PULLBACK EXPONENTIAL ATTRACTORS WITH ADMISSIBLE EXPONENTIAL GROWTH IN THE PAST

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Abstract. For an evolution process we prove the existence of a pullback exponential attractor, a positively invariant family of compact subsets which have a uniformly bounded fractal dimension and pullback attract all bounded subsets at an exponential rate. The construction admits the exponential growth in the past of the sets forming the family and generalizes the known approaches. It also allows to substitute the smoothing property by a weaker requirement without auxiliary spaces. The theory is illustrated with the examples of a nonautonomous Chafee-Infante equation and a time-dependent perturbation of a reaction-diffusion equation improving the results known in the literature.

1. Introduction

In this paper we present a construction of a pullback exponential attractor for an evolution process. A pullback exponential attractor is a positively invariant family of compact subsets which have a uniformly bounded fractal dimension and pullback attract all bounded subsets at an exponential rate. Before the publication of [4] the constructions of a pullback exponential attractor implied that the constructed family is uniformly bounded in the past (see [9], [11], [8]). In [4] it was shown that the family may grow sub-exponentially in the past and still have a uniform bound on the fractal dimension. Below we improve the results of that article by allowing the sets to grow even exponentially in the past and still have a uniformly bounded fractal dimension. There are other differences between the results of [4] and ours. First of all, we do not assume the process to be continuous with respect to all the variables, i.e., the function $\{(t,s)\in\mathbb{R}^2\colon t\geq s\}\times V \ni (t,s,u)\mapsto U(t,s)u\in V$ need not be continuous. However, we assume that the operators $U(t,s)$ are Lipschitz continuous within the positively invariant family of bounded absorbing sets $\{B(t)\colon t\in\mathbb{R}\}$. For convenience, we also require that the sets $B(t)$ are closed subsets of the Banach space V. Contrary to assumptions of [4], the sets $B(t)$ may grow exponentially in the past and the absorption of bounded subsets takes place also only in the past (see assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_3)$ in Section 2). Moreover, we assume that the process decomposes in the past into a contracting part and a smoothing part (see assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ in Section 2). Under these assumptions we prove

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the existence of a pullback exponential attractor, which also pullback attracts the family $\{B(t)\colon t \in \mathbb{R}\}$ (see Theorem 2.2). The same assumptions guarantee also the existence of the pullback global attractor with uniformly bounded fractal dimension (see Corollary 2.8). From the point of view of applications it seems interesting to substitute the smoothing property $(\mathcal{H}_2)$ by a weaker premise, which does not refer to any auxiliary space. This is done in Corollary 2.6, which together with Theorem 2.2 and Corollary 2.8 constitutes the main results of Section 2.

In Section 3 we consider a nonautonomous Chafee-Infante equation with Neumann boundary conditions

$$\begin{cases} \partial_t u = \triangle u + \lambda u - \beta(t)u^3, \ t > s, \ x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \ t > s, \ x \in \partial\Omega, \\ u(s) = u_s, \ x \in \Omega, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^d$. Here we extend the results of [5] by allowing that the real function β tends to 0 in $-\infty$ at an exponential rate. We show that a pullback global attractor and a pullback exponential attractor both exist and have a uniform finite bound on the fractal dimension. However, the diameter of their sections is unbounded in the past and, in the case of a particular β, grows exponentially in the past.

In Section 4 we consider a nonautonomous reaction-diffusion equation with Dirichlet boundary condition

$$\begin{cases} \partial_t u - \triangle u + f(t,u) = g(t), \ t > s, \ x \in \Omega, \\ u = 0, \ t > s, x \in \partial\Omega, \\ u(s) = u_s, \ x \in \Omega, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ under the assumptions on f considered in [2]. However, here we assume that $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ satisfies

$$\|g(t)\|^2_{L^2(\Omega)} \leq M_0 e^{\alpha|t|}, \ t \in \mathbb{R}, \tag{1.1}$$

with $0 \leq \alpha < \lambda_1$ and $M_0 > 0$, where $\lambda_1 > 0$ denotes the first eigenvalue of $-\triangle_D$, where $\triangle_D$ is the Laplace operator in $L^2(\Omega)$ with zero Dirichlet boundary condition, while in [2] the function g could have only a polynomial growth. We prove the existence of a pullback exponential attractor and a pullback global attractor in $H^1_0(\Omega)$, both with uniform bound on fractal dimension of their sections, using Corollary 2.6 without the smoothing property.

2. Construction of pullback exponential attractors

Below we present a construction of a family of sets, called a pullback exponential attractor, for an evolution process (see Theorem 2.2), which is a consequence of similar constructions for a discrete semi-process and a discrete process also provided in this section.

We consider an *evolution process* $\{U(t,s)\colon t \geq s\}$ on a Banach space $(V, \|\cdot\|_V)$, i.e., the family of operators $U(t,s)\colon V \to V$, $t \geq s$, $t,s \in \mathbb{R}$, satisfying the properties

(a) $U(t,s)U(s,r) = U(t,r)$, $t \geq s \geq r$,

(b) $U(t,t) = Id$, $t \in \mathbb{R}$,

where Id denotes the identity operator on V. If X is a normed space we denote by $\mathcal{O}(X)$ the class of all nonempty bounded subsets of X and by $B_R^X(x)$ the open ball in X centered at x of radius $R > 0$.

Definition 2.1. By a *pullback exponential attractor* for the process $\{U(t,s)\colon t \geq s\}$ on V we call a family $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ of nonempty compact subsets of V such that

(i) the family is positively invariant under the process $U(t,s)$, i.e.,

$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t),\ t \geq s,$$

(ii) the fractal dimension in V of the sets forming the family has a uniform bound, i.e., there exists $d \geq 0$ such that

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{M}(t)) \leq d < \infty,$$

(iii) there exists $\omega > 0$ such that every set $D \in \mathcal{O}(V)$ is pullback exponentially attracted at time $t \in \mathbb{R}$ by $\mathcal{M}(t)$ with the rate ω, i.e., for any $D \in \mathcal{O}(V)$ and $t \in \mathbb{R}$ we have

$$\lim_{s\to\infty} e^{\omega s} \operatorname{dist}_V(U(t,t-s)D, \mathcal{M}(t)) = 0, \tag{2.1}$$

where $\operatorname{dist}_V(A,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\|_V$ denotes the Hausdorff semi-distance.

We remark that one may extend (iii) and require attraction of sets D bounded in a given normed space Y provided (2.1) makes sense. This leads to the notion of a $(Y-V)$ *pullback exponential attractor* (cp. [6], [7]).

The construction of a pullback exponential attractor in Theorem 2.2 will be based on the smoothing property (cp. assumption $(\mathcal{H}_2)$ below) and will follow closely the presentation of [4]. Nevertheless, we observe that the smoothing property is not necessary to obtain this result and show the existence of a pullback exponential attractor without the smoothing property in Corollary 2.6. Both results generalize [4] by admitting exponential growth in the past of the diameter of sections of a pullback exponential attractor (cp. assumption $(\mathcal{A}_2)$ below).

We assume that

$(\mathcal{A}_1)$ there exists a family of nonempty closed bounded subsets $B(t)$ of V, $t \in \mathbb{R}$, which is positively invariant under the process, i.e.,

$$U(t,s)B(s) \subset B(t),\ t \geq s,$$

$(\mathcal{A}_2)$ there exist $t_0 \in \mathbb{R}$, $\gamma_0 \geq 0$ and $M > 0$ such that

$$\operatorname{diam}_V(B(t)) < Me^{-\gamma_0 t},\ t \leq t_0,$$

$(\mathcal{A}_3)$ in the past the family $\{B(t)\colon t \in \mathbb{R}\}$ pullback absorbs all bounded subsets of V; that is, for every $D \in \mathcal{O}(V)$ and $t \leq t_0$ there exists $T_{D,t} \geq 0$ such that

$$U(t,t-r)D \subset B(t),\ r \geq T_{D,t},$$

and, additionally, the function $(-\infty, t_0] \ni t \mapsto T_{D,t} \in [0,\infty)$ is nondecreasing for every $D \in \mathcal{O}(V)$; hence, in fact, we have for any $D \in \mathcal{O}(V)$ and $t \leq t_0$

$$U(s,s-r)D \subset B(s),\ s \leq t,\ r \geq T_{D,t}.$$

Note that $(\mathcal{A}_2)$ implies that for any $\gamma > \gamma_0$

$$\operatorname{diam}_V(B(t))e^{\gamma t} \to 0 \text{ as } t \to -\infty,$$

which generalizes the assumption used in [4, Definition 3.1]. In particular, our assumptions admit an exponential growth in the past of the sets forming the pullback absorbing family.

Next, we assume that the semi-process $\{U(t,s)\colon t_0 \geq t \geq s\}$ can be represented as the sum

$$U(t,s) = C(t,s) + S(t,s), \tag{2.2}$$

where $\{C(t,s)\colon t_0 \geq t \geq s\}$ and $\{S(t,s)\colon t_0 \geq t \geq s\}$ are families of operators satisfying the following properties:

$(\mathcal{H}_1)$ there exists $\tilde{t} > 0$ such that $C(t, t-\tilde{t})$ are contractions within the absorbing sets with the contraction constant independent of time, i.e.,

$$\left\|C(t,t-\tilde{t})u - C(t,t-\tilde{t})v\right\|_V \leq \lambda \left\|u-v\right\|_V, \ t \leq t_0, \ u,v \in B(t-\tilde{t}),$$

where $0 \leq \lambda < \frac{1}{2}e^{-\gamma_0 \tilde{t}}$ with $\gamma_0 \geq 0$ from $(\mathcal{A}_2)$,

$(\mathcal{H}_2)$ there exists an auxiliary normed space $(W, \|\cdot\|_W)$ such that V is compactly embedded into W and $\mu > 0$ is such that

$$\|u\|_W \leq \mu \|u\|_V, \ u \in V, \tag{2.3}$$

and there exists $\kappa > 0$ such that $S(t, t-\tilde{t})$ satisfies the smoothing property within the absorbing sets, i.e.,

$$\left\|S(t,t-\tilde{t})u - S(t,t-\tilde{t})v\right\|_V \leq \kappa \left\|u-v\right\|_W, \ t \leq t_0, \ u,v \in B(t-\tilde{t}).$$

Finally, we assume that

$(\mathcal{H}_3)$ the process is Lipschitz continuous within the absorbing sets, i.e., for every $t \in \mathbb{R}$ and $s \in [t, t+\tilde{t}]$ there exists $L_{t,s} > 0$ such that

$$\|U(s,t)u - U(s,t)v\|_V \leq L_{t,s} \|u-v\|_V, \ u,v \in B(t).$$

Indeed, assumption $(\mathcal{H}_3)$ implies that for any $s \geq t$ there exists $L_{t,s} > 0$ such that

$$\|U(s,t)u - U(s,t)v\|_V \leq L_{t,s} \|u-v\|_V, \ u,v \in B(t). \tag{2.4}$$

Note that assumptions $(\mathcal{A}_1)$ and $(\mathcal{H}_3)$ hold for any $t \in \mathbb{R}$, while the rest of assumptions holds only in the past, that is, for $t \leq t_0$.

Theorem 2.2. *If the process $\{U(t,s)\colon t \geq s\}$ on a Banach space V satisfies $(\mathcal{A}_1)$-$(\mathcal{A}_3)$ and $(\mathcal{H}_1)$-$(\mathcal{H}_3)$, then for any $\nu \in (0, \frac{1}{2}e^{-\gamma_0\tilde{t}} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}(t) = \mathcal{M}^\nu(t)\colon t \in \mathbb{R}\}$ in V satisfying the properties:*

(a) *$\mathcal{M}(t)$ is a nonempty compact subset of $B(t)$ for $t \in \mathbb{R}$,*
(b) *$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$, $t \geq s$,*
(c) *the fractal dimension of $\mathcal{M}(t)$ is uniformly bounded w.r.t. $t \in \mathbb{R}$, i.e.,*

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{M}(t)) \leq \frac{-\ln N^W_{\frac{\nu}{\kappa}}(B^V_1(0))}{\ln\left(2(\nu+\lambda)\right) + \gamma_0\tilde{t}},$$

where $N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$ *denotes the smallest number of balls in* W *with radius* $\frac{\nu}{\kappa}$ *and centers in* $B^V_1(0)$ *necessary to cover* $B^V_1(0)$*,*

(d) *for any* $t \in \mathbb{R}$ *there exists* $c_t > 0$ *such that for any* $s \geq \max\{t - t_0, 0\} + 2\tilde{t}$

$$\mathrm{dist}_V(U(t, t-s)B(t-s), \mathcal{M}(t)) \leq c_t e^{-\omega_0 s}, \tag{2.5}$$

where $\omega_0 = -\frac{1}{\tilde{t}}\left(\ln(2(\nu + \lambda)) + \gamma_0 \tilde{t}\right) > 0$*,*

(e) *for any* $0 < \omega < \omega_0$ *we have*

$$\lim_{s\to\infty} e^{\omega s}\,\mathrm{dist}_V(U(t, t-s)D, \mathcal{M}(t)) = 0,\ t \in \mathbb{R},\ D \in \mathcal{O}(V). \tag{2.6}$$

Moreover, if Y *is a normed space such that* $(\mathcal{A}_3)$ *holds for any* $D \in \mathcal{O}(Y)$*, then also* (2.6) *holds for any* $D \in \mathcal{O}(Y)$ *giving rise to a* $(Y - V)$ *pullback exponential attractor.*

The proof of the above theorem is based on the constructions of similar sets for a corresponding discrete semi-process and discrete process. Therefore, for a given $n_0 \in \mathbb{Z}$, we consider a discrete semi-process $\{U_d(n, m) \colon n_0 \geq n \geq m\}$ on a Banach space V, i.e., we have operators $U_d(n, m) \colon V \to V$ satisfying

(a) $U_d(n, m)U_d(m, l) = U_d(m, l)$, $n_0 \geq n \geq m \geq l$, $n, m, l \in \mathbb{Z}$,

(b) $U_d(n, n) = Id$, $n_0 \geq n$, $n \in \mathbb{Z}$.

Furthermore, we require that

$(\mathcal{A}_1')$ there exists a family of nonempty closed bounded subsets $B_d(n)$ of V, $n \leq n_0$, which is positively invariant under the process, i.e.,

$$U_d(n, m)B_d(m) \subset B_d(n),\ n_0 \geq n \geq m,$$

$(\mathcal{A}_2')$ there exist $\gamma_d \geq 0$ and $M_d > 0$ such that

$$\mathrm{diam}_V(B_d(n)) < M_d e^{-\gamma_d n},\ n \leq n_0,$$

$(\mathcal{A}_3')$ the family $\{B_d(n) \colon n \leq n_0\}$ pullback absorbs all bounded subsets of V; that is, for every $D \in \mathcal{O}(V)$ and $n \leq n_0$ there exists $r_{D,n} \in \mathbb{N}$ such that

$$U_d(n, n-r)D \subset B_d(n),\ r \geq r_{D,n},$$

and, additionally, the function $n \mapsto r_{D,n}$ is nondecreasing for any $D \in \mathcal{O}(V)$; hence, in fact, we have for any $D \in \mathcal{O}(V)$ and $n \leq n_0$

$$U_d(m, m-r)D \subset B_d(m),\ m \leq n,\ r \geq r_{D,n}.$$

Next, we assume that the semi-process can be represented as the sum $U_d(n, m) = C_d(n, m) + S_d(n, m)$, where $\{C_d(n, m) \colon n_0 \geq n \geq m\}$ and $\{S_d(n, m) \colon n_0 \geq n \geq m\}$ are families of operators satisfying the following properties:

$(\mathcal{H}_1')$ $C_d(n, n-1)$ are contractions within the absorbing sets with the contraction constant independent of $n \leq n_0$, i.e.,

$$\|C_d(n, n-1)u - C_d(n, n-1)v\|_V \leq \lambda_d \|u - v\|_V\,,\ n \leq n_0,\ u, v \in B_d(n-1),$$

where $0 \leq \lambda_d < \frac{1}{2}e^{-\gamma_d}$ with $\gamma_d \geq 0$ from $(\mathcal{A}_2')$,

($\mathcal{H}_2'$) there exists an auxiliary normed space $(W, \|\cdot\|_W)$ such that V is compactly embedded into W and (2.3) holds with some $\mu > 0$ and there exists $\kappa_d > 0$ such that $S_d(n, n-1)$ satisfies the smoothing property within the absorbing sets, i.e.,

$$\|S_d(n,n-1)u - S_d(n,n-1)v\|_V \le \kappa_d \|u-v\|_W\,,\ n \le n_0,\ u, v \in B_d(n-1).$$

Note that the above assumptions imply that the semi-process is Lipschitz continuous within the absorbing sets, i.e., for any $n_0 \ge n \ge m$

($\mathcal{H}_3'$) $\|U_d(n,m)u - U_d(n,m)v\|_V \le (\kappa_d\mu + \lambda_d)^{n-m} \|u-v\|_V\,,\ u, v \in B_d(m).$

Theorem 2.3. *If the semi-process $\{U_d(n,m)\colon n_0 \ge n \ge m\}$ on a Banach space V satisfies ($\mathcal{A}_1'$)-($\mathcal{A}_3'$) and ($\mathcal{H}_1'$)-($\mathcal{H}_2'$), then for any $\nu \in (0, \frac{1}{2}e^{-\gamma_d} - \lambda_d)$ there exists a family $\{\mathcal{M}_d(n) = \mathcal{M}_d^\nu(n)\colon n \le n_0\}$ in V satisfying the properties:*

(a) *$\mathcal{M}_d(n)$ is a nonempty compact subset of $B_d(n)$ for $n \le n_0$,*

(b) *$U(n,m)\mathcal{M}_d(m) \subset \mathcal{M}_d(n)$, $n_0 \ge n \ge m$,*

(c) *the fractal dimension of $\mathcal{M}_d(n)$ is uniformly bounded, i.e.,*

$$\sup_{n\le n_0} \dim_f^V(\mathcal{M}_d(n)) \le \frac{-\ln N_\nu}{\ln(2(\nu+\lambda_d)) + \gamma_d},$$

where $N_\nu = N^W_{\frac{\nu}{\kappa_d}}(B_1^V(0))$ is the smallest number of balls in W with radius $\frac{\nu}{\kappa_d}$ and centers in $B_1^V(0)$ necessary to cover the unit ball $B_1^V(0)$,

(d) *for any $n \le n_0$ there exists $c_n > 0$ such that for any $k \in \mathbb{N}_0$*

$$\operatorname{dist}_V(U_d(n,n-k)B_d(n-k), \mathcal{M}_d(n)) \le c_n e^{-\omega_d k}, \tag{2.7}$$

where $\omega_d = -(\ln(2(\nu+\lambda_d)) + \gamma_d) > 0$,

(e) *for any $0 < \omega < \omega_d$ we have*

$$\lim_{k\to\infty} e^{\omega k} \operatorname{dist}_V(U_d(n,n-k)D, \mathcal{M}_d(n)) = 0,\ n \le n_0,\ D \in \mathcal{O}(V). \tag{2.8}$$

Moreover, if Y is a normed space such that ($\mathcal{A}_3'$) holds for any $D \in \mathcal{O}(Y)$, then also (2.8) holds for any $D \in \mathcal{O}(Y)$.

Proof. The proof follows the lines of the proof of [4, Theorem 3.3], so some calculations will be omitted here. However, in order to justify and clarify Corollary 2.4 below, we will repeat the main argument here.

By ($\mathcal{A}_2'$) there exist $v_n \in B_d(n)$, $n \le n_0$, such that $B_d(n) \subset B^V_{R_n}(v_n)$ with $R_n = M_d e^{-\gamma_d n}$. We fix $0 < \nu < \frac{1}{2}e^{-\gamma_d} - \lambda_d$ and set $N = N^W_{\frac{\nu}{\kappa_d}}(B_1^V(0))$. Thus we have $w_1, \ldots, w_N \in B_1^V(0)$ such that

$$B_1^V(0) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa_d}}(w_i). \tag{2.9}$$

We define $W^0(n) := \{v_n\}$, $n \le n_0$. By induction, using ($\mathcal{H}_1'$), ($\mathcal{H}_2'$) and ($\mathcal{A}_1'$) we construct $W^k(n)$, $n \le n_0$, with $k \in \mathbb{N}$ such that

($\mathcal{W}_1$) $W^k(n) \subset U_d(n,n-k)B_d(n-k) \subset B_d(n)$,

($\mathcal{W}_2$) $\#W^k(n) \le N^k$,

$(\mathcal{W}_3)$ $U_d(n,n-k)B_d(n-k) \subset \bigcup_{u\in W^k(n)} B^V_{(2(\nu+\lambda_d))^k R_{n-k}}(u)$.

Indeed, let $n \le n_0$ and note that (2.9) implies

$$B_d(n-1) \subset B^V_{R_{n-1}}(v_{n-1}) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa_d}R_{n-1}}(R_{n-1}w_i + v_{n-1}).$$

Due to the smoothing property $(\mathcal{H}_2')$ we get

$$\|S_d(n,n-1)u - S_d(n,n-1)v\|_V \le \kappa_d \|u-v\|_W < 2\nu R_{n-1}$$

for all $u,v \in B_d(n-1) \cap B^W_{\frac{\nu}{\kappa_d}R_{n-1}}(R_{n-1}w_i + v_{n-1})$. This yields

$$S_d(n,n-1)B_d(n-1) \subset \bigcup_{i=1}^N B^V_{2\nu R_{n-1}}(S_d(n,n-1)y_i) \tag{2.10}$$

for some $y_1,\dots,y_N \in B_d(n-1)$. For $u \in B_d(n-1)$ the property $(\mathcal{H}_1')$ now implies

$$\|C_d(n,n-1)u - C_d(n,n-1)y_i\|_V \le \lambda_d \|u-y_i\|_V < 2\lambda_d R_{n-1},\ i=1,\dots,N,$$

and we conclude that

$$C_d(n,n-1)B_d(n-1) \subset B^V_{2\lambda_d R_{n-1}}(C_d(n,n-1)y_i),\ i=1,\dots,N.$$

Hence we obtain

$$\begin{aligned} U_d(n,n-1)B_d(n-1) &\subset \bigcup_{i=1}^N \big(B^V_{2\nu R_{n-1}}(S_d(n,n-1)y_i) + B^V_{2\lambda_d R_{n-1}}(C_d(n,n-1)y_i)\big) \\ &\subset \bigcup_{i=1}^N B^V_{2(\nu+\lambda_d)R_{n-1}}(U_d(n,n-1)y_i) \end{aligned}$$

with centers $U_d(n,n-1)y_i \in U_d(n,n-1)B_d(n-1)$, $i=1,\dots,N$. Denoting this set of centers by $W^1(n)$ it follows that $(\mathcal{W}_1)$-$(\mathcal{W}_3)$ hold for $k=1$.

Let us assume that the sets $W^l(n)$ are already constructed and

$$U_d(n,n-l)B_d(n-l) \subset \bigcup_{u\in W^l(n)} B^V_{(2(\nu+\lambda_d))^l R_{n-l}}(u)$$

for all $l \le k$ and $n \le n_0$. In order to construct a covering of

$$\begin{aligned} &U_d(n,n-(k+1))B_d(n-(k+1)) = U_d(n,n-1)U_d(n-1,n-1-k)B_d(n-1-k) \\ &\subset \bigcup_{u\in W^k(n-1)} U_d(n,n-1)\Big(U_d(n-1,n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u)\Big) \end{aligned}$$

we let $u \in W^k(n-1)$ and proceed as before to conclude from (2.9) that

$$B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa_d}(2(\nu+\lambda_d))^k R_{n-1-k}}(x_i^u),$$

where $x_i^u = (2(\nu+\lambda_d))^k R_{n-1-k}w_i + u$. By the smoothing property $(\mathcal{H}_2')$ we have

$$\|S_d(n,n-1)v - S_d(n,n-1)w\|_V \le \kappa_d \|v-w\|_W < 2\nu(2(\nu+\lambda_d))^k R_{n-1-k}$$

for any $v, w \in U_d(n-1, n-1-k)B_d(n-1-k) \cap B^W_{\frac{\nu}{\kappa_d}(2(\nu+\lambda_d))^k R_{n-1-k}}(x_i^u)$. This yields

$$\begin{aligned} &S_d(n, n-1)(U_d(n-1, n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u)) \\ &\subset \bigcup_{i=1}^N B^V_{2\nu(2(\nu+\lambda_d))^k R_{n-1-k}}(S_d(n, n-1)y_i^u) \end{aligned} \tag{2.11}$$

for some $y_1^u, \dots, y_N^u \in U_d(n-1, n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u)$. Furthermore, the property $(\mathcal{H}_1')$ implies

$$C_d(n, n-1)\left(U_d(n-1, n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u)\right)$$

$$\subset B^V_{2\lambda_d(2(\nu+\lambda_d))^k R_{n-1-k}}(C_d(n, n-1)y_i^u)$$

for every $i = 1, \dots, N$. Consequently, we obtain the covering

$$U_d(n, n-1)(U_d(n-1, n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u))$$

$$\subset \bigcup_{i=1}^N \left(B^V_{2\nu(2(\nu+\lambda_d))^k R_{n-1-k}}(S_d(n, n-1)y_i^u) + B^V_{2\lambda_d(2(\nu+\lambda_d))^k R_{n-1-k}}(C_d(n, n-1)y_i^u)\right)$$

$$\subset \bigcup_{i=1}^N B^V_{(2(\nu+\lambda_d))^{k+1} R_{n-1-k}}(U_d(n, n-1)y_i^u)$$

with $U_d(n, n-1)y_i^u \in U_d(n, n-1-k)B_d(n-1-k)$. Constructing in the same way for every $u \in W^k(n-1)$ such a covering by balls with radius $(2(\nu+\lambda_d))^{k+1} R_{n-1-k}$ in V we obtain a covering of the set $U_d(n, n-1-k)B_d(n-1-k)$ and denote the new set of centers by $W^{k+1}(n)$. This yields $\#W^{k+1}(n) \leq N \# W^k(n-1) \leq N^{k+1}$, by construction the set of the centers $W^{k+1}(n) \subset U_d(n, n-(k+1))B_d(n-(k+1))$ and

$$U_d(n, n-(k+1))B_d(n, n-(k+1)) \subset \bigcup_{u \in W^{k+1}(n)} B^V_{(2(\nu+\lambda_d))^{k+1} R_{n-(k+1)}}(u),$$

which concludes the proof of the properties $(\mathcal{W}_1)$-$(\mathcal{W}_3)$.

Next, we define $E^0(n) = W^0(n)$, $n \leq n_0$, and set

$$E^k(n) := W^k(n) \cup U_d(n, n-1)E^{k-1}(n-1), \; n \leq n_0, \; k \in \mathbb{N}.$$

Then using $(\mathcal{W}_1)$-$(\mathcal{W}_3)$ it follows that the family of sets $E^k(n)$, $k \in \mathbb{N}_0$, satisfies for all $n \leq n_0$ (see [4])

$(\mathcal{E}_1)$ $U_d(n, n-1)E^k(n-1) \subset E^{k+1}(n)$, $E^k(n) \subset U_d(n, n-k)B_d(n-k) \subset B_d(n)$,

$(\mathcal{E}_2)$ $E^k(n) = \bigcup_{l=0}^k U_d(n, n-l)W^{k-l}(n-l), \quad \#E^k(n) \leq \sum_{l=0}^k N^l,$

$(\mathcal{E}_3)$ $U_d(n, n-k)B_d(n-k) \subset \bigcup_{u \in E^k(n)} B^V_{(2(\nu+\lambda_d))^k R_{n-k}}(u).$

We now define

$$\widetilde{\mathcal{M}}_d(n) = \bigcup_{k\in\mathbb{N}_0} E^k(n), \quad \mathcal{M}_d(n) = \mathrm{cl}_V\, \widetilde{\mathcal{M}}_d(n),\ n \le n_0.$$

First, observe that by $(\mathcal{E}_1)$ and the closedness of $B_d(n)$ in V, the set $\mathcal{M}_d(n)$ is a nonempty subset of $B_d(n)$. Moreover, by $(\mathcal{H}_3')$, $(\mathcal{E}_1)$ we have for $n \le n_0$ and $r \in \mathbb{N}$

$$U_d(n,n-r)\mathcal{M}_d(n-r) = \mathrm{cl}_V \bigcup_{k\in\mathbb{N}_0} U_d(n,n-r)E^k(n-r) \subset \mathrm{cl}_V \bigcup_{k\in\mathbb{N}_0} E^{k+r}(n) \subset \mathcal{M}_d(n),$$

which proves the positive invariance of $\{\mathcal{M}_d(n) \colon n \le n_0\}$.

Note that by $(\mathcal{E}_1)$ and $(\mathcal{A}_1')$ for any $l \ge k$, $l,k \in \mathbb{N}_0$ and $n \le n_0$

$$E^l(n) \subset U_d(n,n-k)U_d(n-k,n-l)B_d(n-l) \subset U_d(n,n-k)B_d(n-k).$$

We fix $n \le n_0$. Consequently, for all $k \in \mathbb{N}$ we obtain

$$\widetilde{\mathcal{M}}_d(n) = \bigcup_{l=0}^{k} E^l(n) \cup \bigcup_{l=k+1}^{\infty} E^l(n) \subset \bigcup_{l=0}^{k} E^l(n) \cup U_d(n,n-k)B_d(n-k).$$

Observe that the sequence

$$\mathbb{N} \ni k \mapsto (2(\nu+\lambda_d))^k R_{n-k} = (2(\nu+\lambda_d)e^{\gamma_d})^k M_d e^{-\gamma_d n} \in (0,\infty)$$

is strictly decreasing to 0. Thus for any $\varepsilon > 0$ sufficiently small there exists $k \in \mathbb{N}$ such that

$$(2(\nu+\lambda_d))^k R_{n-k} \le \varepsilon < (2(\nu+\lambda_d))^{k-1} R_{n-k+1}.$$

It follows from $(\mathcal{W}_3)$ that

$$U_d(n,n-k)B_d(n-k) \subset \bigcup_{u\in W^k(n)} B^V_\varepsilon(u).$$

Hence we can estimate the number of ε-balls in V needed to cover $\widetilde{\mathcal{M}}_d(n)$ by

$$N^V_\varepsilon(\widetilde{\mathcal{M}}_d(n)) \le \#\left(\bigcup_{l=0}^{k} E^l(n)\right) + \#W^k(n) \le (k+1)^2 N^k + N^k \le 2(k+1)^2 N^k,$$

where we used $(\mathcal{W}_2)$ and $(\mathcal{E}_2)$. This shows that $\widetilde{\mathcal{M}}_d(n)$ is precompact in V and, since V is a Banach space, its closure $\mathcal{M}_d(n)$ is compact in V. Moreover, we have

$$\dim^V_f(\mathcal{M}_d(n)) = \dim^V_f(\widetilde{\mathcal{M}}_d(n)) = \limsup_{\varepsilon\to 0} \frac{\ln\left(N^V_\varepsilon(\widetilde{\mathcal{M}}_d(n))\right)}{-\ln\varepsilon} \le$$

$$\le \limsup_{k\to\infty} \frac{\ln 2 + 2\ln(k+1) + k\ln N}{-\ln M_d - (k-1)\ln\left(2(\nu+\lambda_d)e^{\gamma_d}\right) + \gamma_d n} = \frac{-\ln N}{\ln\left(2(\nu+\lambda_d)\right) + \gamma_d}.$$

Consequently, the fractal dimension of $\mathcal{M}_d(n)$ in V is uniformly bounded and

$$\sup_{n\le n_0} \dim^V_f(\mathcal{M}_d(n)) \le \frac{-\ln N}{\ln(2(\nu+\lambda_d)) + \gamma_d}.$$

By $(\mathcal{E}_3)$ we have for $n \le n_0$ and $k \in \mathbb{N}_0$

$$\mathrm{dist}_V(U_d(n,n-k)B(n-k), \mathcal{M}_d(n)) \le \mathrm{dist}_V(U_d(n,n-k)B(n-k), E^k(n))$$

$$\leq (2(\nu+\lambda_d))^k R_{n-k} = (2(\nu+\lambda_d)e^{\gamma_d})^k M_d e^{-\gamma_d n} = c_n e^{-\omega_d k}$$

with $c_n = M_d e^{-\gamma_d n}$, which proves (2.7).

We are left to show that the set $\mathcal{M}_d(n)$ pullback exponentially attracts all bounded subsets of V at time $n \leq n_0$. By $(\mathcal{A}_3^{'})$ for any bounded subset D of V and $n \leq n_0$ there exists $r_{D,n} \in \mathbb{N}$ such that

$$U_d(m, m-r)D \subset B_d(m), \ m \leq n, \ r \geq r_{D,n}.$$

We fix $n \leq n_0$ and $D \in \mathcal{O}(V)$. If $k \geq r_{D,n}$, that is, $k = r_{D,n} + k_0$ with some $k_0 \in \mathbb{N}_0$, then by (2.7) we have

$$\operatorname{dist}_V(U_d(n, n-k)D, \mathcal{M}_d(n)) \leq \operatorname{dist}_V(U_d(n, n-k_0)U_d(n-k_0, n-k_0-r_{D,n})D, \mathcal{M}_d(n))$$

$$\leq \operatorname{dist}_V(U_d(n, n-k_0)B_d(n-k_0), \mathcal{M}_d(n)) \leq c_n e^{\omega_d r_{D,n}} e^{-\omega_d k}.$$

Thus (2.8) holds for any $0 < \omega < \omega_d$. If Y is a normed space such that $(\mathcal{A}_3^{'})$ holds for any $D \in \mathcal{O}(Y)$, then also (2.8) holds for any $D \in \mathcal{O}(Y)$. □

It follows from the above proof (see (2.10) and (2.11)) that the smoothing property $(\mathcal{H}_2^{'})$ can be substituted by a more precise requirement.

Corollary 2.4. *Assume that the semi-process $\{U_d(n,m)\colon n_0 \geq n \geq m\}$ on a Banach space V satisfies $(\mathcal{A}_1^{'})$-$(\mathcal{A}_3^{'})$, $(\mathcal{H}_1^{'})$ and is continuous within the absorbing sets. Let $R_n = M_d e^{-\gamma_d n}$, $n \leq n_0$, and $0 < \nu < \frac{1}{2}e^{-\gamma_d} - \lambda_d$ and assume also that*

- (H_2') *there exists $N = N_\nu \in \mathbb{N}$ such that for any $n \leq n_0$, any $k \in \mathbb{N}_0$ and any $u \in U_d(n-1, n-1-k)B_d(n-1-k)$ there exist $z_1, \dots, z_N$ belonging to $S_d(n, n-1)\Big(U_d(n-1, n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u)\Big)$ such that*

$$S_d(n, n-1)\Big(U_d(n-1, n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u)\Big)$$
$$\subset \bigcup_{i=1}^{N} B^V_{2\nu(2(\nu+\lambda_d))^k R_{n-1-k}}(z_i).$$

Then the statements (a)-(e) of Theorem 2.3 hold with $N_\nu = N$ in (c).

We emphasize that the points z_i in (H_2') may depend on n, k, u, but their number N is independent of these variables. Moreover, in the applications, stronger conditions may be verified which imply the precise property (H_2') (see e.g., the smoothing property $(\mathcal{H}_2^{'})$ or Corollary 2.6 and its application in Section 4).

In order to prove the existence of a discrete pullback exponential attractor for a discrete process $\{U_d(n,m)\colon n \geq m\}$ on a Banach space V, i.e., the family of operators $U_d(n,m)\colon V \to V$ satisfying

(a) $U_d(n,m)U_d(m,l) = U_d(m,l)$, $n \geq m \geq l$, $n, m, l \in \mathbb{Z}$,
(b) $U_d(n,n) = Id$, $n \in \mathbb{Z}$,

it is enough to change assumptions $(\mathcal{A}_1^{'})$ and $(\mathcal{H}_3^{'})$, whereas the others remain unchanged and hold for $n \leq n_0$ with some $n_0 \in \mathbb{Z}$. We introduce the assumptions:

($\mathcal{A}_1''$) there exists a family of nonempty closed bounded subsets $B_d(n)$ of V, $n \in \mathbb{Z}$, which is positively invariant under the process, i.e.,

$$U_d(n,m)B_d(m) \subset B_d(n),\ m \le n,$$

($\mathcal{H}_3''$) the operators $U_d(n,m)$ are Lipschitz continuous within the absorbing sets, i.e., for any $n \ge m$ there exists $L_{m,n} > 0$ such that

$$\|U_d(n,m)u - U_d(n,m)v\|_V \le L_{m,n}\,\|u-v\|_V\,,\ u,v \in B_d(m).$$

Theorem 2.5. *If the process $\{U_d(n,m)\colon n \ge m\}$ on a Banach space V satisfies ($\mathcal{A}_1''$), ($\mathcal{A}_2'$), ($\mathcal{A}_3'$), and ($\mathcal{H}_1'$), ($\mathcal{H}_2'$) or (H_2'), ($\mathcal{H}_3''$) with some $n_0 \in \mathbb{Z}$, then for any $\nu \in (0, \frac{1}{2}e^{-\gamma_d} - \lambda_d)$ there exists a family $\{\mathcal{M}_d(n) = \mathcal{M}_d^\nu(n)\colon n \in \mathbb{Z}\}$ in V satisfying the properties:*

(a) *$\mathcal{M}_d(n)$ is a nonempty compact subset of $B_d(n)$ for $n \in \mathbb{Z}$,*
(b) *$U_d(n,m)\mathcal{M}_d(m) \subset \mathcal{M}_d(n)$, $n \ge m$,*
(c) *the fractal dimension of $\mathcal{M}_d(n)$ is uniformly bounded, i.e.,*

$$\sup_{n\in\mathbb{Z}} \dim_f^V(\mathcal{M}_d(n)) \le \frac{-\ln N_\nu}{\ln\left(2(\nu+\lambda_d)\right) + \gamma_d},$$

where $N_\nu = N^W_{\frac{\nu}{\kappa_d}}(B_1^V(0))$ if ($\mathcal{H}_2'$) holds or N_ν comes from (H_2'),
(d) *for any $n \in \mathbb{Z}$ there exists $c_n > 0$ such that for any integer $k \ge \max\{n-n_0, 0\}$*

$$\operatorname{dist}_V(U_d(n,n-k)B_d(n-k), \mathcal{M}_d(n)) \le c_n e^{-\omega_d k}, \tag{2.12}$$

where $\omega_d = -\left(\ln\left(2(\nu+\lambda_d)\right) + \gamma_d\right) > 0$,
(e) *for any $0 < \omega < \omega_d$ we have*

$$\lim_{k\to\infty} e^{\omega k} \operatorname{dist}_V(U_d(n,n-k)D, \mathcal{M}_d(n)) = 0,\ n \in \mathbb{Z},\ D \in \mathcal{O}(V). \tag{2.13}$$

Moreover, if Y is a normed space such that ($\mathcal{A}_3'$) holds for any $D \in \mathcal{O}(Y)$, then also (2.13) *holds for any $D \in \mathcal{O}(Y)$.*

Proof. The assumptions of Theorem 2.3 or Corollary 2.4 are satisfied, hence we have defined sets $\mathcal{M}_d(n)$, $n \le n_0$, with the properties stated in Theorem 2.3. For $n > n_0$ we define

$$\mathcal{M}_d(n) = U_d(n,n_0)\mathcal{M}_d(n_0).$$

By ($\mathcal{A}_1''$) we know that $\mathcal{M}_d(n)$ is a nonempty subset of $B_d(n)$ and by ($\mathcal{H}_3''$) $U_d(n,n_0)$ is continuous on $B_d(n_0)$, so $\mathcal{M}_d(n)$ is also compact for $n > n_0$. Since by ($\mathcal{H}_3''$) the operator $U_d(n,n_0)$ is Lipschitz continuous on $B_d(n_0)$, it follows that $\dim_f^V(\mathcal{M}_d(n)) \le \dim_f^V(\mathcal{M}_d(n_0))$ and thus (c) holds.

To show the positive invariance of the family $\{\mathcal{M}_d(n)\colon n \in \mathbb{Z}\}$ note that if $n \ge m > n_0$ then $U_d(n,m)\mathcal{M}_d(m) = \mathcal{M}_d(n)$ and if $m \le n_0 < n$ then

$$U_d(n,m)\mathcal{M}_d(m) = U_d(n,n_0)U_d(n_0,m)\mathcal{M}_d(m) \subset U_d(n,n_0)\mathcal{M}_d(n_0) = \mathcal{M}_d(n).$$

If $n > n_0$ and $k \ge n - n_0$ we have by (2.7), ($\mathcal{A}_1''$) and ($\mathcal{H}_3''$)

$$\operatorname{dist}_V(U_d(n,n-k)B_d(n-k), \mathcal{M}_d(n))$$

$$= \operatorname{dist}_V(U_d(n,n_0)U_d(n_0,n-k)B_d(n-k), U_d(n,n_0)\mathcal{M}_d(n_0)) \le L_{n_0,n}c_{n_0}e^{\omega_d(n-n_0)}e^{-\omega_d k},$$

which proves (2.12).

We are left to show the pullback exponential attraction for $n > n_0$. In the proof of Theorem 2.3 we have shown that for $m \leq n_0$, $D \in \mathcal{O}(V)$ and $r \geq r_{D,m}$

$$\operatorname{dist}_V(U_d(m, m-r)D, \mathcal{M}_d(m)) \leq c_m e^{\omega_d r_{D,m}} e^{-\omega_d r}.$$

Consider now $n > n_0$, $D \in \mathcal{O}(V)$ and $k \geq n - n_0 + r_{D,n_0}$. We have by $(\mathcal{H}_3'')$

$$\begin{aligned}\operatorname{dist}_V(U_d(n, n-k)D, \mathcal{M}_d(n)) &= \operatorname{dist}_V(U_d(n, n_0)U_d(n_0, n-k)D, U_d(n, n_0)\mathcal{M}_d(n_0)) \\ &\leq L_{n_0,n} \operatorname{dist}_V(U_d(n_0, n_0 - (n_0 - n + k))D, \mathcal{M}_d(n_0)) \leq L_{n_0,n} c_{n_0} e^{\omega_d(n - n_0 + r_{D,n_0})} e^{-\omega_d k},\end{aligned}$$

which proves (2.13) also for $n > n_0$. □

Proof of Theorem 2.2. We define a discrete process

$$U_d(n, m) = U(n\tilde{t}, m\tilde{t}),\ n \geq m,$$

where $\tilde{t} > 0$ comes from $(\mathcal{H}_1)$. We also set $B_d(n) = B(n\tilde{t})$, $n \in \mathbb{Z}$. If $n \geq m$ then by $(\mathcal{A}_1)$

$$U_d(n, m)B_d(m) = U(n\tilde{t}, m\tilde{t})B(m\tilde{t}) \subset B(n\tilde{t}) = B_d(n),$$

which shows $(\mathcal{A}_1'')$. To show $(\mathcal{A}_2')$ we set $n_0 = \left[\frac{1}{\tilde{t}} t_0\right] \in \mathbb{Z}$, $M_d = M > 0$ and $\gamma_d = \gamma_0 \tilde{t} \geq 0$. If $n \leq n_0$ then $n\tilde{t} \leq t_0$ and by $(\mathcal{A}_2)$ we have

$$\operatorname{diam}_V(B_d(n)) = \operatorname{diam}_V(B(n\tilde{t})) < M e^{-\gamma_0 n\tilde{t}} = M_d e^{-\gamma_d n} = R_n$$

and $(\mathcal{A}_2')$ holds. For $D \in \mathcal{O}(V)$ and $n \leq n_0$ we define $r_{D,n} = [\frac{1}{\tilde{t}} T_{D,n\tilde{t}}] + 1 \in \mathbb{N}$ using $(\mathcal{A}_3)$. Then for $r \geq r_{D,n} > \frac{1}{\tilde{t}} T_{D,n\tilde{t}}$ we have

$$U_d(n, n-r)D = U(n\tilde{t}, n\tilde{t} - r\tilde{t})D \subset B(n\tilde{t}) = B_d(n).$$

Moreover, the function $n \mapsto r_{D,n}$ is nondecreasing, which proves $(\mathcal{A}_3')$. We set

$$S_d(n, m) = S(n\tilde{t}, m\tilde{t}),\ C_d(n, m) = C(n\tilde{t}, m\tilde{t}),\ m \leq n \leq n_0.$$

Then $(\mathcal{H}_1')$ follows from $(\mathcal{H}_1)$ with $\lambda_d = \lambda \in [0, \frac{1}{2} e^{-\gamma_d})$ and $(\mathcal{H}_2')$ follows from $(\mathcal{H}_2)$ with $\kappa_d = \kappa > 0$. Finally, by $(\mathcal{H}_3)$ it follows from (2.4) that $(\mathcal{H}_3'')$ also holds. Therefore, all assumptions of Theorem 2.5 are satisfied and for any $v \in (0, \frac{1}{2} e^{-\gamma_0 \tilde{t}} - \lambda)$ there exists a family $\{\mathcal{M}_d(n) = \mathcal{M}_d^{\nu}(n) \colon n \in \mathbb{Z}\}$ of nonempty compact subsets of V satisfying the properties (a)-(e) in Theorem 2.5.

To obtain a pullback exponential attractor for the process $\{U(t, s) \colon t \geq s\}$ we define

$$\mathcal{M}(t) = U(t, n\tilde{t})\mathcal{M}_d(n),\ t \in [n\tilde{t}, (n+1)\tilde{t}),\ n \in \mathbb{Z}.$$

$\mathcal{M}(t)$ is a nonempty subset of $B(t)$ by $(\mathcal{A}_1)$ and by the continuity of $U(t, n\tilde{t})$ on $B(n\tilde{t})$ from $(\mathcal{H}_3)$ we know that $\mathcal{M}(t)$ is compact. The assumption $(\mathcal{H}_3)$ also implies

$$\dim_f^V(\mathcal{M}(t)) \leq \dim_f^V(\mathcal{M}_d(n)) \leq \frac{-\ln N^W_{\frac{\nu}{\kappa}}(B_1^V(0))}{\ln(2(\nu + \lambda)) + \gamma_0 \tilde{t}}$$

for $t \in [n\tilde{t}, (n+1)\tilde{t})$. To show the positive invariance, we consider $t \geq s$ and write $s = k\tilde{t} + s_1$, $t = l\tilde{t} + t_1$ for some $k, l \in \mathbb{Z}$, $k \leq l$ and $s_1, t_1 \in [0, \tilde{t})$. We observe that

$$\begin{aligned}U(t, s)\mathcal{M}(s) &= U(l\tilde{t} + t_1, k\tilde{t} + s_1)U(k\tilde{t} + s_1, k\tilde{t})\mathcal{M}(k\tilde{t}) = U(l\tilde{t} + t_1, k\tilde{t})\mathcal{M}(k\tilde{t}) \\ &= U(l\tilde{t} + t_1, l\tilde{t})U(l\tilde{t}, k\tilde{t})\mathcal{M}(k\tilde{t}) \subset U(l\tilde{t} + t_1, l\tilde{t})\mathcal{M}(l\tilde{t}) = \mathcal{M}(t).\end{aligned}$$

Let $t \in \mathbb{R}$ and $s \geq \max\{t - t_0, 0\} + 2\tilde{t}$. We have $t = n\tilde{t} + t_1$ with $n \in \mathbb{Z}$ and $t_1 \in [0, \tilde{t})$ and $s = k\tilde{t} + \tilde{t} + s_1$ with $k \geq \max\{n - n_0, 1\}$ and $s_1 \in [0, \tilde{t})$, since $s \geq 2\tilde{t}$ and

$$s \geq t - t_0 + 2\tilde{t} = n\tilde{t} + t_1 - t_0 + 2\tilde{t} > (n - n_0 + 1)\tilde{t} + t_1 \geq (n - n_0 + 1)\tilde{t}.$$

Hence we obtain from $(\mathcal{H}_3)$, $(\mathcal{A}_1)$ and (2.12)

$$\begin{aligned}\operatorname{dist}_V(U(t, t-s)B(t-s), \mathcal{M}(t)) &= \operatorname{dist}_V(U(t, n\tilde{t})U(n\tilde{t}, t-s)B(t-s), U(t, n\tilde{t})\mathcal{M}_d(n)) \\ &\leq L_{n\tilde{t},t} \operatorname{dist}_V(U(n\tilde{t}, t-s)B(t-s), \mathcal{M}_d(n)) \\ \leq L_{n\tilde{t},t} \operatorname{dist}_V(U_d(n, n-k)B_d(n-k), \mathcal{M}_d(n)) &\leq L_{n\tilde{t},t} c_n e^{-\omega_d k} \leq L_{n\tilde{t},t} c_n e^{2\omega_d} e^{-\frac{\omega_d}{\tilde{t}} s},\end{aligned}$$

where $\omega_d = -\left(\ln(2(\nu + \lambda)) + \gamma_0 \tilde{t}\right) > 0$, which proves (2.5).

We are left to show that $\mathcal{M}(t)$ pullback exponentially attracts all bounded subsets of V at time $t \in \mathbb{R}$. To this end, we fix $D \in \mathcal{O}(V)$ and $t \in \mathbb{R}$. Let $t = n\tilde{t} + t_1$ with $t_1 \in [0, \tilde{t})$ and $n \in \mathbb{Z}$. Let $s \geq \max\{n - n_0, 0\}\tilde{t} + T_{D, n_0\tilde{t}} + t_1$. Thus we have $s = k\tilde{t} + T_{D, n_0\tilde{t}} + t_1 + s_1$ with $s_1 \in [0, \tilde{t})$ and $k \geq \max\{n - n_0, 0\}$. By $(\mathcal{H}_3)$, $(\mathcal{A}_3)$ and (2.12) it follows that

$$\begin{aligned}\operatorname{dist}_V(U(t, t-s)D, \mathcal{M}(t)) &\leq \operatorname{dist}_V(U(t, n\tilde{t})U(n\tilde{t}, n\tilde{t} - (s - t_1))D, U(t, n\tilde{t})\mathcal{M}_d(n)) \\ &\leq L_{n\tilde{t},t} \operatorname{dist}_V(U(n\tilde{t}, (n-k)\tilde{t})U((n-k)\tilde{t}, (n-k)\tilde{t} - T_{D, n_0\tilde{t}} - s_1)D, \mathcal{M}_d(n)) \\ &\leq L_{n\tilde{t},t} \operatorname{dist}_V(U_d(n, n-k)B_d(n-k), \mathcal{M}_d(n)) \leq L_{n\tilde{t},t} c_n e^{\frac{\omega_d}{\tilde{t}}(T_{D, n_0\tilde{t}} + t_1 + \tilde{t})} e^{-\frac{\omega_d}{\tilde{t}} s},\end{aligned}$$

which ends the proof. □

The smoothing property $(\mathcal{H}_2)$ in Theorem 2.2 may be substituted by less restrictive assumption in order to satisfy (H_2') from Corollary 2.4 for the discrete process U_d in the above proof.

Corollary 2.6. *Assume that the process $\{U(t, s) \colon t \geq s\}$ on a Banach space V satisfies $(\mathcal{A}_1)$-$(\mathcal{A}_3)$, $(\mathcal{H}_3)$ and admits the decomposition* (2.2) *with $(\mathcal{H}_1)$ and let $\nu \in (0, \frac{1}{2} e^{-\gamma_0 \tilde{t}} - \lambda)$. Assume further that*

- (H_2) *there exists $N = N_\nu \in \mathbb{N}$ such that for any $t \leq t_0$, any $R > 0$ and any $u \in B(t - \tilde{t})$ there exist $v_1, \dots, v_N \in V$ such that*

$$S(t, t - \tilde{t})(B(t - \tilde{t}) \cap B_R^V(u)) \subset \bigcup_{i=1}^{N} B_{\nu R}^V(v_i). \tag{2.14}$$

Then there exists a pullback exponential attractor $\{\mathcal{M}(t) = \mathcal{M}^\nu(t) \colon t \in \mathbb{R}\}$ in V satisfying the properties:

- (a) *$\mathcal{M}(t)$ is a nonempty compact subset of $B(t)$ for $t \in \mathbb{R}$,*
- (b) *$U(t, s)\mathcal{M}(s) \subset \mathcal{M}(t)$, $t \geq s$,*
- (c) $\sup_{t \in \mathbb{R}} \dim_f^V(\mathcal{M}(t)) \leq \dfrac{-\ln N_\nu}{\ln(2(\nu + \lambda)) + \gamma_0 \tilde{t}}$,
- (d) *for any $t \in \mathbb{R}$ there exists $c_t > 0$ such that for any $s \geq \max\{t - t_0, 0\} + 2\tilde{t}$* (2.5) *holds with $\omega_0 = -\frac{1}{\tilde{t}}\left(\ln(2(\nu + \lambda)) + \gamma_0 \tilde{t}\right) > 0$,*
- (e) (2.6) *holds for any $0 < \omega < \omega_0$.*

Moreover, if Y is a normed space such that $(\mathcal{A}_3)$ holds for any $D \in \mathcal{O}(Y)$, then also (2.6) *holds for any $D \in \mathcal{O}(Y)$ giving rise to a $(Y - V)$ pullback exponential attractor.*

Proof. The proof starts as the proof of Theorem 2.2, but we note that for $n \le n_0$, $k \in \mathbb{N}_0$ and $u \in U_d(n-1, n-1-k)B_d(n-1-k) \subset B_d(n-1)$ by (H_2) there exist $v_1, \dots, v_N \in V$ such that

$$S_d(n, n-1)(U_d(n-1, n-1-k)B_d(n-1-k) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u))$$

$$\subset S_d(n, n-1)(B_d(n-1) \cap B^V_{(2(\nu+\lambda_d))^k R_{n-1-k}}(u)) \subset \bigcup_{i=1}^{N} B^V_{\nu(2(\nu+\lambda_d))^k R_{n-1-k}}(v_i).$$

By doubling the radius of the balls on the right-hand side, we obtain (H_2'). The result follows from Theorem 2.5 and the rest of the proof of Theorem 2.2. □

Pullback exponential attractors are inseparably connected with the notion of a pullback global attractor (see e.g. [3]).

Definition 2.7. Let $\{U(t, s) \colon t \ge s\}$ be an evolution process on a Banach space V. By a *pullback global attractor* for the process $\{U(t, s) \colon t \ge s\}$ we call a family $\{\mathcal{A}(t) \colon t \in \mathbb{R}\}$ of nonempty compact subsets of V such that

(i) the family is invariant under the process $U(t, s)$, i.e.,

$$U(t, s)\mathcal{A}(s) = \mathcal{A}(t), \ t \ge s,$$

(ii) the family is pullback attracting all bounded subsets of V, i.e., for any $D \in \mathcal{O}(V)$ and $t \in \mathbb{R}$ we have

$$\lim_{s \to \infty} \mathrm{dist}_V(U(t, t-s)D, \mathcal{A}(t)) = 0, \tag{2.15}$$

(iii) the family is minimal in the sense that if another family $\{C(t) \colon t \in \mathbb{R}\}$ of nonempty closed subsets of V pullback attracts all bounded subsets of V, then $\mathcal{A}(t) \subset C(t)$ for $t \in \mathbb{R}$.

In fact, under the assumptions of Theorem 2.2 or Corollary 2.6 it follows that the process is pullback $(\mathcal{O}(V) \cup \{\hat{B}\})$-dissipative with $\hat{B} = \{B(t) \colon t \in \mathbb{R}\}$, pullback $\{\hat{B}\}$-asymptotically closed and pullback $\{\hat{B}\}$-asymptotically compact by (d) of Theorem 2.2 (see e.g. [7] for the definitions) and hence the process possesses a pullback global attractor $\{\mathcal{A}(t) \colon t \in \mathbb{R}\}$ with sections given by

$$\mathcal{A}(t) = \mathrm{cl}_V \bigcup_{D \in \mathcal{O}(V)} \bigcap_{s \le t} \mathrm{cl}_V \bigcup_{r \le s} U(t, r)D, \ t \in \mathbb{R}, \tag{2.16}$$

see [7, Theorem 2.16]. We remark that to show the invariance of the pullback global attractor under the process we use here the positive invariance of the family $\hat{B}$ and $(\mathcal{H}_3)$. We also remark that

$$\mathcal{A}(t) \subset \bigcap_{s \le t} \mathrm{cl}_V \bigcup_{r \le s} U(t, r)B(r) = \omega_V(\hat{B}, t), \ t \in \mathbb{R}$$

and in general this inclusion is proper (cp. [16]). On the other hand, the set on the right-hand side is the section of the pullback $(\mathcal{O}(V) \cup \{\hat{B}\})$-attractor (see [7, Corollary 2.17]) and is contained in $\mathcal{M}(t)$ due to (d) and (e) of Theorem 2.2.

Corollary 2.8. *Under the assumptions of Theorem 2.2 or Corollary 2.6 the process* $\{U(t,s)\colon t \geq s\}$ *possesses a pullback global attractor* $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$ *given by* (2.16), *which is contained in a pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^{\nu}(t)\colon t \in \mathbb{R}\}$ *and thus has a uniformly bounded fractal dimension*

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{A}(t)) \leq \sup_{t\in\mathbb{R}} \dim_f^V(\omega_V(\hat{B},t)) \leq \sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{M}(t)) \leq \frac{-\ln N_\nu}{\ln(2(\nu+\lambda)) + \gamma_0\tilde{t}},$$

where $N_\nu = N^W_{\frac{\nu}{\kappa_d}}(B_1^V(0))$ *if* $(\mathcal{H}_2)$ *holds or* N_ν *comes from* (H_2).

In order to illustrate and better perceive the above corollary, consider the trivial equation $\dot{u} = -u$ for $t \geq s$ with $u(s) = u_s$. The process is given as $U(t,s)u_s = u_s e^{-(t-s)}$, $t \geq s$. In the role of $B(t)$ we can take e.g. $B(t) = [-ce^{-t}, de^{-t}]$ with some $c, d > 0$. Then $\mathcal{A}(t) = \{0\}$, $t \in \mathbb{R}$, and $\omega_{\mathbb{R}}(\hat{B},t) = B(t) = \mathcal{M}(t)$, $t \in \mathbb{R}$, and the family $\hat{B} = \{B(t)\colon t \in \mathbb{R}\}$ is an exponential pullback attractor. In fact, in this example, $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$ is also an exponential pullback attractor, showing again the nonuniqueness of this notion contrary to the pullback global attractor. Of course, virtues of a pullback exponential attractor can be appreciated in the infinite-dimensional evolution processes generated by nonautonomous partial differential equations as will be seen in the next sections.

3. Chafee-Infante equation

We consider the Chafee-Infante equation with the Neumann boundary condition of the form

$$\begin{cases} \partial_t u = \triangle u + \lambda u - \beta(t)u^3, \ t > s, \ x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \ t > s, \ x \in \partial\Omega, \\ u(s) = u_s, \ x \in \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and $s \in \mathbb{R}$, $\lambda \geq 0$ and $\frac{\partial}{\partial\nu}$ denotes the outward unit normal derivative on the boundary $\partial\Omega$.

In this section we extend the results of [5], where the above problem was considered. Taking into account our abstract theory, we admit the situation when the positive function β tends to 0 in $-\infty$ at an exponential rate, including equations to which the results from [4] could not be applied. In the case of similar calculations we omit them here and refer the reader to [5].

In our presentation we assume that $\beta\colon \mathbb{R} \to (0,\infty)$ is a C^1 function such that

(i) $\lim_{t\to-\infty} \beta(t) = 0$,

(ii) there exists $\beta_1 \in \mathbb{R}$ such that
$$\frac{\beta'(t)}{\beta(t)} \leq \beta_1, \ t \in \mathbb{R},$$

(iii) there exist $\gamma_0 > 0$, $K > 0$ and $t_0 \in \mathbb{R}$ such that $\beta(t) \geq Ke^{\gamma_0 t}$ for $t \leq t_0$.

Note that our condition is less restrictive than the condition used in [5], i.e.,

$$\lim_{t\to-\infty}\frac{e^{\gamma t}}{\beta(t)}=0 \text{ for every } \gamma>0.$$

In particular, our assumptions allow to consider $\beta(t)=Ke^{\gamma_0 t}$ with some $K,\gamma_0>0$ for large negative t and extend it to the right so that (ii) holds.

We set $W=C(\overline{\Omega})$ with $\|u\|_W=\sup_{x\in\overline{\Omega}}|u(x)|$, $u\in W$, and consider $-\triangle_N$, minus Laplacian with Neumann boundary condition, in W with

$$D(-\triangle_N)=\{u\in\bigcap_{1\le p<\infty}W^{2,p}_{loc}(\Omega)\colon \frac{\partial u}{\partial\nu}=0 \text{ on } \partial\Omega,\ u,\triangle u\in C(\overline{\Omega})\}.$$

It follows from [14, Corollary 3.1.24] that this operator is sectorial in W in the sense of [10, Definition 1.3.1]. Let $\delta>0$ be such that $A=-\triangle_N+\delta Id$ is a positive sectorial operator. We consider the fractional power spaces $X^\alpha=D(A^\alpha)$ and observe that (see [14, Theorem 3.1.30])

$$X^\alpha\subset\begin{cases}C^{2\alpha}(\overline{\Omega}),\ 0\le\alpha<\frac{1}{2},\\ C^{2\alpha}_N(\overline{\Omega})=\{u\in C^{2\alpha}(\overline{\Omega})\colon \frac{\partial u}{\partial\nu}=0 \text{ on } \partial\Omega\},\ \frac{1}{2}<\alpha<1.\end{cases}\tag{3.2}$$

Moreover, for $\alpha\in(0,1)$ the space X^α is compactly embedded into W and

$$\|u\|_W\le\mu\|u\|_{X^\alpha},\ u\in X^\alpha,\tag{3.3}$$

with some $\mu>0$. The operator $-A$ generates an analytic semigroup $\{e^{-At}\colon t\ge0\}$ in W and for any $\alpha\in[0,1)$ there exists $C_\alpha>0$ such that

$$\left\|e^{-At}\right\|_{\mathcal{L}(W,W)}\le C_0,\ t\ge0,\quad \left\|e^{-At}\right\|_{\mathcal{L}(W,X^\alpha)}\le\frac{C_\alpha}{t^\alpha},\ t>0.\tag{3.4}$$

The problem (3.1) can be considered as an abstract Cauchy problem in W

$$\begin{cases}\partial_t u+Au=F(t,u),\ t>s,\\ u(s)=u_s\end{cases}\tag{3.5}$$

with $F(t,u)=(\lambda+\delta)u-\beta(t)u^3$. Note that for $t_1,t_2\in[s,\infty)$ and $u,v\in W$

$$\begin{aligned}&\|F(t_1,u)-F(t_2,v)\|_W\le(\lambda+\delta)\|u-v\|_W\\&+2\beta(t_1)\|u-v\|_W(\|u\|_W^2+\|v\|_W^2)+|\beta(t_1)-\beta(t_2)|\,\|v\|_W^3.\end{aligned}\tag{3.6}$$

Thus for every $\alpha\in[0,1)$, F is Hölder continuous w.r.t. the first variable and Lipschitz continuous w.r.t. the second variable on every bounded subset of $[s,\infty)\times X^\alpha$. Consequently, for every $\alpha\in[0,1)$ and every $u_s\in X^\alpha$ there exists a unique local X^α solution of (3.5) defined on the maximal interval of existence.

If we multiply the first equation in (3.1) by u^{2m-1} with $m\in\mathbb{N}$ and integrate over Ω we obtain

$$\frac{1}{2m}\frac{d}{dt}\int_\Omega u^{2m}dx=-\frac{2m-1}{m^2}\int_\Omega|\nabla(u^m)|^2dx-\beta(t)\int_\Omega u^{2m+2}dx+\lambda\int_\Omega u^{2m}dx.$$

$$\frac{d}{dt}\|u(t)\|^{2m}_{L^{2m}(\Omega)}\le2m\lambda\|u(t)\|^{2m}_{L^{2m}(\Omega)},\ t>s,$$

and in consequence

$$\|u(t)\|_{L^{2m}(\Omega)} \leq \|u(s)\|_{L^{2m}(\Omega)} e^{\lambda(t-s)},\ t \geq s.$$

Taking $m \to \infty$, we obtain

$$\|u(t)\|_W \leq \|u(s)\|_W e^{\lambda(t-s)},\ t \geq s. \tag{3.7}$$

This implies that X^α solutions exist globally in time for every $\alpha \in [0, 1)$. Consequently, for every $\alpha \in [0, 1)$ and $u_s \in X^\alpha$ there exists a unique global solution of (3.5), i.e.,

$$u \in C([s, \infty); X^\alpha) \cap C((s, \infty); D(A)) \cap C^1((s, \infty); W)$$

satisfying the variation of constants formula

$$u(t) = e^{-A(t-s)}u_s + \int_s^t e^{-A(t-\tau)}F(\tau, u(\tau))d\tau,\ t \geq s. \tag{3.8}$$

We fix $\frac{1}{2} < \alpha < 1$, set $V = X^\alpha$ and define the process $U(t, s)\colon V \to V$ by

$$U(t, s)u_s = u(t),$$

where u is the X^α solution of (3.5) satisfying $u(s) = u_s$.

Set $a > 0$ such that $a^2 \geq \lambda + \frac{\beta_1}{2}$ and observe (cp. [5, Lemma 1]) that then by (ii) the function $c^\star\colon [s, \infty) \times \overline{\Omega} \to (0, \infty)$ given by

$$c^*(t, x) = \frac{a}{\sqrt{\beta(t)}},\ t \geq s,\ x \in \overline{\Omega},$$

is an upper solution for the problem (3.1), whereas $-c^*$ is a lower solution for the problem (3.1).

Proposition 3.1. *The family of nonempty closed subsets of V*

$$\widetilde{B}(t) = \left\{ u \in V\colon\ \|u\|_W \leq \frac{a}{\sqrt{\beta(t)}} \right\},\ t \in \mathbb{R},$$

is positively invariant under the process $\{U(t, s)\colon t \geq s\}$ and in the past pullback absorbs all bounded subsets of V, i.e., for every $D \in \mathcal{O}(V)$ and $t \leq t_0$ there exists $T_{D,t} \geq 0$ such that

$$U(t, t-r)D \subset \widetilde{B}(t),\ r \geq T_{D,t},$$

and the function $t \mapsto T_{D,t}$ is nondecreasing for every $D \in \mathcal{O}(V)$.

Proof. The positive invariance follows from [17, Theorem 2.4.1] and the fact that $-c^*$, c^* are lower and upper solutions, respectively. Let D be a bounded subset of V and $t \leq t_0$. We choose $R = R(D) > 0$ such that $D \subset B^W_R(0)$. Since $\lim_{s \to -\infty} \beta(s) = 0$ by (i), then there exists $s_R \in \mathbb{R}$ such that $\beta(s) \leq \frac{a^2}{R^2}$ for $s \leq s_R$. Consequently, we have $R \leq \frac{a}{\sqrt{\beta(s)}}$ and $D \subset \widetilde{B}(s)$ for $s \leq s_R$. Setting $T_{D,t} = \max\{t - s_R, 0\} \geq 0$ and taking $r \geq T_{D,t}$ we get

$$U(t, t-r)D \subset U(t, t-r)\widetilde{B}(t-r) \subset \widetilde{B}(t),$$

which proves the claim. □

Below we show that the process satisfies the smoothing property.

Proposition 3.2. *These exists a function* $\kappa\colon (0,\infty)\times\mathbb{R}\to(0,\infty)$, *nondecreasing in each variable, such that for any* $T\in\mathbb{R}$

$$\|U(t,s)u-U(t,s)v\|_V \le \kappa(t-s,T)\,\|u-v\|_W\,,\ u,v\in\widetilde{B}(s),\ s<t\le T.$$

Proof. Let $T\in\mathbb{R}$, $s<t\le T$ and $u,v\in\widetilde{B}(s)$. By the variation of constants formula (3.8) and (3.4) we have for $u(t)=U(t,s)u$ and $v(t)=U(t,s)v$

$$\|u(t)-v(t)\|_V \le \frac{C_\alpha}{(t-s)^\alpha}\,\|u-v\|_W + \int_s^t \frac{C_\alpha}{(t-\tau)^\alpha}\,\|F(\tau,u(\tau))-F(\tau,v(\tau))\|_W\,d\tau.$$

Using (3.6), (3.3) and Proposition 3.1 we get

$$\|u(t)-v(t)\|_V \le \frac{C_\alpha}{(t-s)^\alpha}\,\|u-v\|_W + \left(\lambda+\delta+4a^2\right)\int_s^t \frac{C_\alpha\mu}{(t-\tau)^\alpha}\,\|u(\tau)-v(\tau)\|_V\,d\tau.$$

From the Volterra type inequality (see [20, Theorem 1.27]) it follows that

$$\|u(t)-v(t)\|_V \le \kappa(t-s,T)\,\|u-v\|_W\,,\ s<t\le T,$$

where $\kappa\colon(0,\infty)\times\mathbb{R}\to(0,\infty)$ is nondecreasing in each variable. □

We define

$$B(t)=\mathrm{cl}_V\,U(t,t-1)\widetilde{B}(t-1),\ t\in\mathbb{R}.$$

Then $B(t)$ is a nonempty closed subset of V and $B(t)\subset\widetilde{B}(t)$, $t\in\mathbb{R}$. From Proposition 3.2 it follows that for $t\in\mathbb{R}$ we have

$$\mathrm{diam}_V(B(t))=\mathrm{diam}_V(U(t,t-1)\widetilde{B}(t-1))\le\kappa(1,t)\,\mathrm{diam}_W(\widetilde{B}(t-1))\le\frac{2a\kappa(1,t)}{\sqrt{\beta(t-1)}},$$

which shows the boundedness of $B(t)$ in V for every $t\in\mathbb{R}$.

Moreover, from Proposition 3.2 we infer for every $t>s$ and $u,v\in\widetilde{B}(s)$

$$\|U(t,s)u-U(t,s)v\|_V\le\kappa(t-s,t)\,\|u-v\|_W\le\mu\kappa(t-s,t)\,\|u-v\|_V\,,$$

which implies $(\mathcal{H}_3)$. Using this we also have for $t\ge s$

$$U(t,s)B(s)=\mathrm{cl}_V\,U(t,s-1)\widetilde{B}(s-1)\subset\mathrm{cl}_V\,U(t,t-1)\widetilde{B}(t-1)=B(t),$$

which proves $(\mathcal{A}_1)$. By Proposition 3.2 we get for $t\le t_0$

$$\|U(t,t-1)u-U(t,t-1)v\|_V\le\kappa(1,t_0)\,\|u-v\|_W\,,\ u,v\in\widetilde{B}(t-1)$$

and in the consequence of (iii) we obtain for $t\le t_0$

$$\mathrm{diam}_V(B(t))\le\kappa(1,t_0)\,\mathrm{diam}_W(\widetilde{B}(t-1))\le\kappa(1,t_0)\frac{2a}{\sqrt{K}}e^{\frac{\gamma_0}{2}}e^{-\frac{\gamma_0}{2}t}. \tag{3.9}$$

If $D\in\mathcal{O}(V)$ and $t\le t_0$, then for $r\ge T_{D,t}+1$ with $T_{D,t}$ from Proposition 3.1

$$U(t,t-r)D=U(t,t-1)U(t-1,t-1-(r-1))D\subset U(t,t-1)\widetilde{B}(t-1)\subset B(t).$$

The function $t\mapsto T_{D,t}+1$ is nondecreasing and hence the above calculations show that $(\mathcal{A}_2)$, $(\mathcal{A}_3)$, $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ with $C(t,t-1)=0$ and $S(t,t-1)=U(t,t-1)$ are also satisfied.

Therefore, the assumptions of Theorem 2.2 and Corollary 2.8 are verified and we obtain the following result.

Theorem 3.3. *The process* $\{U(t,s)\colon t \geq s\}$ *on* $V = C_N^{2\alpha}(\overline{\Omega})$, *with* $\frac{1}{2} < \alpha < 1$, *generated by problem* (3.1) *possesses a pullback exponential attractor* $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ *in* V. *In particular, there exists a pullback global attractor* $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$ *in* V, *such that for any* $\nu \in (0, \frac{1}{2}e^{-\frac{\gamma_0}{2}})$ *we have*

$$\mathcal{A}(t) \subset \mathcal{M}(t) = \mathcal{M}^\nu(t) \subset B(t) \subset \widetilde{B}(t)$$

and

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{A}(t)) \leq \sup_{t\in\mathbb{R}} \dim_f^V(\mathcal{M}^\nu(t)) \leq \frac{-\ln N^{W}_{\frac{\nu}{\kappa(1,t_0)}}(B_1^V(0))}{\ln(2\nu) + \frac{\gamma_0}{2}},$$

where $W = C(\overline{\Omega})$.

Observe that if u_s is a positive constant function, then (see [12, Proposition 3.1])

$$(U(t,s)u_s)(x) = \frac{e^{\lambda t}}{\sqrt{e^{2\lambda s}u_s^{-2} + 2\int_s^t e^{2\lambda\tau}\beta(\tau)d\tau}},\ t \geq s,\ x \in \overline{\Omega}.$$

Since $\mathcal{A}(t)$ pullback attracts $\{u_s\}$ for every $t \in \mathbb{R}$ and $U(t,s)u_s \to \xi(t)$ in V as $s \to -\infty$, where

$$\xi(t)(x) = \frac{e^{\lambda t}}{\sqrt{2\int_{-\infty}^t e^{2\lambda\tau}\beta(\tau)d\tau}},\ x \in \overline{\Omega},$$

it follows that $\xi(t) \in \mathcal{A}(t)$. The zero solution of (3.1) also belongs to $\mathcal{A}(t)$ and hence

$$\frac{e^{\lambda t}}{\sqrt{2\int_{-\infty}^t e^{2\lambda\tau}\beta(\tau)d\tau}} \leq \operatorname{diam}_V(\mathcal{A}(t)) \leq \operatorname{diam}_V(\mathcal{M}(t)).$$

If $\lambda > 0$ it follows from (i) that

$$\operatorname{diam}_V(\mathcal{A}(t)) \to \infty \text{ and } \operatorname{diam}_V(\mathcal{M}(t)) \to \infty \text{ as } t \to -\infty.$$

In a particular case, when $\beta(t) = Ke^{\gamma_0 t}$, $t \leq t_0$, with $\gamma_0, K > 0$, we have by (3.9)

$$\sqrt{\frac{2\lambda+\gamma_0}{2K}}e^{-\frac{\gamma_0}{2}t} \leq \operatorname{diam}_V(\mathcal{A}(t)) \leq \operatorname{diam}_V(\mathcal{M}(t)) \leq \kappa(1,t_0)\frac{2a}{\sqrt{K}}e^{\frac{\gamma_0}{2}}e^{-\frac{\gamma_0}{2}t},\ t \leq t_0,$$

which shows that $\mathcal{A}(t)$ and $\mathcal{M}(t)$ grow exponentially in the past.

4. Reaction-diffusion equations

Let us consider the initial boundary value problem for the reaction-diffusion equation

$$\begin{cases} \partial_t u - \triangle u + f(t,u) = g(t),\ t > s,\ x \in \Omega, \\ u(t,x) = 0,\ t > s, x \in \partial\Omega, \\ u(s,x) = u_s(x),\ x \in \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$. We assume that $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ and there exist $p \geq 2$, $C_i > 0$, $i = 1, \ldots, 5$ such that

$$C_1 |u|^p - C_2 \leq f(t,u)u \leq C_3 |u|^p + C_4,\ u \in \mathbb{R},\ t \in \mathbb{R}, \tag{4.2}$$

$$f_u(t,u) \geq -C_5,\ u \in \mathbb{R},\ t \in \mathbb{R}, \quad f(t,0) = 0,\ t \in \mathbb{R}. \tag{4.3}$$

The problem of this form was investigated in many research articles. We mention some of them which are related to the question of existence of pullback attractors and their fractal dimension. If f does not depend on time and satisfies the estimate

$$\|g(t)\|^2_{L^2(\Omega)} \leq M_0 e^{\alpha|t|},\ t \in \mathbb{R}, \tag{4.4}$$

with $0 \leq \alpha < \lambda_1$ and $M_0 > 0$, where $\lambda_1 > 0$ denotes the first eigenvalue of $A = -\triangle_D$, where $\triangle_D$ is the Laplace operator in $L^2(\Omega)$ with zero Dirichlet boundary condition, then the existence of a pullback global attractor in $H^1_0(\Omega)$ was shown in [13]. The same result was later obtained in [19] under more general assumption than (4.4)

$$\int_{-\infty}^{t} e^{\lambda_1 s} \|g(s)\|^2_{L^2(\Omega)}\, ds < \infty,\ t \in \mathbb{R}.$$

As refers to the uniform bound of the fractal dimension of its sections, it was proved in $L^2(\Omega)$ in [2], but only under an additional assumption that there exists a positive and nondecreasing function $\xi\colon \mathbb{R} \to (0,\infty)$ such that

$$|f(\tau,u) - f(\tau,v)| \leq \xi(t)\,|u - v|\,,\ \tau \leq t,\ u,v \in \mathbb{R}, \tag{4.5}$$

and under the requirement that there exist $a, b > 0$ and $r \geq 0$ such that

$$\|g(t)\|_{L^2(\Omega)} \leq a\,|t|^r + b,\ t \in \mathbb{R}.$$

Here we improve this result by considering the problem (4.1) with f satisfying (4.2), (4.3), (4.5) and g satisfying (4.4), which admits exponential growth of g in the past and in the future. Below we prove the existence of a pullback exponential attractor in $H^1_0(\Omega)$, which contains a pullback global attractor in $H^1_0(\Omega)$ with fractal dimension uniformly bounded. Note that the existence of a pullback exponential attractor for the problem (4.1) was also considered in [9] and [8] under more restrictive conditions on g than (4.4). Let us denote by $\|\cdot\|$ the $L^2(\Omega)$ norm and by $\|\cdot\|_{H^1_0(\Omega)} = \|\nabla\cdot\|$ the norm in $H^1_0(\Omega)$. Moreover, without loss of generality we assume that $0 < \alpha < \lambda_1$ in (4.4). We recall

Theorem 4.1. *Under the assumptions* (4.2) *and* (4.3) *for every* $s, T \in \mathbb{R}$, $s < T$, $u_s \in L^2(\Omega)$, *there exists a unique solution*

$$u \in C([s,T]; L^2(\Omega)) \cap L^2(s,T; H^1_0(\Omega)) \cap L^p(s,T; L^p(\Omega))$$

of the problem (4.1). *Moreover, for* $u_s, v_s \in L^2(\Omega)$ *we have*

$$\|u(t) - v(t)\| \leq e^{C_5(t-s)}\,\|u_s - v_s\|\,,\ t \in [s,T]. \tag{4.6}$$

If $u_s \in H^1_0(\Omega)$ *and* (4.5) *holds, then*

$$u \in C([s,T]; H^1_0(\Omega)) \cap L^2(s,T; D(A)). \tag{4.7}$$

Furthermore, for $u_s, v_s \in H_0^1(\Omega)$ we have

$$\|u(t) - v(t)\|_{H_0^1(\Omega)} \leq e^{\frac{1}{2}\lambda_1^{-1}\xi^2(t)(t-s)} \|u_s - v_s\|_{H_0^1(\Omega)}, \ t \in [s, T]. \tag{4.8}$$

Proof. The first part can be found e.g. in [18, Theorem 8.4]. If $u_s \in H_0^1(\Omega)$ and (4.5) holds, then the Galerkin approximation is uniformly bounded in $L^2(s, T; D(A))$ with its derivative bounded in $L^2(s, T; L^2(\Omega))$ and (4.7) follows. To show (4.8), we consider the difference $w = u - v$ of two solutions of (4.1) with $u_s, v_s \in H_0^1(\Omega)$. This difference satisfies

$$\partial_\tau w - \triangle w = -(f(\tau, u) - f(\tau, v)), \ \tau > s.$$

Taking the inner product in $L^2(\Omega)$ with $-\triangle w$, we obtain

$$\partial_\tau \left(\|\nabla w\|^2\right) + \|\triangle w\|^2 \leq \int_\Omega |f(\tau, u) - f(\tau, v)|^2 \, dx, \ \tau > s.$$

Using (4.5) and the Poincaré inequality we get

$$\partial_\tau \left(\|\nabla w(\tau)\|^2\right) \leq \lambda_1^{-1}\xi^2(t) \|\nabla w(\tau)\|^2, \ s < \tau \leq t,$$

and thus

$$\|\nabla w(t)\|^2 \leq \|\nabla w(s)\|^2 e^{\lambda_1^{-1}\xi^2(t)(t-s)}, \ t \geq s,$$

which proves (4.8). □

By the above theorem we can define an evolution process $\{U(t, s) \colon t \geq s\}$ in $L^2(\Omega)$ by $U(t, s)u_s = u(t)$. Its restriction to $H_0^1(\Omega)$ also defines an evolution process, which is denoted here also by $U(t, s) \colon H_0^1(\Omega) \to H_0^1(\Omega)$. To show the existence of a pullback absorbing family some standard estimates are used.

Taking the inner product in $L^2(\Omega)$ with u we get from (4.1) and (4.2)

$$\partial_t \left(\|u\|^2\right) + \|\nabla u\|^2 + 2C_1 \|u\|_{L^p(\Omega)}^p \leq 2C_2 |\Omega| + \lambda_1^{-1} \|g(t)\|^2, \ t > s, \tag{4.9}$$

while taking the inner product in $L^2(\Omega)$ with $-\triangle u$ we get from (4.1) and (4.3)

$$\partial_t \left(\|\nabla u\|^2\right) + \lambda_1 \|\nabla u\|^2 \leq 2C_5 \|\nabla u\|^2 + \|g(t)\|^2, \ t > s. \tag{4.10}$$

From (4.9) and (4.4) we get from the Gronwall inequality

$$\|u(t)\|^2 \leq \|u(s)\|^2 e^{-\lambda_1(t-s)} + K_0(t), \ t > s,$$

where

$$K_0(t) = 2C_2 |\Omega| \lambda_1^{-1} + 2\lambda_1^{-1} M_0 (\lambda_1 - \alpha)^{-1} e^{\alpha|t|}.$$

From (4.9) and (4.4) we have

$$\int_t^{t+1} \|\nabla u(\tau)\|^2 \, d\tau \leq \|u(s)\|^2 e^{-\lambda_1(t-s)} + K_1(t+1), \ t > s,$$

where

$$K_1(t) = 2C_2 |\Omega| (\lambda_1^{-1} + 1) + 2\lambda_1^{-1} M_0 e^\alpha ((\lambda_1 - \alpha)^{-1} + \alpha^{-1}) e^{\alpha|t|}.$$

From (4.10) and the Gronwall type inequality (see [15, Lemma 3.3]) we obtain

$$\|\nabla u(t+1)\|^2 \leq \left(2e^{-\frac{\lambda_1}{2}} + 2C_5\right) \int_t^{t+1} \|\nabla u(\tau)\|^2 \, d\tau + \int_t^{t+1} \|g(\tau)\|^2 \, d\tau.$$

Hence we get by the Poincaré inequality

$$\|\nabla u(t)\|^2 \le D_0 \|u(s)\|^2 e^{-\lambda_1(t-s)} + D_1 + D_2 e^{\alpha|t|},\ t-1>s, \tag{4.11}$$

where the positive constants D_0, D_1, D_2 are given as

$$D_0 = e^{\lambda_1}(2e^{-\frac{\lambda_1}{2}} + 2C_5),\ D_1 = 2C_2\,|\Omega|\,(\lambda_1^{-1}+1)(2e^{-\frac{\lambda_1}{2}} + 2C_5),$$

$$D_2 = (2e^{-\frac{\lambda_1}{2}} + 2C_5)(2\lambda_1^{-1}M_0e^{\alpha}((\lambda_1-\alpha)^{-1}+\alpha^{-1})) + 2M_0e^{\alpha}\alpha^{-1}.$$

Let us define

$$\widetilde{B}(t) = \{z \in H_0^1(\Omega)\colon\ \|\nabla z\|^2 \le K_2(t)\},\ t \in \mathbb{R},$$

where $K_2(t) = 2D_1 + 2D_2e^{\alpha|t|}$.

From (4.11) it follows that for every bounded subset D of $L^2(\Omega)$ there exists $r_D \ge 1$ such that

$$U(t,t-r)D \subset \widetilde{B}(t),\ r \ge r_D,\ t \in \mathbb{R}. \tag{4.12}$$

Moreover, there exists $r_0 \ge 1$ such that

$$U(t,t-r)\widetilde{B}(t-r) \subset \widetilde{B}(t),\ r \ge r_0,\ t \in \mathbb{R}, \tag{4.13}$$

since

$$2D_0\lambda_1^{-1}D_1e^{-\lambda_1 r} + 2D_0\lambda_1^{-1}D_2e^{\alpha|t-r|}e^{-\lambda_1 r} \le D_1 + D_2e^{\alpha|t|},\ t \in \mathbb{R},\ r \ge r_0.$$

Our candidate for the pullback absorbing family is

$$B(t) = \mathrm{cl}_{H_0^1(\Omega)} \bigcup_{r \ge r_0} U(t,t-r)\widetilde{B}(t-r),\ t \in \mathbb{R}. \tag{4.14}$$

By (4.13) we know that $B(t) \subset \widetilde{B}(t)$ and thus $B(t)$ is a nonempty closed bounded subset of $H_0^1(\Omega)$. By (4.8) we have for $u, v \in H_0^1(\Omega)$

$$\|U(t,s)u - U(t,s)v\|_{H_0^1(\Omega)} \le e^{\frac{1}{2}\lambda_1^{-1}\xi^2(t)(t-s)} \|u-v\|_{H_0^1(\Omega)},\ t \ge s, \tag{4.15}$$

thus $(\mathcal{H}_3)$ follows. We also have

$$U(t,s)B(s) = \mathrm{cl}_{H_0^1(\Omega)} \bigcup_{r \ge r_0} U(t,t-(t-s+r))\widetilde{B}(t-(t-s+r))$$

$$\subset \mathrm{cl}_{H_0^1(\Omega)} \bigcup_{r \ge r_0} U(t,t-r)\widetilde{B}(t-r) = B(t),$$

which shows $(\mathcal{A}_1)$.

We fix $t_0 \le 0$ and note that

$$\mathrm{diam}_{H_0^1(\Omega)}(B(t)) \le \mathrm{diam}_{H_0^1}(\widetilde{B}(t)) \le 2\sqrt{K_2(t)} < 5\max\{\sqrt{D_1}, \sqrt{D_2}\}e^{-\frac{\alpha}{2}t},\ t \le t_0,$$

so $(\mathcal{A}_2)$ holds with $M = 5\max\{\sqrt{D_1}, \sqrt{D_2}\}$, $\gamma_0 = \frac{\alpha}{2}$. Furthermore, if $D \in \mathcal{O}(L^2(\Omega))$ and $t \le t_0$, then setting $T_{D,t} = r_D + r_0$ and taking $s \ge T_{D,t}$ we get from (4.12)

$$U(t,t-s)D = U(t,t-r_0)U(t-r_0,t-r_0-(s-r_0))D \subset U(t,t-r_0)\widetilde{B}(t-r_0) \subset B(t),$$

which shows that $(\mathcal{A}_3)$ holds for $D \in \mathcal{O}(L^2(\Omega))$.

We are left to prove $(\mathcal{H}_1)$ and (H_2) for $t \le t_0$ for some suitable decomposition of the process.

Let $V_n = \operatorname{span}\{e_1, \dots, e_n\}$ be the linear space spanned by the first n eigenfunctions of $A = -\triangle_D$ in $L^2(\Omega)$ and let $P_n \colon L^2(\Omega) \to V_n$ denote the orthogonal projection and Q_n its complementary projection. For $u \in L^2$ we write $u = P_n u + Q_n u = u_1 + u_2$. We consider the difference $w = u - v$ of two solutions of (4.1) with $u_s, v_s \in H_0^1(\Omega)$. Taking the inner product in $L^2(\Omega)$ with $-\triangle Q_n w = -\triangle w_2$, we obtain

$$\partial_t \left(\|\nabla w_2\|^2\right) + \|\triangle w_2\|^2 \le \int_\Omega |f(t,u) - f(t,v)|^2 \, dx, \ t > s.$$

Note that (4.5) implies for $t \le t_0$

$$|f(t,u) - f(t,v)| \le \xi_0 \, |u - v|$$

with $\xi_0 = \xi(t_0)$ and we get

$$\partial_t \left(\|\nabla w_2\|^2\right) + \|\triangle w_2\|^2 \le \xi_0^2 \, \|w\|^2 , \ s < t \le t_0.$$

Using the properties of the eigenfunctions and (4.6) we obtain

$$\partial_t \left(\|\nabla w_2\|^2\right) + \lambda_{n+1} \|\nabla w_2\|^2 \le \xi_0^2 e^{2C_5(t-s)} \|w(s)\|^2 , \ s < t \le t_0.$$

Integrating and using the Poincaré inequality yields

$$\begin{aligned}\|\nabla w_2(t)\|^2 &\le \|\nabla w_2(s)\|^2 e^{-\lambda_{n+1}(t-s)} + \xi_0^2 \|w(s)\|^2 e^{2C_5(t-s)} \int_s^t e^{-\lambda_{n+1}(t-\tau)} d\tau \\ &\le \|\nabla w(s)\|^2 \left(e^{-\lambda_{n+1}(t-s)} + \lambda_{n+1}^{-1}\lambda_1^{-1}\xi_0^2 e^{2C_5(t-s)}\right), \ s < t \le t_0.\end{aligned}$$

We fix $\tilde t > 0$. Setting $s = t - \tilde t$ we obtain for $u, v \in H_0^1(\Omega)$ and $t \le t_0$

$$\left\|Q_n U(t, t-\tilde t)u - Q_n U(t, t-\tilde t)v\right\|_{H_0^1(\Omega)} \le \|u - v\|_{H_0^1(\Omega)} \left(e^{-\lambda_{n+1}\tilde t} + \lambda_{n+1}^{-1}\lambda_1^{-1}\xi_0^2 e^{2C_5 \tilde t}\right)^{\frac12}.$$

We choose $n \in \mathbb{N}$ so large that

$$\lambda := \left(e^{-\lambda_{n+1}\tilde t} + \lambda_{n+1}^{-1}\lambda_1^{-1}\xi_0^2 e^{2C_5\tilde t}\right)^{\frac12} < \frac12 e^{-\frac{\alpha}{2}\tilde t}.$$

Then $(\mathcal{H}_1)$ is satisfied with $C(t, t-\tilde t) = Q_n U(t, t-\tilde t)$.

To show (H_2) we use a straightforward modification of [1, Lemma 1].

Lemma 4.2. *Let $B_r^{V_n}(a)$ be a ball centered at a of radius $r > 0$ in an n-dimensional space V_n. For any $0 < \mu < r$ the minimum number of balls N_μ of radius μ and centers in $B_r^{V_n}(a)$ which is necessary to cover $B_r^{V_n}(a)$ is less or equal to $(1 + \frac{2r}{\mu})^n$.*

Observe that from (4.15) we have

$$\left\|U(t, t-\tilde t)u - U(t, t-\tilde t)v\right\|_{H_0^1(\Omega)} \le L \, \|u - v\|_{H_0^1(\Omega)} , \ u, v \in H_0^1(\Omega), \ t \le t_0,$$

with $L = e^{\frac12 \lambda_1^{-1}\xi_0^2 \tilde t}$. This implies that for any $0 < \nu < L$, $R > 0$ and $u \in B(t - \tilde t)$

$$P_n U(t, t-\tilde t)(B(t-\tilde t) \cap B_R^{H_0^1(\Omega)}(u)) \subset B_{LR}^{V_n}(P_n U(t, t-\tilde t)u) \subset \bigcup_{i=1}^{N_\nu} B_{\nu R}^{V_n}(v_i)$$

with $v_i \in V_n \subset H_0^1(\Omega)$ and by Lemma 4.2

$$N_\nu \le \left(1 + \frac{2L}{\nu}\right)^n \le \left(\frac{3L}{\nu}\right)^n ,$$

which shows (H_2). Therefore, all assumptions of Corollary 2.6 and Corollary 2.8 are satisfied.

Theorem 4.3. *Under the assumptions* (4.2), (4.3), (4.5) *and* (4.4) *the process* $\{U(t,s)\colon t \ge s\}$ *on* $H^1_0(\Omega)$ *generated by the problem* (4.1) *possesses a pullback exponential attractor* $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ *in* $H^1_0(\Omega)$ *such that*

(a) $\mathcal{M}(t)$ *is a nonempty compact subset of* $B(t) \subset \widetilde{B}(t)$ *for* $t \in \mathbb{R}$ *and*

$$\operatorname{diam}_{H^1_0}(\mathcal{M}(t)) \le \operatorname{diam}_{H^1_0(\Omega)}(\widetilde{B}(t)) \le 4\max\{\sqrt{D_1}, \sqrt{D_2}\}e^{\frac{\alpha}{2}|t|},\ t \in \mathbb{R},$$

(b) $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t),\ t \ge s,$

(c) *if* $\tilde{t} > 0$ *and* $t_0 \le 0$ *and*

$$\lambda := \left(e^{-\lambda_{n+1}\tilde{t}} + \lambda_{n+1}^{-1}\lambda_1^{-1}\xi^2(t_0)e^{2C_5\tilde{t}}\right)^{\frac{1}{2}} < \frac{1}{2}e^{-\frac{\alpha}{2}\tilde{t}} \tag{4.16}$$

for some $n \in \mathbb{N}$ *and* $\nu \in (0, \frac{1}{2}e^{-\frac{\alpha}{2}\tilde{t}} - \lambda)$, *then* $\mathcal{M}(t) = \mathcal{M}^\nu(t)$ *and*

$$\sup_{t\in\mathbb{R}} \dim_f^{H^1_0(\Omega)}(\mathcal{M}^\nu(t)) \le \frac{-n\ln\left(1 + 2\nu^{-1}e^{\frac{1}{2}\lambda_1^{-1}\xi^2(t_0)\tilde{t}}\right)}{\ln(2(\nu+\lambda)) + \frac{\alpha}{2}\tilde{t}}, \tag{4.17}$$

(d) $\{\mathcal{M}(t)\colon t \in \mathbb{R}\}$ *pullback exponentially attracts* $\{B(t)\colon t \in \mathbb{R}\}$ *and every bounded subset of* $L^2(\Omega)$ *in the Hausdorff semidistance in* $H^1_0(\Omega)$.

Moreover, the process possesses a pullback global attractor $\{\mathcal{A}(t)\colon t \in \mathbb{R}\}$ *contained in the pullback exponential attractor* $\{\mathcal{M}(t) = \mathcal{M}^\nu(t)\colon t \in \mathbb{R}\}$ *and thus has a uniformly bounded fractal dimension in* $H^1_0(\Omega)$.

Remark 4.4. We remark that estimating the bound in (4.17) and taking the limit $\nu \to 0$ we get

$$\sup_{t\in\mathbb{R}} \dim_f^{H^1_0(\Omega)}(\mathcal{A}(t)) \le n,$$

where $n \in \mathbb{N}$ satisfies (4.16) and can be further estimated from above using (4.16) and the arguments as in [2, p. 220].

References

[1] F. Balibrea, J. Valero, On dimension of attractors of differential inclusions and reaction-diffusion equations, *Discrete Contin. Dyn. Syst.* 5 (3) (1999), 515–528.

[2] T. Caraballo, J. A. Langa, J. Valero, The dimension of attractors of nonautonomous partial differential equations, *ANZIAM J.* 45 (2003), 207–222.

[3] A. N. Carvalho, J. A. Langa, J. C. Robinson, *Attractors for infinite-dimensional non-autonomous dynamical systems*, Springer, New York, 2013.

[4] A. N. Carvalho, S. Sonner, Pullback exponential attractors for evolution processes in Banach spaces: theoretical results, *Commun. Pure Appl. Anal.* 12 (6) (2013), 3047–3071.

[5] A. N. Carvalho, S. Sonner, Pullback exponential attractors for evolution processes in Banach spaces: properties and applications, preprint.

[6] J. W. Cholewa, R. Czaja, G. Mola, Remarks on the fractal dimension of bi-space global and exponential attractors, *Boll. Unione Mat. Ital. (9)* 1 (2008), 121–145.

[7] R. Czaja, Bi-space pullback attractors for closed processes, *São Paulo J. Math. Sci.* 6 (2) (2012), 227–246.

[8] R. Czaja, M. Efendiev, Pullback exponential attractors for nonautonomous equations Part I: Semilinear parabolic problems, *J. Math. Anal. Appl.* 381 (2011), 748–765.

[9] M. Efendiev, A. Miranville, S. Zelik, Exponential attractors and finite-dimensional reduction for non-autonomous dynamical systems, *Proc. Roy. Soc. Edinburgh Sect. A* 135 (2005), 703–730.
[10] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.
[11] J. A. Langa, A. Miranville, J. Real, Pullback exponential attractors, *Discrete Contin. Dyn. Syst.* 26 (4) (2010), 1329–1357.
[12] J. A. Langa, J. C. Robinson, A. Suárez, Stability, instability, and bifurcation phenomena in non-autonomous differential equations, *Nonlinearity* 15 (2002), 887–903.
[13] Y. Li, C. Zhong, Pullback attractors for the norm-to-weak continuous process and application to the nonautonomous reaction-diffusion equations, *Appl. Math. Comput.* 190 (2007), 1020–1029.
[14] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.
[15] G. Łukaszewicz, On pullback attractors in H_0^1 for nonautonomous reaction-diffusion equations, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 20 (9) (2010), 2637–2644.
[16] P. Marín-Rubio, J. Real, On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, *Nonlinear Anal.* 71 (2009), 3956–3963.
[17] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.
[18] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge Univerisity Press, Cambridge, 2001.
[19] H. Song, Pullback attractors of non-autonomous reaction-diffusion equations in H_0^1, *J. Differential Equations* 249 (2010), 2357–2376.
[20] A. Yagi, *Abstract Parabolic Evolution Equations and their Applications*, Springer, Berlin, 2010.

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