x_0 - n, \ x_0 < 0, \ t \in [-x_0 + n - 1, -x_0 + n), \ n \in \mathbb{N}, \\ (x_0 - 1)e^{-t} + 1, \ x_0 \geqslant 0, \ t \in [0, \infty). \end{cases} \tag{1.4}$$

We can see that the dynamics is quite different, since there appeared the "impulsive periodic orbit" $[-1, 0)$. *Note that in this case there is no subset of* $\mathbb{R}$ *satisfying all the conditions of Definition 1.7. But we can distinguish some interesting sets:*

- *The set* $\mathcal{A}_1 = [-1, 0) \cup \{1\}$ *is* $\tilde{\pi}-$*invariant and* $\tilde{\pi}-$*attracting bounded sets,* $\mathcal{A}_1 \cap M = \varnothing$, *but* $\mathcal{A}_1$ *is not compact.*
- *The set* $\mathcal{A}_2 = [-1, 0] \cup \{1\}$ $\tilde{\pi}-$*attracts bounded sets,* $\mathcal{A}_2$ *is compact, but* $\mathcal{A}_2 \cap M \neq \varnothing$ *and* $\mathcal{A}_2$ *is neither* $\tilde{\pi}-$*positively nor* $\tilde{\pi}-$*negatively invariant.*
- *The set* $\mathcal{A}_3 = [-1, 1]$ $\tilde{\pi}-$*attracts bounded sets,* $\mathcal{A}_3$ *is compact, it is* $\tilde{\pi}-$*positively invariant, but it is not* $\tilde{\pi}-$*negatively invariant and* $\mathcal{A}_3 \cap M \neq \varnothing$.

If one is familiar with the theory of the semigroups, the global attractor in this context is characterized as the union of all bounded global solutions of π, and this property is closely related with the invariance of the global attractor. Hence, looking at the three sets above, one can conjecture that the set $\mathcal{A}_1$ is a natural candidate for the global attractor of this impulsive system, since it is the only $\tilde{\pi}-$invariant among the three.

We will show that in fact this is the case. Also, if we recall the definition of an impulsive trajectory, we can see that through points of M there can be no global solutions of $\tilde{\pi}$, hence it is natural to assume that if the invariance is the property we seek for our global attractor, then no point of M can be in it; therefore the hypothesis $\mathcal{A} \cap M = \varnothing$ in Definition 1.7 needs to be maintained and this implies a direct consequence: the hypothesis of compactness needs to be weakened. The set $\mathcal{A}_1$ above is not compact, but it is precompact and moreover $\mathcal{A}_1 = \overline{\mathcal{A}_1} \setminus M$.

All of these arguments lead us to our definition of a global attractor.

Definition 1.9. *A subset $\mathcal{A} \subset X$ will be called a* **global attractor** *for the IDS (X, π, M, I) if it satisfies the following conditions:*

(i) *$\mathcal{A}$ is precompact and $\mathcal{A} = \overline{\mathcal{A}} \setminus M$;*
(ii) *$\mathcal{A}$ is $\tilde{\pi}-$invariant;*
(iii) *$\mathcal{A}$ $\tilde{\pi}-$attracts bounded subsets of X.*

With this definition, we can see that the set $\mathcal{A}_1$ is the global attractor for the IDS in Example 1.8, and although it is a quite simple example its dynamics is much richer than the dynamics arising from continuous differential equations, this attractor is disconnected and it is the union of two disjoint isolated invariant sets with no connection between them; also, solutions reach the periodic orbit $[-1, 0)$ in finite time (hence there is no backward uniqueness in general). This simple example shows us that a very large amount of systems is not present in the context of [7]. We aim to describe the asymptotic dynamics of a larger class of impulsive dynamical systems. To this end, our work is divided in three main sections.

Section 2 is one of technical nature, and deals with "tube conditions" for an impulsive dynamical system. The main purpose of this section is to develop a result that enables us to overcome the difficulty found in the previous theory: the negative invariance of impulsive $\omega-$limits. This result (Proposition 2.6) states that the impulsive flow $\tilde{\pi}(t)$ cannot reach the "right side" of the impulsive set M for large values of t.

In Section 3, we discuss the impulsive $\omega-$limits of bounded subsets of X. In this section we define *bounded dissipative* impulsive dynamical systems and we study the properties of the impulsive $\omega-$limits for such impulsive systems. We prove the positive invariance (Proposition 3.7) and, using the result of Section 2, the negative invariance (Proposition 3.12). Also, we prove the attraction of the impulsive ω-limits (Proposition 3.14).

In Section 4 we show the existence a global attractor for a strongly bounded dissipative impulsive dynamical system, using impulsive $\omega-$limits (Theorem 4.7), and we present three examples, in increasing order of complexity: a simple planar impulsive system, an impulsive ODE and finally, an impulsive PDE. In each one of these three cases we are able to show the existence of a global attractor for the generated impulsive dynamical system.

2. Tube conditions on impulsive dynamical systems

In this section we deal with several technicalities that may arise when we are working with impulses. In order to obtain invariance results, we must ensure that the original semiflow $\{\pi(t) \colon t \geqslant 0\}$ behaves nicely near the impulsive set M; and for this we mean that the flow has a well defined direction when crossing M. Therefore we define "tube conditions", which guarantee a nice behavior of the semigroup $\{\pi(t) \colon t \geqslant 0\}$ near the impulsive set M (see [16] for more details).

Definition 2.1. *Let $\{\pi(t) \colon t \geqslant 0\}$ be a semigroup on X. A closed set S containing $x \in X$ is called a* **section** *through x if there exists $\lambda > 0$ and a closed subset L of X such that:*

(a) *$F(L, \lambda) = S$;*
(b) *$F(L, [0, 2\lambda])$ contains a neighborhood of x;*
(c) *$F(L, \nu) \cap F(L, \zeta) = \varnothing$, if $0 \leqslant \nu < \zeta \leqslant 2\lambda$.*

*We say that the set $F(L, [0, 2\lambda])$ is a $\lambda-$**tube** (or simply **tube**) and the set L is a **bar**.*

Lemma 2.2. *If S is a section and $\lambda > 0$ is given as in the previous definition, then any $0 < \mu \leqslant \lambda$ satisfies conditions (a), (b) and (c) above with L replaced with $L_\mu = F(L, \lambda - \mu)$ and λ replaced with μ.*

Proof: See Lemma 1.9 in [19].

Definition 2.3. *Let (X, π, M, I) be an IDS. We say that a point $x \in M$ satisfies the **strong tube condition** (**STC**), if there exists a section S through x such that $S = F(L, [0, 2\lambda]) \cap M$. Also, we say that a point $x \in M$ satisfies the **special strong tube condition** (**SSTC**) if it satisfies STC and the λ-tube $F(L, [0, 2\lambda])$ is such that $F(L, [0, \lambda]) \cap I(M) = \varnothing$.*

Theorem 2.4. *Let (X, π, M, I) be an IDS such that each point of M satisfies STC. Then ϕ is upper semicontinuous in X and it is continuous in $X \setminus M$. Moreover, if there are no initial points in M and ϕ is continuous at x then $x \notin M$.*

Proof: See Theorem 3.8 in [16].

Remark 2.5. *Before continuing, we note that if we assume that $I(M) \cap M = \varnothing$, then no point $x \in M$ is in any impulsive $\tilde{\pi}-$trajectory, except if the trajectory starts at x. This is a simple consequence of the definition of impulsive trajectories and this fact will be used in what follows.*

This following proposition is the main result of this section, and will later help us with negatively $\tilde{\pi}-$invariant sets, giving us a better understanding about the behavior of impulsive trajectories near the impulsive set M.

Proposition 2.6. *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$ and let $y \in M$ satisfy SSTC with λ-tube $F(L, [0, 2\lambda])$. Then $\tilde{\pi}(t)X \cap F(L, [0, \lambda]) = \varnothing$ for $t > \lambda$.*

Proof: Suppose contrary to the claim that there exist $t > \lambda$ and $z = \tilde{\pi}(t)x \in F(L, [0, \lambda])$ for some $x \in X$. Hence there exists $\mu \in [0, \lambda]$ such that $\pi(\mu)z \in L$. If $\mu = \lambda$, then $z \in S \subseteq M$, which is not possible by Remark 2.5, and hence $\mu \in [0, \lambda)$.

If $t < \phi(x)$, then $z = \tilde{\pi}(t)x = \pi(t)x$ and we consider $w = \pi(t - (\lambda - \mu))x$. We have that $\pi(\lambda)w = \pi(t + \mu)x = \pi(\mu)\pi(t)x = \pi(\mu)z \in L$. Thus $w \in S \subseteq M$, a contradiction with the fact that $t - (\lambda - \mu) < \phi(x)$.

Consider now the case $z = \tilde{\pi}(t)x = \pi(t')x^+$, where $x^+ \in I(M)$ and $t' \in [0, \phi(x^+))$. If $t' \geqslant \lambda - \mu$, then we consider $w = \pi(t' - (\lambda - \mu))x^+$. We have

$$\pi(\lambda)w = \pi(\mu)\pi(t')x^+ = \pi(\mu)z \in L$$

and consequently $w \in S \subseteq M$, a contradiction with the fact that $t' < \phi(x^+)$.

Finally, if $t' \in [0, \lambda - \mu)$, then

$$\pi(t' + \mu)x^+ = \pi(\mu)\pi(t')x^+ = \pi(\mu)z \in L$$

and $0 \leqslant \mu \leqslant t' + \mu < \lambda$, so $x^+ \in F(L, [0, \lambda)) \cap I(M)$, which is a contradiction with SSTC. ◻

3. Impulsive $\omega-$limits

We now want to give necessary and sufficient conditions to ensure the existence of a global attractor for an IDS (X, π, M, I) as defined in Definition 1.9. To begin our study, we define the notion of an impulsive $\omega-$limit.

Definition 3.1. *We represent the* **impulsive positive orbit** *of $x \in X$ starting at $s \geqslant 0$ by the set*

$$\tilde{\gamma}^+_s(x) = \{\tilde{\pi}(t)x \colon t \geqslant s\}.$$

Also we set $\tilde{\gamma}^+(x) = \gamma^+_0(x)$.

Given a subset $B \subseteq X$ we define $\tilde{\gamma}^+_s(B) = \bigcup_{x \in B} \tilde{\gamma}^+_s(x)$ and we define the **impulsive $\omega-$limit** *of B as the set*

$$\tilde{\omega}(B) = \bigcap_{t \geqslant 0} \overline{\tilde{\gamma}^+_t(B)}.$$

Analogously to the case of semigroups, we have the following characterization result for impulsive $\omega-$limits.

Lemma 3.2. *We have*

$$\tilde{\omega}(B) = \{x \in X \colon \text{ there exist sequences } \{x_n\}_{n \in \mathbb{N}} \subseteq B \text{ and } \{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \\ \text{with } t_n \to \infty \text{ such that } \tilde{\pi}(t_n)x_n \to x \text{ as } n \to \infty\},$$

and $\tilde{\omega}(B)$ is closed for every subset $B \subseteq X$.

Proof: If $x \in \tilde{\omega}(B)$ then given $n \in \mathbb{N}$, $x \in \overline{\tilde{\gamma}^+_n(B)}$, which implies that there exist $x_n \in B$ and $t_n \geqslant n$ such that $d(x, \tilde{\pi}(t_n)x_n) < \frac{1}{n}$ and therefore $t_n \to \infty$ and $\tilde{\pi}(t_n)x_n \to x$ as $n \to \infty$. Conversely, given $t \geqslant 0$, we have $\tilde{\pi}(t_n)x_n \in \tilde{\gamma}^+_t(B)$ for $t_n \geqslant t$ and hence $x \in \overline{\tilde{\gamma}^+_t(B)}$.

Finally, $\tilde{\omega}(B)$ is closed, since it is an intersection of closed sets, which concludes the proof. ◻

To continue with a more detailed description of the properties of impulsive $\omega-$limits, we will need a dissipativity condition on the IDS (X, π, M, I).

Definition 3.3. *An IDS (X, π, M, I) is called* **bounded dissipative** *if there exists a precompact set $K \subseteq X$ with $K \cap M = \varnothing$ that $\tilde{\pi}-$attracts all bounded subsets of X. Any set K satisfying these conditions will be called a* **pre-attractor***.*

Proposition 3.4. *If (X, π, M, I) is a bounded dissipative IDS with a pre-attractor K, then for any nonempty bounded subset B of X the impulsive $\omega-$limit $\tilde{\omega}(B)$ is nonempty, compact and $\tilde{\omega}(B) \subseteq \overline{K}$.*

Proof: If $x \in \tilde{\omega}(B)$, then $\tilde{\pi}(t_n)x_n \to x$ for some $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $t_n \to \infty$, and we have

$$\mathrm{d}_H(\tilde{\pi}(t_n)x_n, K) \leq \mathrm{d}_H(\tilde{\pi}(t_n)B, K) \to 0 \text{ as } n \to \infty,$$

since K $\tilde{\pi}-$attracts B. Hence $x \in \overline{K}$ and thus $\tilde{\omega}(B) \subseteq \overline{K}$. Since $\tilde{\omega}(B)$ is closed and contained in the compact set $\overline{K}$, $\tilde{\omega}(B)$ is compact.

If $\{t_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathbb{R}_+$, with $t_n \to \infty$, and $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in B, the attraction property of K ensures that there exists a convergent subsequence of $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ and hence $\tilde{\omega}(B)$ is nonempty. □

Proposition 3.5. *If (X, π, M, I) is a bounded dissipative IDS with a pre-attractor K then, for any nonempty bounded subset B of X, the impulsive $\omega-$limit $\tilde{\omega}(B)$ $\tilde{\pi}-$attracts B.*

Proof: Suppose contrary to the claim that there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$, $t_n \to \infty$, $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\epsilon_0 > 0$ such that $\mathrm{d}_H(\tilde{\pi}(t_n)x_n, \tilde{\omega}(B)) > \epsilon_0$. We know that $\mathrm{d}_H(\tilde{\pi}(t_n)x_n, K) \to 0$ as $n \to \infty$, so for some subsequence there exists $x \in \overline{K}$ such that $\tilde{\pi}(t_{n_k})x_{n_k} \to x$, which implies that $0 = \mathrm{d}_H(x, \tilde{\omega}(B)) \geqslant \epsilon_0$, which is a contradiction. □

3.1. **Positive invariance of impulsive $\omega-$limits.** In this subsection we establish the positive invariance for impulsive $\omega-$limits. We will need one auxiliary result:

Lemma 3.6. *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$ and each point of M satisfies STC. Let also $x \in X \setminus M$ and $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $z_n \to x$. Then, given $t \geqslant 0$, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\eta_n \to 0$ and $\tilde{\pi}(t + \eta_n)z_n \to \tilde{\pi}(t)x$.*

Proof: If $\phi(x) = \infty$, it follows by continuity of ϕ on $X \setminus M$ that for a given $t \in [0, \infty)$ there is a natural number $n_0 > 0$ such that $\phi(z_n) > t$, for all $n \geqslant n_0$. Consequently, for $n \geqslant n_0$, $\tilde{\pi}(t)z_n = \pi(t)z_n$, and the result follows from the continuity of $\pi(t)$ by setting $\eta_n = 0$, $n \in \mathbb{N}$.

Now, let us assume that $\phi(x) < \infty$. Since ϕ is continuous on $X \setminus M$, we may assume that $\phi(z_n) < \infty$ for all $n \in \mathbb{N}$.

Case 1: $0 \leqslant t < \phi(x)$.

In this case, consider $0 < \epsilon < \phi(x) - t$. By continuity of ϕ there is $n_0 \in \mathbb{N}$ such that $\phi(x) - \epsilon < \phi(z_n)$ for all $n \geqslant n_0$. Then $t < \phi(z_n)$ and $\tilde{\pi}(t)z_n = \pi(t)z_n$ for all $n \geqslant n_0$. Taking $\eta_n = 0$, $n \in \mathbb{N}$, it follows that

$$\tilde{\pi}(t + \eta_n)z_n = \pi(t)z_n \to \pi(t)x = \tilde{\pi}(t)x.$$

Case 2: $t = \phi(x)$.

Note that $\tilde{\pi}(t)x = \tilde{\pi}(\phi(x))x = x_1^+$. Thus

$$(z_n)_1 = \pi(\phi(z_n))z_n \to \pi(\phi(x))x = x_1.$$

Since I is continuous on M we have

$$(z_n)_1^+ = I((z_n)_1) \to I(x_1) = x_1^+.$$

By the continuity of ϕ we have that $\phi(z_n) = \phi(x) + \eta_n = t + \eta_n$, where $\{\eta_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\eta_n \to 0$. Hence,

$$\tilde{\pi}(t+\eta_n)z_n = \tilde{\pi}(\phi(z_n))z_n = (z_n)_1^+ \to x_1^+ = \tilde{\pi}(t)x.$$

Case 3: $t > \phi(x)$.

In this case, there exists $m \in \mathbb{N}$ such that $t = \sum_{i=0}^{m-1} \phi(x_i^+) + t'$ with $0 \leqslant t' < \phi(x_m^+)$. Define $(z_n)_i$ by

$$(z_n)_1 = \pi(\phi(z_n))z_n \text{ and } (z_n)_{i+1} = \pi((z_n)_i^+))(z_n)_i^+, \ i = 1, \dots, m-1.$$

Set $t_n = \sum_{i=0}^{m-1} \phi((z_n)_i^+)$. Since $\phi(z_n) \to \phi(x)$, we have

$$(z_n)_1 = \pi(\phi(z_n))z_n \to \pi(\phi(x))x = x_1.$$

By continuity of I we have

$$(z_n)_1^+ = I((z_n)_1) \to I(x_1) = x_1^+.$$

Now, since $\phi((z_n)_1^+) \to \phi(x_1^+)$, because $x_1^+ \notin M$, we get

$$(z_n)_2 = \pi(\phi((z_n)_1^+))(z_n)_1^+ \to \pi(\phi(x_1^+))x_1^+ = x_2.$$

By continuing with this process, we obtain

$$(z_n)_i \to x_i \quad \text{and} \quad (z_n)_i^+ \to x_i^+, \quad \text{for all} \quad i = 1, \dots, m.$$

Thus $\sum_{i=0}^{m-1} \phi((z_n)_i^+) \to \sum_{i=0}^{m-1} \phi(x_i^+)$. Define the sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ by $\eta_n = t_n + t' - t$. Note that $\eta_n \to 0$ and $t + \eta_n = t_n + t' \geqslant 0$. Then, since $t' < \phi((z_n)_m^+)$ for a large n, we get

$$\tilde{\pi}(t+\eta_n)z_n = \pi(t')(z_n)_m^+ \to \pi(t')x_m^+ = \tilde{\pi}(t)x.$$

Therefore, the result is proved. □

Proposition 3.7 (**Positive invariance**)**.** *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$ and each point of M satisfies STC. Then for any nonempty bounded subset B of X the set $\tilde{\omega}(B) \setminus M$ is positively $\tilde{\pi}-$invariant.*

Proof: Let $x \in \tilde{\omega}(B) \setminus M$ and $t \geqslant 0$. Then there exist $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $t_n \to \infty$ such that $\tilde{\pi}(t_n)x_n \to x$. Since $x \notin M$ and M is closed, we can assume that $\tilde{\pi}(t_n)x_n \notin M$. Therefore by Lemma 3.6 there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\eta_n \to 0$ and

$$\tilde{\pi}(t_n + t + \eta_n)x_n = \tilde{\pi}(t+\eta_n)\tilde{\pi}(t_n)x_n \to \tilde{\pi}(t)x \text{ as } n \to \infty.$$

Hence $\tilde{\pi}(t)x \in \tilde{\omega}(B)$. Observe that $\tilde{\pi}(t)x \notin M$, since any impulsive trajectory starting at a point of $X \setminus M$ never reaches M in finite time (note that $I(M) \cap M = \varnothing$). This shows positive $\tilde{\pi}$-invariance of $\tilde{\omega}(B) \setminus M$. □

3.2. **Negative invariance of impulsive $\omega-$limits.** The negative invariance of $\omega-$limits is harder to obtain. To this end, we must perform a deep analysis of the behavior of the impulsive semiflow $\tilde{\pi}$ near the impulsive set M. In [7], the authors consider the global attractor away from the impulsive set and avoided this study, but now, with a finer study, we are able to complete the theory in order to allow the attractor to be near M.

We begin with a sequence of results which, when put together, will give us the desired negative invariance.

Lemma 3.8. *Let (X,π,M,I) be an IDS such that each point in M satisfies STC, $z \notin M$ and $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $X \setminus M$ such that $z_n \to z$. Then if $\alpha_n \to 0$ and $\alpha_n \geqslant 0$, for all $n \in \mathbb{N}$, we have $\tilde{\pi}(\alpha_n)z_n \to z$.*

Proof: Since $z \notin M$, by the continuity of the function ϕ in $X \setminus M$, we can assume that $\frac{\phi(z)}{2} < \phi(z_n) < \frac{3\phi(z)}{2}$ and since $\alpha_n \to 0$ we can assume that $0 \leqslant \alpha_n < \phi(z_n)$ for all $n \in \mathbb{N}$. Hence

$$\tilde{\pi}(\alpha_n)z_n = \pi(\alpha_n)z_n \to \pi(0)z = z,$$

by the joint continuity of the semigroup π. $\square$

Corollary 3.9. *Under the assumptions of Lemma 3.6, there exists a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subseteq [0,\infty)$ such that $\epsilon_n \to 0$ and $\tilde{\pi}(t+\epsilon_n)z_n \to \tilde{\pi}(t)z$.*

Proof: By Lemma 3.6 we know that there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\eta_n \to 0$ and $\tilde{\pi}(t+\eta_n)z_n \to \tilde{\pi}(t)z \notin M$. Thus from Lemma 3.8 we have

$$\tilde{\pi}(t+\eta_n+|\eta_n|)z_n = \tilde{\pi}(|\eta_n|)\tilde{\pi}(t+\eta_n)z_n \to \tilde{\pi}(t)z$$

and the claim follows by setting $\epsilon_n = \eta_n + |\eta_n|$. $\square$

Lemma 3.10. *Let (X,π,M,I) be an IDS and let $x \in M$ satisfy STC with $\lambda-$tube $F(L,[0,2\lambda])$. Assume that there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that $z_n \in F(L,(\lambda,2\lambda])$ and $z_n \to x$. Then there exist a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ and a sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ such that $\epsilon_k > 0$ and $\epsilon_k \to 0$ as $k \to \infty$, $x_k = \pi(\epsilon_k)z_{n_k} \in M$, $\phi(z_{n_k}) = \epsilon_k$ and $x_k \to x$.*

Proof: There exist $\lambda < \lambda_n \leqslant 2\lambda$ such that $\pi(\lambda_n)z_n \in L$. We can choose a convergent subsequence $\lambda_{n_k} \to \bar{\lambda} \in [\lambda, 2\lambda]$. By the continuity of π, we have $\pi(\lambda_{n_k})z_{n_k} \to \pi(\bar{\lambda})x \in L$, since L is closed. Thus $x \in F(L,\bar{\lambda}) \cap F(L,\lambda)$, which implies $\bar{\lambda} = \lambda$. Set $\epsilon_k = \lambda_{n_k} - \lambda > 0$ and consider $x_k = \pi(\epsilon_k)z_{n_k}$. We have $\pi(\lambda)x_k = \pi(\lambda_{n_k})z_{n_k} \in L$, so $x_k \in S \subseteq M$. Moreover, $\epsilon_k \to 0$ and $x_k \to x$, by the continuity of π. If $w_{n_k} = \pi(t_0)z_{n_k} \in M$ for some $t_0 \in (0,\epsilon_k)$, then $\pi(\lambda_{n_k} - t_0)w_{n_k} \in L$ and $w_{n_k} \in F(L,[0,2\lambda]) \cap M = S$. Hence $w_{n_k} \in F(L,\lambda) \cap F(L,\lambda_{n_k} - t_0)$ and thus $t_0 = \lambda_{n_k} - \lambda = \epsilon_k$. This contradiction shows that $\phi(z_{n_k}) = \epsilon_k$. $\square$

Lemma 3.11. *Let (X,π,M,I) be an IDS with $I(M) \cap M = \varnothing$. Assume that every point from M satisfies SSTC and let $B \subseteq X$. If $y \in \tilde{\omega}(B) \cap M$ then $I(y) \in \tilde{\omega}(B) \setminus M$.*

Proof: Let $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $t_n \to \infty$, be such that $\tilde{\pi}(t_n)x_n \to y \in \tilde{\omega}(B)\cap M$. We can assume that $y_n = \tilde{\pi}(t_n)x_n \in F(L,[0,2\lambda])$ and $t_n > \lambda$ for some λ-tube $F(L,[0,2\lambda])$ through y. Since $\tilde{\pi}(t)X \cap F(L,[0,\lambda]) = \varnothing$ for $t > \lambda$ (see Proposition 2.6), it follows that $y_n \in F(L,(\lambda,2\lambda])$.

Lemma 3.10 then implies that there exist a positive sequence $\epsilon_k \to 0$ such that $\pi(\epsilon_k)y_{n_k} \in M$ for some subsequence $\{y_{n_k}\}$ and $\pi(\epsilon_k)y_{n_k} \to y$. By continuity of I, we have

$$\tilde{\pi}(t_{n_k}+\epsilon_k)x_{n_k} = \tilde{\pi}(\epsilon_k)y_{n_k} = I(\pi(\epsilon_k)y_{n_k}) \to I(y).$$

Therefore $I(y) \in \tilde{\omega}(B)\setminus M$, since $I(M)\cap M = \varnothing$. □

Now with all these results at hand, we are able to prove the negative invariance for the impulsive $\omega-$limit.

Proposition 3.12 (**Negative invariance**). *Let (X,π,M,I) be an IDS such that $I(M)\cap M = \varnothing$ and each point from M satisfies SSTC and let $B \subseteq X$. If $\tilde{\omega}(B)$ is compact and $\tilde{\pi}-$attracts B, then $\tilde{\omega}(B)\setminus M$ is negatively $\tilde{\pi}-$invariant.*

Proof: Let $x \in \tilde{\omega}(B)\setminus M$ and $t \geqslant 0$. Then there exist sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $t_n \to \infty$ such that

$$\tilde{\pi}(t_n)x_n \to x \text{ as } n \to \infty.$$

Now, since $\tilde{\omega}(B)$ is compact and $\tilde{\pi}-$attracts B, we can assume that $\{\tilde{\pi}(t_n - t)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence (which we denote the same, and we already assumed that $t_n > t$, since $t_n \to \infty$ and t is fixed). Thus $\tilde{\pi}(t_n - t)x_n \to y \in \tilde{\omega}(B)$.

• Case 1: $y \in M$.

In this case, using Proposition 2.6, we can assume that all points $y_n := \tilde{\pi}(t_n - t)x_n$ are in $F(L,(\lambda,2\lambda])$, where $F(L,[0,2\lambda])$ is a $\lambda-$tube through y. Hence there exists a sequence $\epsilon_n \to 0$, $\epsilon_n > 0$, such that $z_n := \pi(\epsilon_n)y_n \in M$ and $z_n \to y$, by Lemma 3.10, renaming the sequence if necessary. By the continuity of I, we have $z_n^+ := \tilde{\pi}(\epsilon_n)y_n = I(z_n) \to I(y) =: z \in \tilde{\omega}(B)\setminus M$, using Lemma 3.11.

Now, by Corollary 3.9, there exists a non-negative sequence $\alpha_n \to 0$ such that

$$\tilde{\pi}(t+\alpha_n)z_n^+ \to \tilde{\pi}(t)z.$$

But $\tilde{\pi}(t+\alpha_n)z_n^+ = \tilde{\pi}(t_n+\epsilon_n+\alpha_n)x_n$ and again, using Lemma 3.8, we have $\tilde{\pi}(t+\alpha_n)z_n^+ \to x$, therefore $x = \tilde{\pi}(t)z \in \tilde{\pi}(t)(\tilde{\omega}(B)\setminus M)$.

• Case 2: $y \notin M$.

We know that there exists a non-negative sequence $\epsilon_n \to 0$ such that

$$\tilde{\pi}(t+\epsilon_n)y_n \to \tilde{\pi}(t)y.$$

But $\tilde{\pi}(t+\epsilon_n)y_n = \tilde{\pi}(t_n+\epsilon_n)x_n$, and using again Lemma 3.8 we know that $\tilde{\pi}(t+\epsilon_n)y_n \to x$. Therefore $x = \tilde{\pi}(t)y \in \tilde{\pi}(t)(\tilde{\omega}(B)\setminus M)$. □

3.3. **Attraction.** We already know (see Proposition 3.5) that if the IDS (X, π, M, I) is bounded dissipative and B is a nonempty subset of X, then $\tilde{\omega}(B)$ $\tilde{\pi}-$attracts B. But $\tilde{\omega}(B)$ can possess points in M, but with our definition of a global attractor, we do not want this to happen and hence we need to be able to prove that $\tilde{\omega}(B) \setminus M$ also $\tilde{\pi}-$attracts B. This subsection is devoted to this goal.

Lemma 3.13. *Let (X, π, M, I) be a bounded dissipative IDS with a pre-attractor K such that $I(M) \cap M = \varnothing$ and every point from M satisfies SSTC. Assume that there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. If B is a nonempty bounded subset of X, then $\tilde{\omega}(B) \cap M \subseteq \overline{\tilde{\omega}(B) \setminus M}$.*

Proof: Let $x \in \tilde{\omega}(B) \cap M$. Then there exist sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $t_n \to \infty$ such that $\tilde{\pi}(t_n)x_n \to x$. By Proposition 2.6 we can assume that $z_n = \tilde{\pi}(t_n)x_n \in F(L, (\lambda, 2\lambda])$, where $F(L, [0, 2\lambda])$ is a λ-tube through x. We can choose a subsequence if necessary, which we will call the same, $\{\lambda_n\}_{n\in\mathbb{N}} \subset (\lambda, 2\lambda]$ such that $\lambda_n \to \lambda$ and $\pi(\lambda_n - \lambda)z_n \in M$, as in Lemma 3.10. We may also assume that $0 < \epsilon_n := \lambda_n - \lambda < \frac{\xi}{2}$ for $n \in \mathbb{N}$.

Recall that there exists $\epsilon_x > 0$ such that $F(x, (0, \epsilon_x)) \cap M = \varnothing$. Let $m_0 \in \mathbb{N}$ be such that $\frac{1}{m_0} < \min\{\epsilon_x, \frac{\xi}{2}\}$. For each $m \geqslant m_0$ we consider the sequence $w_n^m = \tilde{\pi}(t_n - \frac{1}{m})x_n$, $n \in \mathbb{N}$. By the bounded dissipativity we can assume that $w_n^m \to y_m \in \tilde{\omega}(B)$ as $n \to \infty$, for each $m \geqslant m_0$.

We claim that $\phi(w_n^m) > \frac{1}{m}$ for all $n \in \mathbb{N}$ and $m \geqslant m_0$. Indeed, suppose that $\phi(w_n^m) \leqslant \frac{1}{m}$ for some $n \in \mathbb{N}$ and $m \geqslant m_0$. This means that $\pi(\phi(w_n^m))w_n^m \in M$ and $v_n^m = \tilde{\pi}(\phi(w_n^m))w_n^m \in I(M)$. We have

$$\pi(\epsilon_n + 1/m - \phi(w_n^m))v_n^m = \pi(\epsilon_n)\pi(1/m - \phi(w_n^m))v_n^m = \pi(\epsilon_n)z_n \in M$$

since $1/m - \phi(w_n^m) < 1/m < \xi$, and this is a contradiction, since $0 < \epsilon_n + 1/m - \phi(w_n^m) < \epsilon_n + 1/m < \xi$ and $v_n^m \in I(M)$.

This shows that for $n \in \mathbb{N}$ and $m \geqslant m_0$

$$\pi(1/m)w_n^m = \tilde{\pi}(1/m)w_n^m = \tilde{\pi}(1/m)\tilde{\pi}(t_n - 1/m)x_n = \tilde{\pi}(t_n)x_n.$$

By the continuity of π we get $\pi(1/m)y_m = x \in M$. Since $1/m < \epsilon_x$, we obtain $y_m \in \tilde{\omega}(B) \setminus M$.

If $\{y_m\}$ does not converge to x, then we can choose a convergent subsequence $\{y_{m_l}\}$ to a point $x_0 \neq x$, but $x = \pi(1/m_l)y_{m_l} \to \pi(0)x_0 = x_0$, which gives us a contradiction and proves that $y_m \to x$. □

Proposition 3.14 (**Attraction**)**.** *With the hypotheses of Lemma 3.13, if $\tilde{\omega}(B)$ $\tilde{\pi}-$attracts B, then $\tilde{\omega}(B) \setminus M$ $\tilde{\pi}-$attracts B.*

Proof: By Lemma 3.13, $\tilde{\omega}(B) = \overline{\tilde{\omega}(B) \setminus M}$ and the result follows directly by the definition of attraction. □

4. Global attractors for impulsive dynamical systems

We would like to formulate a theorem on the existence of global attractors for impulsive dynamical systems. In [7, Theorem 3.7] the authors considered global attractors as in Definition 1.9, which are compact instead of only precompact in X. They showed that the existence of the compact global attractor is equivalent to the existence of a compact subset K of X such that $K \cap M = \varnothing$ and K $\tilde{\pi}-$attracts all bounded subsets of X. This excludes important classes of dynamical systems with impulse, e.g., those containing a periodic orbit with points from M in its ω-limit set (cp. Example 1.8). Moreover, the long-time dynamics in their case does not differ from the dynamics of the original semiflow π. Since our definition of a global attractor implies that the impulsive dynamical system is bounded dissipative (with the global attractor being the pre-attractor), we make it a starting assumption for the existence of a global attractor as in Definition 1.9.

Before proving the existence result, we first give some characterizations of the global attractor which are very useful and are analogous to the characterizations we already have for the case with no impulse.

Proposition 4.1. *With Definition 1.9, if $\mathcal{A}$ exists, it is uniquely determined.*

Proof: Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy Definition 1.9. Then by (ii) and (iii) $\mathrm{d}_H(\mathcal{A}_1, \mathcal{A}_2) = 0 = \mathrm{d}_H(\mathcal{A}_2, \mathcal{A}_1)$ and hence $\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}$. Therefore, by (i)

$$\mathcal{A}_1 = \overline{\mathcal{A}_1} \setminus M = \overline{\mathcal{A}_2} \setminus M = \mathcal{A}_2.$$

□

Definition 4.2. *We say that a function $\psi\colon \mathbb{R} \to X$ is a* **global solution** *of $\tilde{\pi}$ if*

$$\tilde{\pi}(t)\psi(s) = \psi(t+s), \text{ for all } t \geqslant 0 \text{ and } s \in \mathbb{R}.$$

Moreover, if $\psi(0) = x$ we say that ψ is a **global solution through** *x.*

Proposition 4.3. *With Definition 1.9, if the IDS (X, π, M, I) has a global attractor $\mathcal{A}$ and $I(M) \cap M = \varnothing$ then*

$$\mathcal{A} = \{x \in X\colon \text{there exists a bounded global solution of } \tilde{\pi} \text{ through } x\}.$$

Proof: If $\psi(\cdot)$ is a bounded global solution of $\tilde{\pi}$ then $\psi(\mathbb{R}) \cap M = \varnothing$, since if $\psi(t_0) \in M$ for some $t_0 \in \mathbb{R}$ then $\tilde{\pi}(s)\psi(t_0 - s) = \psi(t_0) \in M$ for each $s > 0$ which cannot happen (the impulsive semiflow from x cannot reach M in positive time for any $x \in X$, because $I(M) \cap M = \varnothing$).

Hence $\psi(\mathbb{R}) \cap M = \varnothing$; by its invariance we can see that $\psi(\mathbb{R}) \subseteq \overline{\mathcal{A}}$ and therefore $\psi(\mathbb{R}) \subseteq \overline{\mathcal{A}} \setminus M = \mathcal{A}$.

For the reverse inclusion, if $x \in \mathcal{A}$ then $x \in \tilde{\pi}(1)\mathcal{A}$ and there exists $x_{-1} \in \mathcal{A}$ such that $\tilde{\pi}(1)x_{-1} = x$. Again, since $x_{-1} \in \mathcal{A}$ there exists $x_{-2} \in \mathcal{A}$ such that $\tilde{\pi}(1)x_{-2} = x_{-1}$. Inductively, we can construct a sequence $\{x_{-n}\}_{n\in\mathbb{N}}$ such that $\tilde{\pi}(1)x_{-n-1} = x_{-n}$ for all $n \geqslant 0$, with $x_0 = x$. Then we can define

$$\psi(t) = \begin{cases} \tilde{\pi}(t+n)x_{-n}, & \text{if } t \in [-n, -n+1],\ n \in \mathbb{N}, \\ \tilde{\pi}(t)x_0, & \text{if } t \geqslant 0. \end{cases}$$

It is clear that $\psi(\mathbb{R}) \subseteq \mathcal{A}$ hence it is bounded and ends the proof of our claim. □

Proposition 4.4. *With Definition 1.9, if the IDS (X, π, M, I) has a global attractor $\mathcal{A}$ and $I(M) \cap M = \varnothing$, then, denoting by $\mathcal{B}(X)$ the collection of all bounded subsets of X, we have*

$$\mathcal{A} = \bigcup_{B\in\mathcal{B}(X)} \left(\tilde{\omega}(B) \setminus M\right).$$

Proof: Let B be a bounded subset of X. By Proposition 3.4, $\tilde{\omega}(B) \subseteq \overline{\mathcal{A}}$ and hence $\tilde{\omega}(B) \setminus M \subseteq \overline{\mathcal{A}} \setminus M = \mathcal{A}$, which proves one inclusion.

For the other one, choose $x_0 \in \mathcal{A}$. Then $x_0 \notin M$ and using Proposition 4.3 we consider a bounded global solution ψ of $\tilde{\pi}$ through x_0 and an arbitrary sequence $t_n \to \infty$. We have $\tilde{\pi}(t_n)\psi(-t_n) = x_0$ and thus $x_0 \in \tilde{\omega}(\psi(\mathbb{R}))$. Concluding, we obtain $x_0 \in \tilde{\omega}(\psi(\mathbb{R})) \setminus M$. □

Proposition 4.5. *With Definition 1.9, if the IDS (X, π, M, I) has a global attractor $\mathcal{A}$ then it is the minimal subset among all subsets $K \subset X$ with $K = \overline{K} \setminus M$ which $\tilde{\pi}-$attracts all bounded subsets of X.*

Proof: By the invariance of $\mathcal{A}$, if K is such a subset, we have

$$\mathrm{d}_H(\mathcal{A}, K) = \mathrm{d}_H(\tilde{\pi}(t)\mathcal{A}, K) \to 0 \text{ as } t \to \infty.$$

Hence $\overline{\mathcal{A}} \subseteq \overline{K}$ and therefore $\mathcal{A} \subseteq K$. □

An important class of results for continuous semigroups consists of theorems on existence of global attractors. Below we show a counterpart of [30, Theorem 10.5] for impulsive dynamical systems.

Definition 4.6. *An impulsive dynamical system (X, π, M, I) is called **strongly bounded dissipative** if there exists a nonempty precompact set K in X such that $K \cap M = \varnothing$ and $\tilde{\pi}$-absorbs all bounded subsets of X, i.e., for any bounded subset B of X there exists $t_B \geqslant 0$ such that $\tilde{\pi}(t)B \subseteq K$ for all $t \geqslant t_B$.*

Note that if (X, π, M, I) is strongly bounded dissipative, then it is bounded dissipative.

Theorem 4.7. *Let (X, π, M, I) be a strongly bounded dissipative IDS with $\tilde{\pi}-$absorbing set K, such that $I(M) \cap M = \varnothing$, every point in M satisfies SSTC and there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. Then (X, π, M, I) has a global attractor $\mathcal{A}$ and we have $\mathcal{A} = \tilde{\omega}(K) \setminus M$.*

Proof: By Propositions 3.7 and 3.12 we know that $\tilde{\omega}(K) \setminus M$ is $\tilde{\pi}$-invariant. Since K is nonempty, it follows from Proposition 3.4 that $\tilde{\omega}(K)$ is a nonempty compact subset of $\overline{K}$. If $\tilde{\omega}(K) \cap M = \varnothing$, then $\tilde{\omega}(K) \setminus M$ is nonempty. If $\tilde{\omega}(K) \cap M \neq \varnothing$, then by Lemma 3.11 $\tilde{\omega}(K) \setminus M$ is nonempty.

Moreover, we have

$$\overline{\tilde{\omega}(K) \setminus M} \subseteq \overline{\tilde{\omega}(K)} = \tilde{\omega}(K),$$

so $\tilde{\omega}(K) \setminus M$ is a precompact subset of X and by Lemma 3.13, $\tilde{\omega}(K) = \overline{\tilde{\omega}(K) \setminus M}$ which implies that

$$\tilde{\omega}(K) \setminus M = \overline{\tilde{\omega}(K) \setminus M} \setminus M.$$

We are left to show that $\tilde{\omega}(K) \setminus M$ $\tilde{\pi}-$attracts all bounded subsets of X. First, observe that for any bounded subset B of X we have $\tilde{\omega}(B) \subseteq \tilde{\omega}(K)$. Indeed, if $x \in \tilde{\omega}(B)$, then there exist sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $t_n \to \infty$ such that $\tilde{\pi}(t_n)x_n \to x$. From the strong bounded dissipativity we know that $\tilde{\pi}(t_B)x_n \in K$ and $\tilde{\pi}(t_n - t_B)\tilde{\pi}(t_B)x_n \to x$, so $x \in \tilde{\omega}(K)$.

Since $\tilde{\omega}(K) \setminus M$ contains $\tilde{\omega}(B) \setminus M$ for every bounded $B \subseteq X$ and by Proposition 3.14, $\tilde{\omega}(B) \setminus M$ $\tilde{\pi}-$attracts B, it follows that $\tilde{\omega}(K) \setminus M$ $\tilde{\pi}-$attracts all bounded subsets of X, which concludes the proof. □

4.1. **Example 1.** In this subsection we present a simple example to illustrate the theory described above.

Consider the impulsive dynamical system in $X = \mathbb{R}^2$ given by

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = -y, \\ (x(0), y(0)) = (x_0, y_0), \\ I\colon M \to I(M), \end{cases} \tag{4.1}$$

where $M = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$, $I(M) \subset \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 9\}$ and the function $I\colon M \to I(M)$ is given as follows: given $(x, y) \in M$ we consider the line segment $\Gamma_{(x,y)}$ that connects the points (x, y) and $(3, y)$. The point $I(x, y)$ is the point in the intersection $\Gamma_{(x,y)} \cap I(M)$, as we can see in Figure 1 below.

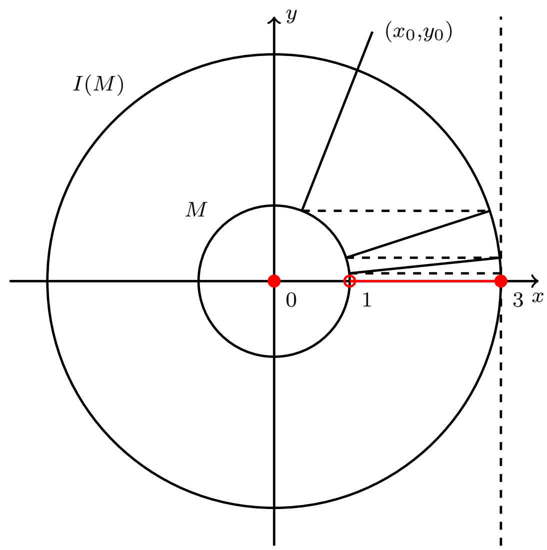

Figure 1: Impulsive trajectory of $(x_0, y_0) \in \mathbb{R}^2$.

Let $\{\pi(t)\colon t \geqslant 0\}$ be the semigroup in $\mathbb{R}^2$ generated by (4.1) with no impulse; that is, $\pi(t)(x_0, y_0) = (x_0 e^{-t}, y_0 e^{-t})$ and consider the IDS (X, π, M, I). It is easy to see that each point of M satisfies SSTC, $I(M) \cap M = \varnothing$, there exists $\xi > 0$ such that $\phi(x, y) \geqslant \xi$ for all $(x, y) \in I(M)$.

If we let $K = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leqslant 9\} \setminus M$, it is clear that K is a precompact subset of $\mathbb{R}^2$, $K \cap M = \varnothing$ and K $\tilde{\pi}-$absorbs all bounded subsets of X, hence (X, π, M, I) is strongly bounded dissipative with $\tilde{\pi}-$absorbing set K and Theorem 4.7 ensures that (X, π, M, I) has a global attractor $\mathcal{A} = \tilde{\omega}(K) \setminus M$.

We can see that $\tilde{\omega}(K) = \{(0, 0)\} \cup \{(x, 0)\colon x \in [1, 3]\}$ and hence $\mathcal{A} = \{(0, 0)\} \cup \{(x, 0)\colon x \in (1, 3]\}$, which is the part in red in Figure 1 above.

4.2. **Example 2.** Consider the dynamical system generated by

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0, \end{cases} \tag{4.2}$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. We suppose that all the solutions of (4.2) are defined in the whole real line and give rise to a semigroup π on $\mathbb{R}^n$.

Let $M \subseteq \mathbb{R}^n$ be an impulsive set and $I\colon M \to \mathbb{R}^n$ be an impulse function. We consider M and I such that $I(M) \cap M = \varnothing$, each point of M satisfies SSTC, there exists $\xi > 0$ such that $\phi(x) \geqslant \xi$ for all $x \in M$ and the conditions (1.1) and (G) are satisfied.

Then, we consider the associated impulsive system

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0, \\ I\colon M \to \mathbb{R}^n. \end{cases} \tag{4.3}$$

Now, let $V \in C^1(\mathbb{R}^n, \mathbb{R})$ be a function satisfying the following conditions:

(i) $\nabla V(x) \cdot f(x) \leqslant \alpha_1 - \alpha_2 V(x)$, for all $x \in \mathbb{R}^n$,

(ii) $V(I(x)) \leqslant \mu$, for all $x \in M$,

(iii) $V^{-1}((-\infty, \mu + \frac{\alpha_1}{\alpha_2}])$ is bounded,

where $\alpha_1, \alpha_2 > 0$ and $\mu > 0$.

In the sequel, we shall provide conditions for the system (4.3) to be strongly bounded dissipative.

Lemma 4.8. *If $z \in I(M)$ then $V(\tilde{\pi}(t)z) \leqslant \mu + \frac{\alpha_1}{\alpha_2}$ for all $t \geqslant 0$.*

Proof: Let $z \in I(M)$ and $0 \leqslant t \leqslant \phi(z)$ (if $\phi(z) = \infty$ we take $0 \leqslant t < \phi(z)$). Then, by (i), we have

$$\frac{d}{dt} V(\pi(t)z) = \nabla V(\pi(t)z) \cdot f(\pi(t)z) \leqslant \alpha_1 - \alpha_2 V(\pi(t)z).$$

Therefore

$$V(\pi(t)z) \leqslant e^{-\alpha_2 t} V(z) + \frac{\alpha_1}{\alpha_2} \leqslant V(z) + \frac{\alpha_1}{\alpha_2} \leqslant \mu + \frac{\alpha_1}{\alpha_2}, \text{ for all } 0 \leqslant t \leqslant \phi(z),$$

which implies that

$$V(\tilde{\pi}(t)z) \leqslant \mu + \frac{\alpha_1}{\alpha_2}, \text{ for all } 0 \leqslant t < \phi(z).$$

If $\phi(z) = \infty$, we are done. Otherwise, since $z_1^+ := \tilde{\pi}(\phi(z))z \in I(M)$, we can repeat the process above starting from z_1^+ and inductively we obtain the desired result. □

Theorem 4.9. *The system* (4.3) *is strongly bounded dissipative.*

Proof: Let $K = V^{-1}((-\infty, \mu + \frac{\alpha_1}{\alpha_2}]) \setminus M$. Then K is a precompact set (since K is bounded in $\mathbb{R}^n$ by (iii)) and $K \cap M = \varnothing$. We will show that K $\tilde{\pi}-$absorbs bounded subsets of $\mathbb{R}^n$ and it is sufficient to prove that for each $x \in \mathbb{R}^n$ there exists $\delta_x > 0$ and $T = T(x, \delta_x) \geqslant 0$ such that

$$\tilde{\pi}(t)y \in K, \text{ for all } y \in B_{\delta_x}(x) \text{ and } t \geqslant T.$$

We have some cases to consider:

<u>Case 1:</u> $x \notin M$ and $\phi(x) = \infty$.

First choose a ball B centered in x and set $\beta = \max_{y \in \overline{B}} V(y)$. Let $k > \max\{0, -\alpha_2^{-1} \ln(\frac{\mu}{|\beta|+1})\}$ be given, and by the continuity of ϕ, there exists $\delta = \delta(x, k) > 0$ such that $\phi(y) > k$ for all $y \in B_\delta(x) \subset \overline{B}$. Now we can split the ball $B_\delta(x)$ in $B_1 \cup B_2$ where $B_1 = \{y \in B_\delta(x) : \phi(y) = \infty\}$ and $B_2 = \{y \in B_\delta(x) : k < \phi(y) < \infty\}$. In B_1, using the arguments from the proof of Lemma 4.8 we have

$$V(\pi(t)y) \leqslant e^{-\alpha_2 t} V(y) + \frac{\alpha_1}{\alpha_2} \leqslant e^{-\alpha_2 t} \beta + \frac{\alpha_1}{\alpha_2},$$

Therefore, if $T := \max\{0, -\alpha_2^{-1} \ln(\frac{\mu}{|\beta|+1})\}$ then

$$V(\pi(t)y) \leqslant \mu + \frac{\alpha_1}{\alpha_2}, \text{ for all } t \geqslant T \text{ and } y \in B_1.$$

In B_2, we see that

$$V(\pi(t)y) \leqslant \mu + \frac{\alpha_1}{\alpha_2}, \text{ for all } k \leqslant t \leqslant \phi(y) \text{ and } y \in B_2,$$

and Lemma 4.8 shows that $V(\tilde{\pi}(t)y) \leqslant \mu + \frac{\alpha_1}{\alpha_2}$ for all $t \geqslant \phi(y)$ and $y \in B_\delta(x)$.
Case 2: $x \notin M$ and $\phi(x) < \infty$.

By the continuity of ϕ, there exists $\delta_x > 0$ such that $|\phi(y) - \phi(x)| < 1$ if $y \in B_{\delta_x}(x)$. Thus if $T \geqslant \sup\{\phi(y) \colon y \in B_{\delta_x}(x)\}$ then the conclusion follows from Lemma 4.8.
Case 3: $x \in M$ and $\phi(x) = \infty$.

The point x satisfies SSTC with $\lambda-$tube $F(L, [0, 2\lambda])$ and a section S through x. Since the tube contains a neighborhood of x, there is $\epsilon > 0$ such that

$$B_\epsilon(x) \subset F(L, [0, 2\lambda]).$$

Set

$$H_1 = F(L, (\lambda, 2\lambda]) \cap B_\epsilon(x) \qquad \text{and} \qquad H_2 = F(L, [0, \lambda]) \cap B_\epsilon(x).$$

Observe that $\phi(y) \leqslant \lambda$ for all $y \in H_1$. Indeed, if $y \in H_1$ then $\pi(\mu)y \in L$ for some $\mu \in (\lambda, 2\lambda]$ and thus $\pi(\mu - \lambda)y \in F(L, \lambda) \subseteq M$. Hence we get $\phi(y) \leqslant \mu - \lambda \leqslant \lambda$.

By the proof of Case 2, we obtain

$$V(\tilde{\pi}(t)y) \leqslant \mu + \frac{\alpha_1}{\alpha_2},$$

for all $y \in H_1$ and for all $t \geqslant \lambda$.

On the other hand, since $\phi(x) = \infty$, for any $k > 0$ there exists $0 < \delta = \delta(x, k, \epsilon) < \epsilon$ such that $\phi(y) > k$ for $y \in B_\delta(x) \cap H_2$. Indeed, assume contrary to the claim that there exists $k_0 > 0$ and a sequence $x_n \in H_2$ such that $x_n \to x$ and $\phi(x_n) \leqslant k_0$. Then $\pi(\lambda_n)x_n \in L$ for some $\lambda_n \in [0, \lambda]$. Choosing a subsequence if necessary, we can assume that $\lambda_n \to \lambda$ and $\phi(x_n) \to 0$, since $\phi(x) = \infty$. Hence we have for large n

$$\phi(x_n) < \lambda \quad \text{and} \quad \pi(\phi(x_n))x_n \in F(L, [0, 2\lambda]) \cap M = S = F(L, \lambda).$$

We obtain $\lambda_n = \lambda + \phi(x_n)$, which contradicts the fact that $\lambda_n \in [0, \lambda]$.

Therefore, in the same way as in the proof of Case 1, we can choose $\delta > 0$ and $T > 0$ such that

$$V(\tilde{\pi}(t)y) \leqslant \mu + \frac{\alpha_1}{\alpha_2}$$

for all $y \in B_\delta(x) \cap H_2$ and for all $t \geqslant T$.

Thus we have

$$V(\tilde{\pi}(t)y) \leqslant \mu + \frac{\alpha_1}{\alpha_2}$$

for all $y \in B_\delta(x)$ and for all $t \geqslant \max\{\lambda, T\}$.
Case 4: $x \in M$ and $\phi(x) < \infty$.

The proof follows the idea of the proof of Case 3. The only difference is that we use the upper semicontinuity of ϕ at $x \in M$ and the proof of Case 2 to show that there exist $\delta > 0$ and $T = T(x, \delta) > 0$ such that

$$V(\tilde{\pi}(t)y) \leqslant \mu + \frac{\alpha_1}{\alpha_2}$$

for all $y \in B_\delta(x) \cap H_2$ and for all $t \geqslant T$. When we work in H_1, the proof is the same as the proof presented in Case 3.

This concludes all the possible cases and proves that system (4.3) is strongly bounded dissipative.

□

Corollary 4.10. *The system* (4.3) *has a global attractor* $\mathcal{A}$.

4.3. **Example 3.** Consider the following nonlinear reaction-diffusion initial boundary value problem

$$\begin{cases} u_t - \Delta u = f(u), \ t > 0, \\ u|_{\partial\Omega} = 0, \ t > 0, \\ u(0) = u_0 \in L^2(\Omega), \end{cases} \tag{4.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, Ω is a bounded smooth domain of $\mathbb{R}^n$ with smooth boundary and Δ is the Laplace operator in Ω.

We assume the following conditions:

a) $|f(u) - f(v)| \leqslant c|u - v|(1 + |u|^{\rho-1} + |v|^{\rho-1})$, for all $u, v \in \mathbb{R}$, where $c > 0$ is a constant and $1 \leqslant \rho \leqslant 1 + \frac{4}{n}$;

b) $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplace operator in $L^2(\Omega)$ with Dirichlet boundary condition.

It follows that for any initial data $u_0 \in L^2(\Omega)$ there exists a unique solution u of (4.4) such that

$$u \in C([0, \infty); L^2(\Omega)),$$

and the mapping $u_0 \mapsto u(t)$ is continuous in $L^2(\Omega)$. Moreover, it takes bounded subsets of $L^2(\Omega)$ into precompact sets in $L^2(\Omega)$. Thus $\pi(t)u_0 := u(t)$, $t \geqslant 0$, where u is the solution of (4.4) defines a compact semigroup π on $X = L^2(\Omega)$. The reader may see [2, 3] for more details.

Let $M \subseteq L^2(\Omega)$ be an impulsive set and $I\colon M \to L^2(\Omega)$ be an impulse function. We consider M and I such that $I(M) \cap M = \varnothing$, each point from M satisfies SSTC, there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$.

We also assume the following additional condition:

c) $\|I(u)\|_2^2 \leqslant \mu$ for all $u \in M$, where $\mu > 0$.

Let $\tilde{u}(t) = \tilde{\pi}(t)u_0$ be the associated impulsive solution of

$$\begin{cases} u_t - \Delta u = f(u), \ t > 0, \\ u|_{\partial\Omega} = 0, \ t > 0, \\ u(0) = u_0, \\ I \colon M \to L^2(\Omega). \end{cases} \tag{4.5}$$

Lemma 4.11. *There exist constants $C, \eta > 0$ such that if $z \in I(M)$ then $\|\tilde{\pi}(t)z\|_2^2 \leqslant \mu + \frac{C|\Omega|}{\eta}$ for all $t \geqslant 0$.*

Proof: Consider the equation $u_t - \Delta u = f(u)$ and take the scalar product with $u \in L^2(\Omega)$. Then

$$\frac{1}{2}\frac{d\|u\|_2^2}{dt} = (f(u) + \Delta u, u).$$

Note that there exist constants $C, \eta > 0$ such that $f(s)s \leqslant (\lambda_1 - \eta)s^2 + C$, for all $s \in \mathbb{R}$ and we have

$$\frac{1}{2}\frac{d\|u\|_2^2}{dt} = (f(u) - (\lambda_1 - \eta)u, u) + ((\lambda_1 - \eta)u + \Delta u, u),$$

and hence, using the Poincaré inequality,

$$\frac{d\|u\|_2^2}{dt} \leqslant -2\eta\|u\|_2^2 + 2C|\Omega|, \ t > 0. \tag{4.6}$$

Define $y(t) = \|u(t)\|_2^2$ and note that $u(t)$ and $y(t)$ depend on the initial data, that is, $u(t) = u(t; 0, u_0)$ and $y(t) = y(t; 0, \|u_0\|_2^2)$. By (4.6), we have

$$y'(t) + 2\eta y(t) \leqslant 2C|\Omega|, \ t > 0. \tag{4.7}$$

By inequality (4.7), if $z \in I(M)$ and $0 \leqslant t < \phi(z)$, we have by c)

$$\|\widetilde{\pi}(t)z\|_2^2 = \|\pi(t)z\|_2^2 = y(t; 0, \|z\|_2^2) \leqslant \|z\|_2^2 e^{-2\eta t} + \frac{C|\Omega|}{\eta}(1 - e^{-2\eta t}) \leqslant \mu + \frac{C|\Omega|}{\eta}.$$

If $t = \phi(z)$, we get with $z_1 = \pi(\phi(z))z \in M$

$$\|\widetilde{\pi}(\phi(z))z\|_2^2 = \|I(z_1)\|_2^2 \leqslant \mu \leqslant \mu + \frac{C|\Omega|}{\eta}.$$

In this way, we can repeat the process above starting in $z_1^+ = I(z_1)$ and inductively we obtain the desired result. ∎

We define from now on $\gamma := \mu + \frac{C|\Omega|}{\eta}$.

Lemma 4.12. *Let $B \subseteq L^2(\Omega)$ be a bounded set. Then there is $T = T(B) > 0$ such that $\|\tilde{\pi}(t)z\|_2^2 \leqslant \gamma$ for all $t \geqslant T$ and for all $z \in B$.*

Proof: Let B be a bounded set in $L^2(\Omega)$. Then there is $L > 0$ such that $\|u\|_2^2 \leqslant L$ for all $u \in B$. We may decompose the set B in the following way

$$B = B_1 \cup B_2,$$

where $B_1 = \{u \in B \colon M^+(u) \neq \varnothing\}$ and $B_2 = \{u \in B \colon M^+(u) = \varnothing\}$.

If $z \in B_2$ then $\widetilde{\pi}(t)z = \pi(t)z$ for all $t \geqslant 0$. Set $T_2 = \max\left\{0, -\frac{1}{2\eta}\ln\left(\frac{\mu}{L}\right)\right\}$. If $t \geqslant T_2$ we have

$$\|\widetilde{\pi}(t)z\|_2^2 = \|\pi(t)z\|_2^2 = y(t;0,\|z\|_2^2) \leqslant \|z\|_2^2 e^{-2\eta t} + \frac{C|\Omega|}{\eta}(1-e^{-2\eta t}) \leqslant Le^{-2\eta t} + \frac{C|\Omega|}{\eta} \leqslant \gamma,$$

for all $z \in B_2$.

Now, we can write $B_1 = B_1^+ \cup B_1^-$, where

$$B_1^+ = \{u \in B_1 \colon \phi(u) > 1\} \text{ and } B_1^- = \{u \in B_1 \colon \phi(u) \leqslant 1\}.$$

It is clear from Lemma 4.11 that $\|\tilde{\pi}(t)u\|_2^2 \leqslant \gamma$ if $u \in B_1^-$ and $t \geqslant 1$. Since $\pi(\tau)$ is a compact map for each $\tau > 0$, hence if $\alpha \in (0,1)$ it follows that $\tilde{\pi}(\alpha)B_1^+ = \pi(\alpha)B_1^+$ is precompact in $L^2(\Omega)$.

As in the proof of Theorem 4.9 of the previous example, we can prove that for any point $x \in \overline{\tilde{\pi}(\alpha)B_1^+}$, there exists a ball B_x containing x and a time $T_x > 0$ such that $\|\tilde{\pi}(t)y\|_2^2 \leqslant \gamma$ for all $t \geqslant T_x$ and $y \in B_x \cap \overline{\tilde{\pi}(\alpha)B_1^+}$. This, together with the precompactness of $\tilde{\pi}(\alpha)B_1^+$ shows that there exists a time $T_1^+ > 0$ such that $\|\tilde{\pi}(t)u\|_2^2 \leqslant \gamma$ for all $t \geqslant T_1^+$ and $u \in B_1^+$.

The result now follows by taking $T \geqslant \max\{1, T_1^+, T_2\}$. □

Lemma 4.13. *Let (X,π,M,I) be an IDS such that there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. If G is a precompact subset of X and $\alpha \in [0,\frac{\xi}{2})$, then $\tilde{\pi}(\alpha)G$ is precompact in X.*

Proof: For $\alpha = 0$ the result is trivial. Now assume that $\alpha > 0$ and since each subset of a precompact set is also precompact, we can only consider the case when $\phi(x) \leqslant \alpha$ for all $x \in G$ with $\alpha \in (0,\frac{\xi}{2})$ (we could write $G = G_1 \cup G_2$, where $\phi|_{G_1} \leqslant \alpha$ and $\phi|_{G_2} > \alpha$, and in the latter the precompactness follows from the continuity of π). In this case, if $B := \bigcup_{s\in[0,\alpha]} \pi(s)G$, we have

$$\tilde{\pi}(\alpha)G \subseteq \bigcup_{s\in[0,\alpha]} \pi(s)(I(B\cap M)),$$

and by the joint continuity of the semigroup π and the continuity of I, the precompactness of $\tilde{\pi}(\alpha)G$ follows. □

Theorem 4.14. *The system* (4.5) *is strongly bounded dissipative.*

Proof: Let $B_0 = \{u \in L^2(\Omega) \colon \|u\|_2^2 \leqslant \gamma\}$. From Lemma 4.12, we see that B_0 $\tilde{\pi}-$absorbs all bounded subsets of $L^2(\Omega)$. We claim that $K_t := \tilde{\pi}(t)B_0$ is precompact for some $t > 0$ and to this end, we write $B_0 = G_1 \cup G_2 \cup G_3$ where

$$G_1 = \{u \in B_0 \colon M^+(u) = \varnothing\},$$

$$G_2 = \left\{u \in B_0 \colon M^+(u) \neq \varnothing \text{ and } \phi(u) > \frac{\xi}{2}\right\},$$

$$G_3 = \left\{u \in B_0 \colon M^+(u) \neq \varnothing \text{ and } \phi(u) \leqslant \frac{\xi}{2}\right\}.$$

Let $0 < \alpha < \frac{\xi}{2}$. Then,

$$\widetilde{\pi}\left(\alpha + \frac{\xi}{2}\right) B_0 = \pi\left(\alpha + \frac{\xi}{2}\right) G_1 \cup \tilde{\pi}(\alpha)\pi\left(\frac{\xi}{2}\right) G_2 \cup \pi(\alpha)\tilde{\pi}\left(\frac{\xi}{2}\right) G_3,$$

because $\phi(x) \geqslant \xi$ for all $x \in I(M)$.

Since G_1 and $\tilde{\pi}\left(\frac{\xi}{2}\right) G_3$ are bounded by Lemma 4.11, the sets $\pi\left(\alpha + \frac{\xi}{2}\right) G_1$ and $\pi(\alpha)\tilde{\pi}\left(\frac{\xi}{2}\right) G_3$ are precompact in X. Now, since $\pi\left(\frac{\xi}{2}\right) G_2$ is precompact in X, it follows by Lemma 4.13 that $\tilde{\pi}(\alpha)\pi\left(\frac{\xi}{2}\right) G_2$ is also precompact in X.

Therefore, $K := K_{\alpha+\frac{\xi}{2}}$ is precompact in X, $K \cap M = \varnothing$ and we can easily see that K $\tilde{\pi}-$absorbs bounded sets of $L^2(\Omega)$. ∎

Corollary 4.15. *The system* (4.5) *has a global attractor* $\mathcal{A}$.

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