Pullback exponential attractors for nonautonomous equations Part II: Applications to reaction-diffusion systems☆

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Abstract

The existence of a pullback exponential attractor being a family of compact and positively invariant sets with a uniform bound on their fractal dimension which at a uniform exponential rate pullback attract bounded subsets of the phase space under the evolution process is proved for the nonautonomous logistic equation and a system of reaction-diffusion equations with time-dependent external forces including the case of the FitzHugh-Nagumo system.

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1. Pullback exponential and global attractors for semilinear parabolic problems

In Part I of this work (see [5]) we have constructed a *pullback exponential attractor* for an evolution process. By this we mean a family of compact and positively invariant sets with uniformly bounded fractal dimension which under the evolution process at a uniform exponential rate pullback attract bounded subsets of the phase space. We have also compared this object with a better known notion of a *pullback global attractor* (see for example [2], [3]) being a minimal family of compact invariant sets under the process and pullback attracting each bounded subset of the phase space. Moreover, we have formulated conditions under

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which the mentioned abstract results apply to nonautonomous semilinear parabolic problems. For completeness we recall here the main result (see [5, Theorem 3.6]) and refer the reader for the proof and details to Part I of this work.

We consider a positive sectorial operator $A\colon X \supset D(A) \to X$ in a Banach space X having a compact resolvent (see [7]). Denoting by X^γ, $\gamma \geq 0$, the associated fractional power spaces, we fix $\alpha \in [0,1)$ and consider a function $F\colon \mathbb{R} \times X^\alpha \to X$ satisfying the following assumption

$$\begin{gathered}\forall_{G\subset X^\alpha,\ \text{bounded}} \exists_{0<\theta=\theta(G)\leq 1} \forall_{T_1,T_2\in\mathbb{R},T_1<T_2} \exists_{L=L(T_2-T_1,G)>0} \forall_{\tau_1,\tau_2\in[T_1,T_2]} \forall_{u_1,u_2\in G} \\ \|F(\tau_1,u_1)-F(\tau_2,u_2)\|_X \leq L(|\tau_1-\tau_2|^\theta + \|u_1-u_2\|_{X^\alpha}).\end{gathered} \tag{F1}$$

Note that L depends only on the difference $T_2 - T_1$ and on G. Under this assumption for any $\sigma \in \mathbb{R}$ and $u_0 \in X^\alpha$ there exists a unique (forward) local X^α solution to the problem

$$\begin{cases} u_\tau + Au = F(\tau,u),\ \tau > \sigma, \\ u(\sigma) = u_0, \end{cases} \tag{1.1}$$

defined on the maximal interval of existence $[\sigma, \tau_{max})$, i.e. a function

$$u \in C([\sigma,\tau_{max}),X^\alpha) \cap C((\sigma,\tau_{max}),X^1) \cap C^1((\sigma,\tau_{max}),X)$$

satisfying (1.1) in X and such that either $\tau_{max} = \infty$ or $\tau_{max} < \infty$ and in the latter case

$$\limsup_{\tau\to\tau_{max}} \|u(\tau)\|_{X^\alpha} = \infty.$$

Furthermore, we denote

$$\mathcal{T} = \{\tau \in \mathbb{R}\colon \tau \leq \tau_0\}$$

with $\tau_0 \leq \infty$ fixed and assume that for some $M > 0$

$$\sup_{\tau\in\mathcal{T}} \|F(\tau,0)\|_X \leq M. \tag{F2}$$

In order to prove that the local solutions can be extended globally (forward) in time and obtain the existence of a bounded absorbing set in X^α in specific examples we will verify an appropriate a priori estimate. Here we assume that

$$\text{each local solution can be extended globally (forward) in time, i.e. } \tau_{max} = \infty, \tag{F3a}$$

there exists a constant $\omega > 0$ and a nondecreasing function $Q\colon [0,\infty) \to [0,\infty)$ (both independent of σ) such that

$$\|u(\tau)\|_{X^\alpha} \leq Q(\|u_0\|_{X^\alpha})e^{-\omega(\tau-\sigma)} + R_0,\ \sigma \leq \tau,\ \tau \in \mathcal{T}, \tag{F3b}$$

holds with a constant $R_0 = R_0(\tau_0) > 0$ independent of σ, τ and u_0 and (in case $\tau_0 < \infty$) for any $T > 0$ there exists $R_{T,\sigma} > 0$ and a nondecreasing function $\widetilde{Q}_{T,\sigma}\colon [0,\infty) \to [0,\infty)$ such that

$$\|u(\tau)\|_{X^\alpha} \le \widetilde{Q}_{T,\sigma}(\|u_0\|_{X^\alpha}) + R_{T,\sigma},\ \tau \in [\sigma, \sigma + T]. \tag{F3c}$$

Note that hypotheses (F3a)–(F3c) can be replaced by a single stronger requirement that (1.1) admits the following dissipativity condition in X^α

$$\|u(\tau)\|_{X^\alpha} \le Q(\|u_0\|_{X^\alpha})e^{-\omega(\tau-\sigma)} + R(\tau),\ \tau \in [\sigma, \tau_{max}), \tag{F3}$$

where $\omega > 0$, $Q\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function and $R\colon \mathbb{R} \to [0,\infty)$ is a continuous function such that for some positive constant R_0 (independent of u_0, σ, τ)

$$\sup_{\tau\in\mathcal{T}} R(\tau) \le R_0.$$

Because of (F3a) we define the evolution process $\{U(\tau,\sigma)\colon \tau \ge \sigma\}$ on X^α by

$$U(\tau,\sigma)u_0 := u(\tau),\ \tau \ge \sigma,\ u_0 \in X^\alpha, \tag{1.2}$$

where $u(\tau)$ is the value at time τ of the X^α solution of (1.1) starting at time σ from u_0. Thus we have

$$U(\tau,\sigma)U(\sigma,\rho) = U(\tau,\rho),\ \tau \ge \sigma \ge \rho,\ \tau,\sigma,\rho \in \mathbb{R}, \quad U(\tau,\tau) = I,\ \tau \in \mathbb{R}, \tag{1.3}$$

where I denotes an identity operator on X^α.

Theorem 1.1. *Under the conditions stated above for any $\beta \in (\alpha, 1)$ there exists a family $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ of nonempty compact subsets of X^β such that*

(i) $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ is positively invariant under the process $U(\tau,\sigma)$, i.e.

$$U(\tau,\sigma)\mathcal{M}(\sigma) \subset \mathcal{M}(\tau),\ \tau \ge \sigma,$$

(ii) $\mathcal{M}(\tau)$ has a finite fractal dimension in X^β uniformly with respect to $\tau \in \mathbb{R}$, i.e. there exists $d < \infty$ such that

$$d_f^{X^\alpha}(\mathcal{M}(\tau)) \le d_f^{X^\beta}(\mathcal{M}(\tau)) \le d,\ \tau \in \mathbb{R},$$

(iii) $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ has the property of pullback exponential attraction, i.e.

$$\exists_{\varphi>0}\forall_{B_1\subset X^\beta,\ bounded}\ \forall_{\tau\in\mathbb{R}}\quad \lim_{t\to\infty} e^{\varphi t}\,\mathrm{dist}_{X^\beta}(U(\tau,\tau-t)B_1, \mathcal{M}(\tau)) = 0$$

and if $\tau_0 = \infty$, the pullback attraction is uniform with respect to τ

$$\exists_{\varphi>0}\forall_{B_1\subset X^\beta,\ bounded}\quad \lim_{t\to\infty} e^{\varphi t}\sup_{\tau\in\mathbb{R}} \operatorname{dist}_{X^\beta}(U(\tau,\tau-t)B_1,\mathcal{M}(\tau))=0.$$

This property is equivalent to the uniform forwards exponential attraction

$$\exists_{\varphi>0}\forall_{B_1\subset X^\beta,\ bounded}\quad \lim_{t\to\infty} e^{\varphi t}\sup_{\tau\in\mathbb{R}} \operatorname{dist}_{X^\beta}(U(t+\tau,\tau)B_1,\mathcal{M}(t+\tau))=0.$$

Furthermore, the pullback exponential attractor $\{\mathcal{M}(\tau)\colon \tau\in\mathbb{R}\}$ contains a (finite dimensional) pullback global attractor $\{\mathcal{A}(\tau)\colon \tau\in\mathbb{R}\}$, i.e. a family of nonempty compact subsets of X^β, invariant under the process $\{U(\tau,\sigma)\colon \tau\geq\sigma\}$

$$U(\tau,\sigma)\mathcal{A}(\sigma)=\mathcal{A}(\tau),\ \tau\geq\sigma,$$

pullback attracting all bounded subsets of X^β

$$\forall_{B_1\subset X^\beta,\ bounded}\,\forall_{\tau\in\mathbb{R}}\quad \lim_{t\to\infty}\operatorname{dist}_{X^\beta}(U(\tau,\tau-t)B_1,\mathcal{A}(\tau))=0$$

and minimal in the sense that if $\{\widetilde{\mathcal{A}}(\tau)\colon \tau\in\mathbb{R}\}$ is a family of closed sets in X^β pullback attracting all bounded subsets of X^β, then $\mathcal{A}(\tau)\subset\widetilde{\mathcal{A}}(\tau)$, $\tau\in\mathbb{R}$.

In this paper we apply Theorem 1.1 to nonautonomous reaction-diffusion equations and systems. In Section 2 we verify the above hypotheses in an introductory example of the nonautonomous logistic equation with Dirichlet boundary condition and in Section 3 we consider a system of reaction-diffusion equations perturbed by a time-dependent external forces. This system satisfies an anisotropic dissipativity condition that holds, for example, for the FitzHugh-Nagumo system or in some chemical reaction systems (see Remark 3.1).

2. Nonautonomous logistic equation

We consider Dirichlet boundary problem for the nonautonomous logistic equation (cf. [8]) in a sufficiently smooth bounded domain $\Omega\subset\mathbb{R}^N$, $N\leq 3$, of the form

$$\begin{cases}\partial_\tau u=\triangle_D u+\lambda u-b(\tau)u^3,\ \tau>\sigma,\ x\in\Omega,\\ u(\sigma,x)=u_0(x),\ x\in\Omega,\quad u(\tau,x)=0,\ \tau\geq\sigma,\ x\in\partial\Omega.\end{cases}\tag{2.1}$$

Here $u=u(\tau,x)$ is an unknown function, $\lambda\in\mathbb{R}$ and b is Hölder continuous on $\mathbb{R}$ with exponent $\theta\in(0,1]$ and satisfies

$$0<b(\tau)\leq M,\ \tau\in\mathbb{R},\tag{2.2}$$

for some positive M. Moreover, we assume that there exist $\tau_0 \leq \infty$ and $m > 0$ such that

$$m \leq b(\tau), \ \tau \in \mathcal{T}, \tag{2.3}$$

where we denoted $\mathcal{T} = \{\tau \in \mathbb{R} \colon \tau \leq \tau_0\}$. We rewrite the problem (2.1) as an abstract Cauchy problem (1.1), where $A = -\triangle_D$ in $X = L^2(\Omega)$ with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is a positive sectorial operator with compact resolvent. We also consider its fractional power spaces and have for $\alpha \in \left(\frac{1}{4}, 1\right)$

$$X^\alpha = H_0^{2\alpha}(\Omega) = \{\phi \in H^{2\alpha}(\Omega) \colon \phi|_{\partial\Omega} = 0\}.$$

Observe that $F \colon \mathbb{R} \times X^{\frac{1}{2}} \to X$ given as $F(\tau, u) = \lambda u - b(\tau)u^3$ is well defined and by (2.2) we have for u_1, u_2 from a bounded subset G of $X^{\frac{1}{2}} = H_0^1(\Omega)$ and $\tau_1, \tau_2 \in \mathbb{R}$

$$\|F(\tau_1, u_1) - F(\tau_2, u_2)\|_{L^2(\Omega)} \leq c_1 |\tau_2 - \tau_1|^\theta + c_2 \|u_1 - u_2\|_{H_0^1(\Omega)}.$$

This shows, in particular, that assumption (F1) is satisfied with $\alpha = \frac{1}{2}$.

Moreover, we have $\|F(\tau, 0)\|_{L^2(\Omega)} = 0$ for $\tau \in \mathbb{R}$. Hence (F2) is satisfied trivially.

Finally, we verify that (F3) also holds. Multiplying the first equation in (2.1) by u and integrating over Ω we get

$$\frac{1}{2}\frac{d}{dt} \|u\|_{L^2(\Omega)}^2 = -\, \||\nabla u|\|_{L^2(\Omega)}^2 + \int_\Omega \lambda u^2 - b(t)u^4 dx.$$

Note that by the Cauchy inequality we have

$$\frac{d}{dt} \|u\|_{L^2(\Omega)}^2 + 2\, \||\nabla u|\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\lambda^2 |\Omega| \frac{1}{b(t)}. \tag{2.4}$$

Observe that by the Poincaré inequality we obtain

$$\frac{d}{dt} \|u\|_{L^2(\Omega)}^2 + 2\lambda_1 \|u\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\lambda^2 |\Omega| \frac{1}{b(t)},$$

where $\lambda_1 > 0$ is the principal eigenvalue of $-\triangle_D$. Integrating over the time interval from σ to τ we get

$$\|u(\tau)\|_{L^2(\Omega)}^2 \leq \|u(\sigma)\|_{L^2(\Omega)}^2 e^{-2\lambda_1(\tau-\sigma)} + \frac{1}{2}\lambda^2 |\Omega| \int_\sigma^\tau \frac{e^{-2\lambda_1(\tau-t)}}{b(t)} dt. \tag{2.5}$$

Now we proceed to obtain the a priori estimate in $H_0^1(\Omega)$. We multiply the first equation in (2.1) by $-\triangle_D u$, integrate over Ω and use integration by parts to get

$$\frac{1}{2}\frac{d}{dt} \||\nabla u|\|_{L^2(\Omega)}^2 + \|\triangle_D u\|_{L^2(\Omega)}^2 = \int_\Omega (\lambda - 3b(t)u^2) |\nabla u|^2 dx.$$

Because b is a positive function, we obtain

$$\frac{d}{dt}\||\nabla u|\|_{L^2(\Omega)}^2 \leq 2|\lambda|\,\||\nabla u|\|_{L^2(\Omega)}^2. \tag{2.6}$$

We add to both sides $\lambda_1\||\nabla u|\|_{L^2(\Omega)}^2$, multiply by $e^{\lambda_1 t}$ and integrate from σ to τ to obtain

$$\||\nabla u(\tau)|\|_{L^2(\Omega)}^2 \leq \||\nabla u(\sigma)|\|_{L^2(\Omega)}^2 e^{-\lambda_1(\tau-\sigma)} + (2|\lambda|+\lambda_1)\int_\sigma^\tau \||\nabla u(t)|\|_{L^2(\Omega)}^2 e^{\lambda_1(t-\tau)}dt. \tag{2.7}$$

We return now to (2.4) and use the Poincaré inequality to get

$$\frac{d}{dt}\|u\|_{L^2(\Omega)}^2 + \lambda_1\|u\|_{L^2(\Omega)}^2 + \||\nabla u|\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\lambda^2|\Omega|\frac{1}{b(t)}.$$

Multiplying by $e^{\lambda_1 t}$ and integrating from σ to τ we conclude that

$$\int_\sigma^\tau \||\nabla u(t)|\|_{L^2(\Omega)}^2 e^{\lambda_1(t-\tau)}dt \leq \|u(\sigma)\|_{L^2(\Omega)}^2 e^{-\lambda_1(\tau-\sigma)} + \frac{1}{2}\lambda^2|\Omega|\int_\sigma^\tau \frac{e^{-\lambda_1(\tau-t)}}{b(t)}dt. \tag{2.8}$$

Combining (2.5), (2.7) and (2.8) and using (2.3) we get

$$\|u(\tau)\|_{H_0^1(\Omega)} \leq \sqrt{1+2|\lambda|+\lambda_1}\,\|u(\sigma)\|_{H_0^1(\Omega)} e^{-\frac{\lambda_1}{2}(\tau-\sigma)} + R(\tau), \tag{2.9}$$

where

$$R(\tau) = R_0\left(\lambda_1 m\int_{-\infty}^\tau \frac{e^{-\lambda_1(\tau-t)}}{b(t)}dt\right)^{\frac{1}{2}},\ \tau\in\mathbb{R}, \tag{2.10}$$

and

$$R_0 = \sqrt{\frac{(1+2|\lambda|+\lambda_1)\lambda^2|\Omega|}{2\lambda_1 m}}.$$

Note that the function R is well defined and $R(\tau)\leq R_0$ for $\tau\in\mathcal{T}$. This shows that assumption (F3) holds with $\alpha=\frac{1}{2}$. Therefore we may apply Theorem 1.1 and obtain the following

Corollary 2.1. *If* (2.2) *and* (2.3) *hold, then the problem* (2.1) *generates an evolution process* $\{U(\tau,\sigma)\colon \tau\geq\sigma\}$ *in* $H_0^1(\Omega)$ *and for any* $\beta\in(\frac{1}{2},1)$ *there exists a family* $\{\mathcal{M}(\tau)\colon \tau\in\mathbb{R}\}$ *of nonempty compact subsets of* $H_0^{2\beta}(\Omega)$ *with the following properties:*

(i) $\{\mathcal{M}(\tau)\colon\tau\in\mathbb{R}\}$ *is positively invariant under the process* $U(\tau,\sigma)$*, i.e.*

$$U(\tau,\sigma)\mathcal{M}(\sigma)\subset\mathcal{M}(\tau),\ \tau\geq\sigma,$$

(ii) $\mathcal{M}(\tau)$ has a finite fractal dimension in $H_0^{2\beta}(\Omega)$ uniformly w.r.t. $\tau \in \mathbb{R}$, i.e.

$$d_f^{H_0^1(\Omega)}(\mathcal{M}(\tau)) \le d_f^{H_0^{2\beta}(\Omega)}(\mathcal{M}(\tau)) \le d < \infty,\ \tau \in \mathbb{R},$$

(iii) $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ has the property of pullback exponential attraction, i.e.

$$\exists_{\varphi>0}\forall_{B_1\subset H_0^{2\beta}(\Omega),\ bounded}\forall_{\tau\in\mathbb{R}}\quad \lim_{t\to\infty} e^{\varphi t}\operatorname{dist}_{H_0^{2\beta}(\Omega)}(U(\tau,\tau-t)B_1,\mathcal{M}(\tau)) = 0$$

and if $\tau_0 = \infty$, the pullback attraction is uniform w.r.t. $\tau \in \mathbb{R}$

$$\exists_{\varphi>0}\forall_{B_1\subset H_0^{2\beta}(\Omega),\ bounded}\quad \lim_{t\to\infty} e^{\varphi t}\sup_{\tau\in\mathbb{R}}\operatorname{dist}_{H_0^{2\beta}(\Omega)}(U(\tau,\tau-t)B_1,\mathcal{M}(\tau)) = 0.$$

Furthermore, the pullback exponential attractor $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ contains a (finite dimensional) pullback global attractor $\{\mathcal{A}(\tau)\colon \tau \in \mathbb{R}\}$, i.e. a family of nonempty compact subsets of $H_0^{2\beta}(\Omega)$, invariant under the process $\{U(\tau,\sigma)\colon \tau \ge \sigma\}$

$$U(\tau,\sigma)\mathcal{A}(\sigma) = \mathcal{A}(\tau),\ \tau \ge \sigma,$$

and pullback attracting all bounded subsets of $H_0^{2\beta}(\Omega)$

$$\forall_{B_1\subset H_0^{2\beta}(\Omega),\ bounded}\forall_{\tau\in\mathbb{R}}\quad \lim_{t\to\infty}\operatorname{dist}_{H_0^{2\beta}(\Omega)}(U(\tau,\tau-t)B_1,\mathcal{A}(\tau)) = 0.$$

3. Anisotropic nonautonomous reaction-diffusion systems

Following [6] we consider the nonautonomous reaction-diffusion system

$$\begin{cases}\partial_\tau u + Au = f(u) + g(\tau),\ \tau > \sigma,\ x \in \Omega,\\ u(\sigma,x) = u_0(x),\ x \in \Omega, \quad u(\tau,x) = 0,\ \tau \ge \sigma,\ x \in \partial\Omega,\end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with $\partial\Omega \in C^{2+\eta}$. Here $u(\tau,x) = (u_1(\tau,x),\dots,u_k(\tau,x))$ is an unknown function and $f(u) = (f_1(u),\dots,f_k(u))$ and $g(\tau,x) = (g_1(\tau,x),\dots,g_k(\tau,x))$ are given functions. We suppose that A is a second order elliptic differential operator of the form $Au = (A_1u_1,\dots,A_ku_k)$, where

$$A_l u_l(x) = \sum_{i,j=1}^{3} \partial_{x_i}(a_{ij}^l(x)\partial_{x_j}u_l(x)),\ x \in \Omega,\ l = 1,\dots,k, \tag{3.2}$$

with the functions $a_{ij}^l = a_{ji}^l$ from $C^{1+\eta}(\overline{\Omega})$ and satisfying uniformly strong ellipticity condition

$$\exists_{\nu>0}\forall_{l=1,\dots,k}\forall_{x\in\Omega}\forall_{\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3}\quad -\sum_{i,j=1}^{3} a_{ij}^l(x)\xi_i\xi_j \ge \nu\,|\xi|^2. \tag{3.3}$$

We also assume that for the nonlinear term $f \in C(\mathbb{R}^k, \mathbb{R}^k)$ there exist constants $p_1, \dots, p_k \geq 0$ and $q_1, \dots, q_k \geq 0$ such that f satisfies the growth assumption

$$\exists_{c>0} \forall_{u=(u_1,\dots,u_k), v=(v_1,\dots,v_k)\in\mathbb{R}^k} \ |f(u) - f(v)|^2 \leq c \sum_{l=1}^{k} |u_l - v_l|^2 \left(1 + |u_l|^{p_l} + |v_l|^{p_l}\right) \tag{3.4}$$

and the anisotropic dissipativity assumption

$$\exists_{C>0} \forall_{u=(u_1,\dots,u_k)\in\mathbb{R}^k} \ \sum_{l=1}^{k} f_l(u) u_l |u_l|^{q_l} \leq C. \tag{3.5}$$

The restrictions on the range of constants will be imposed later. As refers to the time-dependent perturbation we assume that

$$g \colon \mathbb{R} \to [L^2(\Omega)]^k \text{ is globally Hölder continuous with exponent } \theta \in (0, 1] \tag{3.6}$$

and there is $\tau_0 \leq \infty$ such that

$$\sup_{\tau \in \mathcal{T}} \|g(\tau)\|_{[L^2(\Omega)]^k} < \infty, \tag{3.7}$$

where we denoted $\mathcal{T} = \{\tau \in \mathbb{R} \colon \tau \leq \tau_0\}$.

Below in Remark 3.1 we present two particular cases of the system (3.1) concerning time-perturbed systems of two coupled reaction-diffusion equations.

Remark 3.1. If $k = 2$, we consider the perturbed FitzHugh-Nagumo system modelling transmission of nerve impulses in axons, i.e. for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\varepsilon > 0$

$$f_1(u_1, u_2) = \alpha u_1 + \beta u_1^2 - u_1^3 - \gamma u_2, \quad f_2(u_1, u_2) = \delta u_1 - \varepsilon u_2. \tag{3.8}$$

Note that the following inequality holds

$$\forall_{q\geq 0} \exists_{C>0} \forall_{(u_1,u_2)\in\mathbb{R}^2} \ \sum_{l=1}^{2} f_l(u_1, u_2) u_l |u_l|^q \leq C. \tag{3.9}$$

Indeed, by the Young inequality it follows that for some positive c_1

$$(\alpha u_1 + \beta u_1^2 - u_1^3 - \gamma u_2) u_1 |u_1|^q + (\delta u_1 - \varepsilon u_2) u_2 |u_2|^q \leq c_1 |u_1|^{2+q} + |\beta| |u_1|^{3+q} - |u_1|^{4+q}.$$

Applying again the Young inequality, we obtain (3.9).

Note that there are positive c_2, c_3 such that for $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ we have

$$|f(u) - f(v)|^2 \leq c_2 |u_1 - v_1|^2 (1 + |u_1|^4 + |v_1|^4) + c_3 |u_2 - v_2|^2 .$$

Thus both assumptions (3.4) and (3.5) are satisfied with $p_1 = 4$, $p_2 = 0$ and $q_1 = q_2 = q$, where $q \geq 0$ is arbitrary.

We also consider the following chemical reaction nonlinearity

$$f_1(u_1, u_2) = u_2 - u_1^3, \quad f_2(u_1, u_2) = u_1^3 - u_2. \tag{3.10}$$

Observe that by the Young inequality we have

$$(u_2 - u_1^3)u_1 |u_1|^4 + (u_1^3 - u_2)u_2 |u_2|^{\frac{2}{3}} \leq |u_2| |u_1|^5 - |u_1|^8 + |u_1|^3 |u_2|^{\frac{5}{3}} - |u_2|^{\frac{8}{3}} \leq 0$$

and

$$|f(u) - f(v)|^2 \leq 18 |u_1 - v_1|^2 (|u_1|^4 + |v_1|^4) + 4 |u_2 - v_2|^2 .$$

This means that assumptions (3.4) and (3.5) are satisfied with $p_1 = 4$, $p_2 = 0$ and $q_1 = 4$, $q_2 = \frac{2}{3}$. Note also that the usual dissipativity assumption ($q_1 = q_2 = 0$) is not satisfied in this case, since the expression $(u_2 - u_1^3)u_1 + (u_1^3 - u_2)u_2 = (u_2 - u_1)(u_1^3 - u_2)$ can be made arbitrarily large.

We consider (3.1) as an abstract semilinear parabolic Cauchy problem (1.1) in the space $X = [L^2(\Omega)]^k$ with $F(\tau, u) = f(u) + g(\tau)$. Note that A is a sectorial operator in X with the domain $D(A) = [H^2(\Omega) \cap H_0^1(\Omega)]^k$ (see [7, Example 1.3(3)], [1, Theorem 1.6.1], [4, Proposition 1.2.3]) and has a compact resolvent and the fractional power spaces are described as follows

$$X^\alpha = [X, D(A)]_\alpha = [H_0^{2\alpha}(\Omega)]^k = [\{\phi \in H^{2\alpha}(\Omega) \colon \phi|_{\partial\Omega} = 0\}]^k, \alpha \in \left(\frac{1}{4}, 1\right)$$

(cf. [1, Proposition 2.3.3], [4, Section 1.3]). We fix $\alpha = \frac{1}{2}$ and have $X^{\frac{1}{2}} = [H_0^1(\Omega)]^k$. Below we show that $F \colon \mathbb{R} \times X^{\frac{1}{2}} \to X$ is well defined and assumption (F1) is satisfied in $X^{\frac{1}{2}}$ when we suitably restrict the range of constants p_l.

Proposition 3.2. *If $0 \leq p_l \leq 4$, $l = 1, \ldots, k$, then there exists $\theta \in (0, 1]$ such that for any bounded subset G of $X^{\frac{1}{2}}$ there exists $L > 0$ such that for any $u, v \in G$ and $\tau_1, \tau_2 \in \mathbb{R}$ we have*

$$\|F(\tau_1, u) - F(\tau_2, v)\|_{[L^2(\Omega)]^k} \leq L(|\tau_1 - \tau_2|^\theta + \|u - v\|_{X^{\frac{1}{2}}}).$$

Proof. We have

$$\|F(\tau_1,u)-F(\tau_2,v)\|_{[L^2(\Omega)]^k} \le \|f(u)-f(v)\|_{[L^2(\Omega)]^k} + \|g(\tau_1)-g(\tau_2)\|_{[L^2(\Omega)]^k}. \tag{3.11}$$

Since by assumption we know that

$$\|g(\tau_1)-g(\tau_2)\|_{[L^2(\Omega)]^k} \le L_1 |\tau_1-\tau_2|^\theta,\ \tau_1,\tau_2\in\mathbb{R},$$

it is enough to estimate the first term in (3.11). Indeed, using (3.4) and the Hölder inequality in case $p_l > 0$, we obtain

$$\|f(u)-f(v)\|^2_{[L^2(\Omega)]^k} \le \widetilde{c}\sum_{l=1}^k \|u_l-v_l\|^2_{L^{p_l+2}(\Omega)}(1+\|u_l\|^{p_l}_{L^{p_l+2}(\Omega)}+\|v_l\|^{p_l}_{L^{p_l+2}(\Omega)}). \tag{3.12}$$

Hence we have

$$\|f(u)-f(v)\|^2_{[L^2(\Omega)]^k} \le \widetilde{c}\,\|u-v\|^2_{[L^{p_l+2}(\Omega)]^k}\sum_{l=1}^k(1+\|u_l\|^{p_l}_{L^{p_l+2}(\Omega)}+\|v_l\|^{p_l}_{L^{p_l+2}(\Omega)}),$$

If $0\le p_l\le 4$, $l=1,\dots,k$, then $H^1_0(\Omega)\hookrightarrow L^{p_l+2}(\Omega)$ and in consequence for any bounded subset G of $X^{\frac12}=[H^1_0(\Omega)]^k$ we have

$$\|f(u)-f(v)\|_{[L^2(\Omega)]^k} \le L_G\|u-v\|_{X^{\frac12}},\ u,v\in G.$$

This proves the claim. □

Thus if $0\le p_l\le 4$, $l=1,\dots,k$, then for any $\sigma\in\mathbb{R}$ and $u_0\in X^{\frac12}$ there exists a unique (forward) $X^{\frac12}$ solution to (3.1) defined on the maximal interval of existence $[\sigma,\tau_{max})$, i.e.

$$u\in C([\sigma,\tau_{max}),[H^1_0(\Omega)]^k)\cap C((\sigma,\tau_{max}),[H^2(\Omega)\cap H^1_0(\Omega)]^k)\cap C^1((\sigma,\tau_{max}),[L^2(\Omega)]^k)$$

and either $\tau_{max}=\infty$ or $\tau_{max}<\infty$ and in the latter case

$$\limsup_{\tau\to\tau_{max}}\|u(\tau)\|_{[H^1_0(\Omega)]^k}=\infty. \tag{3.13}$$

Note that assumption (F2) is also clearly satisfied, since by (3.7) we have

$$\sup_{\tau\in\mathcal{T}}\|F(\tau,0)\|_{[L^2(\Omega)]^k}\le\|f(0)\|_{[L^2(\Omega)]^k}+\sup_{\tau\in\mathcal{T}}\|g(\tau)\|_{[L^2(\Omega)]^k}<\infty.$$

Now we will show that under certain constraints on p_l and q_l assumptions (F3a)–(F3c) also hold. To this end, we develop some a priori estimates following [6].

Lemma 3.3. *For any $\gamma > 0$ there exists $C_\gamma > 0$ such that for any $h > 0$, any real $\tau \geq \sigma + h$ and any nonnegative integrable function z on $[\sigma, \tau]$ we have*

$$\int_\sigma^\tau z(t)dt \leq C_\gamma \sup_{t\in[\sigma+h,\tau]} \left(e^{\frac{\gamma}{2}\frac{\tau-t}{h}} \int_{t-h}^t z(s)ds \right). \tag{3.14}$$

Proof. Observe that

$$\int_\sigma^\tau z(t)dt \leq e^{\frac{\gamma}{2}} \left(1 + e^{-\frac{\gamma}{2}} + e^{-\gamma} + \ldots + e^{-\frac{\gamma}{2}[\frac{\tau-\sigma}{h}]} \right) \sup_{t\in[\sigma+h,\tau]} \left(e^{\frac{\gamma}{2}\frac{\tau-t}{h}} \int_{t-h}^t z(s)ds \right),$$

since $e^{-\frac{\gamma}{2}(\frac{\tau-\sigma}{h}-1)} \leq e^{-\frac{\gamma}{2}[\frac{\tau-\sigma}{h}]} e^{\frac{\gamma}{2}}$. This leads to (3.14) with $C_\gamma = e^{\frac{\gamma}{2}}(1-e^{-\frac{\gamma}{2}})^{-1}$. □

We also adapt the following lemma from [10, Proposition 3].

Lemma 3.4. *Assume that a continuous function $z\colon [a,b) \to [0,\infty)$, $a < b \leq \infty$, satisfies*

$$z(\tau) \leq D_0 e^{-\beta(\tau-a)} + D_1 + \mu \sup_{s\in[a,\tau]} \{e^{-\gamma(\tau-s)} z(s)\},\ a \leq \tau < b \tag{3.15}$$

with $\beta \geq \gamma > 0$, $D_0, D_1 \geq 0$ and $0 \leq \mu < 1$. Then we have

$$z(\tau) \leq D_0(1-\mu)^{-1} e^{-\gamma(\tau-a)} + D_1(1-\mu)^{-1},\ a \leq \tau < b. \tag{3.16}$$

Proof. Fix any $a < T < b$. From (3.15) it follows that

$$z(\tau) \leq D_0 e^{-\beta(\tau-a)} + D_1 + \mu \sup_{s\in[a,T]} \{e^{-\gamma|\tau-s|} z(s)\},\ \tau \in [a,T]. \tag{3.17}$$

Let us fix $\rho \in [a,T]$. Then we multiply the above equation by $e^{-\gamma|\rho-\tau|}$ and take the supremum with respect to $\tau \in [a,T]$

$$\sup_{\tau\in[a,T]} e^{-\gamma|\rho-\tau|} z(\tau) \leq D_0 \sup_{\tau\in[a,T]} e^{-\beta(\tau-a)-\gamma|\rho-\tau|} + D_1 + \mu \sup_{\tau\in[a,T]} \sup_{s\in[a,T]} \{e^{-\gamma(|\tau-s|+|\rho-\tau|)} z(s)\}.$$

Note that we have $\sup_{\tau\in[a,T]} e^{-\beta(\tau-a)-\gamma|\rho-\tau|} = e^{-\gamma(\rho-a)}$ and

$$\sup_{\tau\in[a,T]} \sup_{s\in[a,T]} \{e^{-\gamma(|\tau-s|+|\rho-\tau|)} z(s)\} = \sup_{s\in[a,T]} e^{-\gamma|\rho-s|} z(s).$$

Concluding, we get

$$\sup_{s\in[a,T]} e^{-\gamma|\rho-s|} z(s) \leq D_0 e^{-\gamma(\rho-a)} + D_1 + \mu \sup_{s\in[a,T]} e^{-\gamma|\rho-s|} z(s).$$

Since $0 \leq \mu < 1$ and $\sup\limits_{s\in[a,T]} e^{-\gamma|\rho-s|}z(s) < \infty$, we obtain

$$\sup_{s\in[a,T]} e^{-\gamma|\rho-s|}z(s) \leq D_0(1-\mu)^{-1}e^{-\gamma(\rho-a)} + D_1(1-\mu)^{-1},\ \rho \in [a,T].$$

We apply this estimate to (3.17). From the arbitrary choice of $T < b$ we get (3.16). □

Proposition 3.5. *Let $u = (u_1, \dots, u_k)$ be an $X^{\frac{1}{2}}$ solution of (3.1) on $[\sigma, \tau_{max})$.*

If $\tau_{max} < \infty$, then with $h > 0$ such that $\sigma < \sigma + h < \tau_{max}$ we have for $\sigma \leq \tau < \tau_{max}$

$$\begin{aligned}\sum_{l=1}^{k} \|u_l(\tau)\|^{2+q_l}_{L^{2+q_l}(\Omega)} \leq 2e^{\frac{\lambda_1\nu}{2}h} \sum_{l=1}^{k} \|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)} e^{-\frac{\lambda_1\nu}{2}(\tau-\sigma)} + \\ + C_8\left(\sum_{l=1}^{k}\left(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\right)^{q_l+2} + 1\right),\end{aligned} \tag{3.18}$$

and for $\sigma + h \leq \tau < \tau_{max}$

$$\begin{aligned}\nu\int_{\tau-h}^{\tau}\sum_{l=1}^{k}\left\|\left|\nabla(|u_l(s)|^{\frac{q_l+2}{2}})\right|\right\|^2_{L^2(\Omega)} ds \leq 2e^{\frac{\lambda_1\nu}{2}h}\sum_{l=1}^{k}\|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)} e^{-\frac{\lambda_1\nu}{2}(\tau-\sigma)} + \\ + C_8\left(\sum_{l=1}^{k}\left(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\right)^{q_l+2} + 1\right),\end{aligned} \tag{3.19}$$

with $E = [\sigma, \tau_{max})$, where $C_8 = C_8(h)$ is a positive constant.

If $\tau_{max} = \infty$, then we choose $h = 1$ and (3.18) holds with $E = (-\infty, \tau_0 + 2)$ for $\sigma \leq \tau$, $\tau \in \mathcal{T}$, whereas (3.19) holds with $E = (-\infty, \tau_0 + 2)$ for $\sigma + 1 \leq \tau$, $\tau \in \mathcal{T}$.

If $\tau_{max} = \infty$, then for any $T > 0$ we choose $0 < h < T$ and (3.18) holds with $E = [\sigma, \sigma+T]$ for $\sigma \leq \tau \leq \sigma + T$, while (3.19) holds with $E = [\sigma, \sigma + T]$ for $\sigma + h \leq \tau \leq \sigma + T$.

Proof. For each $l = 1, \dots, k$ we multiply the l-th equation in (3.1) by $u_l\,|u_l|^{q_l}$ and integrate over Ω

$$\int_\Omega (\partial_t u_l)u_l\,|u_l|^{q_l}\,dx + \int_\Omega (A_l u_l)u_l\,|u_l|^{q_l}\,dx = \int_\Omega f_l(u)u_l\,|u_l|^{q_l}\,dx + \int_\Omega g_l(t)u_l\,|u_l|^{q_l}\,dx.$$

Note that

$$\int_\Omega (\partial_t u_l)u_l\,|u_l|^{q_l}\,dx = \frac{1}{q_l+2}\partial_t\,\|u_l\|^{2+q_l}_{L^{2+q_l}(\Omega)}$$

and by integration by parts and (3.3) we have

$$\int_\Omega (A_l u_l) u_l |u_l|^{q_l} \, dx = -(q_l+1) \sum_{i,j=1}^{3} \int_\Omega a_{ij}^l \partial_{x_j} u_l \, |u_l|^{q_l} \, \partial_{x_i} u_l dx.$$

Thus if $q_l > 0$ then

$$\int_\Omega (A_l u_l) u_l |u_l|^{q_l} \, dx = -\frac{4(q_l+1)}{(q_l+2)^2} \sum_{i,j=1}^{3} \int_\Omega a_{ij}^l \partial_{x_i} \left(|u_l|^{\frac{q_l+2}{2}} \right) \partial_{x_j} \left(|u_l|^{\frac{q_l+2}{2}} \right) dx \geq$$

$$\geq \frac{4(q_l+1)}{(q_l+2)^2} \nu \int_\Omega \sum_{i=1}^{3} \left| \partial_{x_i} \left(|u_l|^{\frac{q_l+2}{2}} \right) \right|^2 dx = \frac{4(q_l+1)}{(q_l+2)^2} \nu \left\| \left| \nabla (|u_l|^{\frac{q_l+2}{2}}) \right| \right\|_{L^2(\Omega)}^2$$

and if $q_l = 0$ then we have

$$\int_\Omega (A_l u_l) u_l |u_l|^{q_l} \, dx \geq \nu \, \| |\nabla u_l| \|_{L^2(\Omega)}^2 .$$

Since $\frac{4(q_l+1)}{q_l+2} \geq 2$, we obtain

$$\partial_t \|u_l\|_{L^{2+q_l}(\Omega)}^{2+q_l} + \nu \left\| \left| \nabla (|u_l|^{\frac{q_l+2}{2}}) \right| \right\|_{L^2(\Omega)}^2 \leq (q_l+2) \left(\int_\Omega f_l(u) u_l |u_l|^{q_l} \, dx + \int_\Omega g_l(t) u_l |u_l|^{q_l} \, dx \right)$$

omitting the modulus under the gradient when $q_l = 0$. We set

$$F_u(t) = \sum_{l=1}^{k} \|u_l\|_{L^{2+q_l}(\Omega)}^{2+q_l}, \quad \Phi_u(t) = \sum_{l=1}^{k} \left\| \left| \nabla (|u_l|^{\frac{q_l+2}{2}}) \right| \right\|_{L^2(\Omega)}^2, \quad G_u(t) = \sum_{l=1}^{k} \int_\Omega g_l(t) u_l |u_l|^{q_l} \, dx.$$

We add the obtained inequalities and use (3.5) to get

$$\partial_t F_u(t) + \nu \Phi_u(t) \leq (q+2) \left(C |\Omega| + G_u(t) \right), \tag{3.20}$$

where $q = \max\{q_1, \ldots, q_k\}$. We use the Poincaré inequality $\lambda_1 \|\phi\|_{L^2(\Omega)}^2 \leq \| |\nabla \phi| \|_{L^2(\Omega)}^2$ with $\phi = |u_l|^{\frac{q_l+2}{2}}$ if $q_l > 0$ or $\phi = u_l$ if $q_l = 0$ and thus obtain

$$\partial_t F_u(t) + \lambda_1 \nu F_u(t) \leq (q+2) \left(C |\Omega| + G_u(t) \right).$$

We multiply by $e^{\lambda_1 \nu t}$ and integrate from σ to τ to get with $C_1 > 0$

$$F_u(\tau) \leq F_u(\sigma) e^{-\lambda_1 \nu (\tau - \sigma)} + C_1 + (q+2) \int_\sigma^\tau G_u(t) e^{-\lambda_1 \nu (\tau - t)} dt, \; \sigma \leq \tau < \tau_{max}. \tag{3.21}$$

Let $h > 0$ be such that $\sigma < \sigma + h < \tau_{max}$. Assume now that $\sigma + h \le \tau < \tau_{max}$. We integrate (3.20) from $\tau - h$ to $s \le \tau$ and in consequence we get

$$\sup_{s\in[\tau-h,\tau]} F_u(s) + \nu \int_{\tau-h}^{\tau} \Phi_u(t)dt \le F_u(\tau-h) + (q+2)C\,|\Omega|\,h + (q+2)\int_{\tau-h}^{\tau} |G_u(t)|\,dt.$$

Combining this estimate with (3.21) we obtain

$$\sup_{s\in[\tau-h,\tau]} F_u(s) + \nu \int_{\tau-h}^{\tau} \Phi_u(t)dt \le F_u(\sigma)e^{-\lambda_1\nu(\tau-\sigma-h)} + C_2 + C_3 \int_{\sigma}^{\tau} |G_u(t)|\, e^{-\lambda_1\nu(\tau-t)}dt, \quad (3.22)$$

where $C_2 = C_2(h)$ and $C_3 = C_3(h)$ are positive constants.

We estimate the last term using Lemma 3.3 with $\gamma = \lambda_1\nu h$ and get with $C_4 = C_4(h) > 0$

$$C_3 \int_{\sigma}^{\tau} |G_u(t)|\, e^{-\lambda_1\nu(\tau-t)}dt \le C_4 \sup_{t\in[\sigma+h,\tau]} \left(e^{\frac{\lambda_1\nu}{2}(\tau-t)} \int_{t-h}^{t} |G_u(s)|\, e^{-\lambda_1\nu(\tau-s)}ds \right). \qquad (3.23)$$

Moreover, it follows that

$$e^{\frac{\lambda_1\nu}{2}(\tau-t)} \int_{t-h}^{t} |G_u(s)|\, e^{-\lambda_1\nu(\tau-s)}ds \le e^{-\frac{\lambda_1\nu}{2}(\tau-t)} \sum_{l=1}^{k} \int_{t-h}^{t} \left| \int_{\Omega} g_l(s) u_l\, |u_l|^{q_l}\, dx \right| ds. \qquad (3.24)$$

Observe that by Schwarz and Young inequalities we have

$$\int_{t-h}^{t} \left| \int_{\Omega} g_l(s) u_l\, |u_l|^{q_l}\, dx \right| ds \le \sup_{s\in[\sigma,\tau]} \|g(s)\|_{[L^2(\Omega)]^k}\, \|u_l\|^{q_l+1}_{L^{q_l+1}([t-h,t],L^{2(q_l+1)}(\Omega))} \le$$

$$\le \mu\, \|u_l\|^{q_l+2}_{L^{q_l+1}([t-h,t],L^{2(q_l+1)}(\Omega))} + C_\mu \left(\sup_{s\in[\sigma,\tau]} \|g(s)\|_{[L^2(\Omega)]^k} \right)^{q_l+2},$$

where $\mu > 0$ and C_μ is independent of l. Note that

$$\|u_l\|^{q_l+2}_{L^{q_l+1}([t-h,t],L^{2(q_l+1)}(\Omega))} \le \widetilde{C}\, \|u_l\|^{q_l+2}_{L^{\frac{r(q_l+2)}{2}}([t-h,t],L^{r(q_l+2)}(\Omega))} = \widetilde{C} \left\| |u_l|^{\frac{q_l+2}{2}} \right\|^2_{L^r([t-h,t],L^{2r}(\Omega))},$$

since $q_l + 1 < \frac{7}{6}(q_l+2) = \frac{r}{2}(q_l+2)$ with $r = \frac{7}{3}$ and $\widetilde{C}$ does not depend on l and t.

Observe that by interpolation inequalities we have

$$\left\| |u_l|^{\frac{q_l+2}{2}} \right\|_{L^r([t-h,t],L^{2r}(\Omega))} \le \widehat{C} \left\| |u_l|^{\frac{q_l+2}{2}} \right\|^{1-\theta_0}_{L^\infty([t-h,t],L^2(\Omega))} \left\| |u_l|^{\frac{q_l+2}{2}} \right\|^{\theta_0}_{L^2([t-h,t],H^1_0(\Omega))},$$

where $\theta_0 = \frac{6}{7}$, since by [9, §4.3.1, Theorem 2]

$$[L^2(\Omega), H^1(\Omega)]_{\theta_0} = H^{\theta_0}(\Omega) \hookrightarrow L^{\frac{6}{3-2\theta_0}}(\Omega) = L^{2r}(\Omega)$$

and by [9, §1.18.4(10)]

$$[L^\infty([t-h,t], L^2(\Omega)), L^2([t-h,t], H^1(\Omega))]_{\theta_0} = L^r([t-h,t], [L^2(\Omega), H^1(\Omega)]_{\theta_0}).$$

Hence we get with $C_5 = C_5(h) > 0$

$$\|u_l\|^{q_l+2}_{L^{q_l+1}([t-h,t],L^{2(q_l+1)}(\Omega))} \le C_5 \left(\sup_{s\in[t-h,t]} F_u(s) + \nu \int_{t-h}^{t} \Phi_u(s) ds \right),$$

where we used the Young inequality again.

Summarizing, we get

$$\int_{t-h}^{t} \left| \int_\Omega g_l(s) u_l \, |u_l|^{q_l} \, dx \right| ds \le$$

$$\le \mu C_5 \Big(\sup_{s\in[t-h,t]} F_u(s) + \nu \int_{t-h}^{t} \Phi_u(s) ds \Big) + C_\mu \big(\sup_{s\in[\sigma,\tau]} \|g(s)\|_{[L^2(\Omega)]^k} \big)^{q_l+2}.$$

Applying this estimate to (3.24) we obtain with $C_6 = C_6(h) > 0$

$$e^{\frac{\lambda_1 \nu}{2}(\tau - t)} \int_{t-h}^{t} |G_u(s)| \, e^{-\lambda_1 \nu (\tau - s)} ds \le \mu C_6 e^{-\frac{\lambda_1 \nu}{2}(\tau - t)} Z_u(t) + C_\mu \sum_{l=1}^{k} \big(\sup_{s\in[\sigma,\tau]} \|g(s)\|_{[L^2(\Omega)]^k} \big)^{q_l+2},$$

where

$$Z_u(t) = \sup_{s\in[t-h,t]} F_u(s) + \nu \int_{t-h}^{t} \Phi_u(s) ds.$$

Therefore, it follows from (3.23) that

$$C_3 \int_\sigma^\tau |G_u(t)| \, e^{-\lambda_1 \nu (\tau - t)} dt \le \mu C_7 \sup_{t\in[\sigma+h,\tau]} \big(e^{-\frac{\lambda_1 \nu}{2}(\tau - t)} Z_u(t) \big) + \widetilde{C}_\mu \sum_{l=1}^{k} \big(\sup_{s\in[\sigma,\tau]} \|g(s)\|_{[L^2(\Omega)]^k} \big)^{q_l+2}$$

with $C_7 = C_7(h) > 0$. Applying this estimate to (3.22) we finally obtain for any $\mu > 0$

$$\begin{aligned} Z_u(\tau) &\le F_u(\sigma) e^{-\lambda_1 \nu (\tau - \sigma - h)} + \mu C_7 \sup_{t\in[\sigma+h,\tau]} \Big(e^{-\frac{\lambda_1 \nu}{2}(\tau - t)} Z_u(t) \Big) + \\ &+ \widehat{C}_\mu \Big(\sum_{l=1}^{k} \big(\sup_{s\in[\sigma,\tau]} \|g(s)\|_{[L^2(\Omega)]^k} \big)^{q_l+2} + 1 \Big), \ \sigma + h \le \tau < \tau_{max}. \end{aligned} \tag{3.25}$$

If $\tau_{max} < \infty$, then we choose $\mu = \frac{1}{2C_7}$ and use Lemma 3.4 to see that for $\sigma + h \leq \tau < \tau_{max}$

$$Z_u(\tau) \leq 2F_u(\sigma)e^{-\frac{\lambda_1 \nu}{2}(\tau-\sigma-h)} + C_8\Big(\sum_{l=1}^{k}(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k})^{q_l+2} + 1\Big), \tag{3.26}$$

where $E = [\sigma, \tau_{max})$. It follows immediately that (3.19) holds with $E = [\sigma, \tau_{max})$ and $\sigma + h \leq \tau < \tau_{max}$. Moreover, we know in particular that

$$\sup_{s\in[\sigma,\sigma+h]} F_u(s) \leq 2F_u(\sigma) + C_8\Big(\sum_{l=1}^{k}(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k})^{q_l+2} + 1\Big) \tag{3.27}$$

with $E = [\sigma, \tau_{max})$. This implies (3.18) with $E = [\sigma, \tau_{max})$ and for $\sigma \leq \tau < \tau_{max}$.

If $\tau_{max} = \infty$, then we set $h = 1$ and apply Lemma 3.4 to (3.25) and in case $\sigma + 1 < \tau_0$ we obtain (3.26) with $E = (-\infty, \tau_0 + 2)$ for $\sigma + 1 \leq \tau$, $\tau \in \mathcal{T}$ and (3.27) with $E = (-\infty, \tau_0 + 2)$. This implies that (3.19) holds with $h = 1$, $E = (-\infty, \tau_0 + 2)$ for $\sigma + 1 \leq \tau$, $\tau \in \mathcal{T}$ and (3.18) with $E = (-\infty, \tau_0 + 2)$ for $\sigma \leq \tau$, $\tau \in \mathcal{T}$. Moreover, in case $\sigma + 1 \geq \tau_0$ and $\sigma \leq \tau$, $\tau \in \mathcal{T}$, we know that (3.26) holds with $h = 1$, $E = (-\infty, \tau_0 + 2)$ for $\sigma + 1 \leq \tau < \tau_0 + 2$ and hence (3.18) holds with $E = (-\infty, \tau_0 + 2)$ for $\sigma \leq \tau$, $\tau \in \mathcal{T}$ also in this case.

Finally, suppose that $\tau_{max} = \infty$ and let $T > 0$. We choose $0 < h < T$ and apply Lemma 3.4 to (3.25) in order to obtain (3.26) and thus (3.19) with $E = [\sigma, \sigma + T]$ for $\sigma + h \leq \tau \leq \sigma + T$. Moreover, (3.27) holds with $E = [\sigma, \sigma + T]$ and hence (3.18) with $E = [\sigma, \sigma + T]$ for $\sigma \leq \tau \leq \sigma + T$. □

As follows from the above proposition we will obtain below a priori estimates in the following three cases:

$$\begin{aligned}
&1)\quad \tau_{max} < \infty,\ \sigma < \sigma + h < \tau_{max},\ E = J = [\sigma, \tau_{max}),\ J_h = [\sigma + h, \tau_{max}),\\
&2)\quad \tau_{max} = \infty,\ T > 0,\ 0 < h < T,\ E = J = [\sigma, \sigma + T],\ J_h = [\sigma + h, \sigma + T],\\
&3)\quad \tau_{max} = \infty,\ h = 1,\ E = (-\infty, \tau_0 + 2),\ J = \{\tau \in \mathbb{R}\colon \sigma \leq \tau, \tau \in \mathcal{T}\},\\
&\qquad J_h = \{\tau \in \mathbb{R}\colon \sigma + 1 \leq \tau, \tau \in \mathcal{T}\}.
\end{aligned} \tag{3.28}$$

Proposition 3.6. *Let $q_l \geq p_l$, $l = 1, \dots, k$ and $u = (u_1, \dots, u_k)$ be an $X^{\frac{1}{2}}$ solution of* (3.1) *on $[\sigma, \tau_{max})$. We have for $\tau \in J$*

$$\|F(\tau, u(\tau))\|^2_{[L^2(\Omega)]^k} \leq c_4 \sum_{l=1}^{k} \|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}\, e^{-\frac{\lambda_1 \nu}{2}(\tau-\sigma)} + P\Big(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\Big), \tag{3.29}$$

where $c_4 = c_4(h) > 0$ is a constant and $P = P(h)$ is a nondecreasing positive function in any of the three cases stated in (3.28).

Proof. We have

$$\|F(\tau,u)\|^2_{[L^2(\Omega)]^k} \le 2\,\|f(u)\|^2_{[L^2(\Omega)]^k} + 2\,\|g(\tau)\|^2_{[L^2(\Omega)]^k}\,. \tag{3.30}$$

We estimate using (3.12)

$$\|f(u)\|^2_{[L^2(\Omega)]^k} \le 2\widetilde{c}\sum_{l=1}^{k} \|u_l\|^2_{L^{2+p_l}(\Omega)}\left(1+\|u_l\|^{p_l}_{L^{2+p_l}(\Omega)}\right) + 2\,\|f(0)\|^2_{[L^2(\Omega)]^k} \le$$

$$\le c_1\left(1+\sum_{l=1}^{k}\|u_l\|^{2+p_l}_{L^{2+p_l}(\Omega)}\right) \le c_2\left(1+\sum_{l=1}^{k}\|u_l\|^{2+q_l}_{L^{2+q_l}(\Omega)}\right).$$

Combining it with (3.30) and applying (3.18) on an appropriate interval J with the corresponding set E, we obtain (3.29). □

Proposition 3.7. *Let $q_l \ge p_l$, $l = 1,\ldots,k$ and $u = (u_1,\ldots,u_k)$ be an $X^{\frac{1}{2}}$ solution of* (3.1) *on $[\sigma,\tau_{max})$. Then we have for $\tau \in J$*

$$\|u(\tau)\|^2_{[L^2(\Omega)]^k} \le c_9 \sum_{l=1}^{k} \|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}\, e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)} + Q(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}), \tag{3.31}$$

and for $\tau \in J_h$

$$\int_{\tau-h}^{\tau}\sum_{l=1}^{k} \||\nabla u_l(s)|\|^2_{L^2(\Omega)}\,ds \le c_9 \sum_{l=1}^{k} \|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}\, e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)} + Q(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}), \tag{3.32}$$

where $c_9 = c_9(h) > 0$ is a constant, $Q = Q(h)$ is a nondecreasing positive function and J, J_h and E come from each of the three cases in (3.28).

Proof. For each $l = 1,\ldots,k$ we multiply the l-th equation in (3.1) by u_l, integrate over Ω and add the equations. Integrating by parts and using the Schwarz inequality we obtain

$$\frac{1}{2}\partial_t \|u(t)\|^2_{[L^2(\Omega)]^k} - \sum_{l=1}^{k}\sum_{i,j=1}^{3}\int_\Omega a^l_{ij}\partial_{x_i}u_l\partial_{x_j}u_l dx \le \sum_{l=1}^{k}\|F_l(t,u)\|_{L^2(\Omega)}\,\|u_l\|_{L^2(\Omega)}\,.$$

By (3.3) and the Cauchy inequality we get for any $\varepsilon > 0$

$$\frac{1}{2}\partial_t \|u(t)\|^2_{[L^2(\Omega)]^k} + \nu\sum_{l=1}^{k}\||\nabla u_l(t)|\|^2_{L^2(\Omega)} \le \frac{\varepsilon}{2}\,\|u(t)\|^2_{[L^2(\Omega)]^k} + \frac{1}{2\varepsilon}\,\|F(t,u)\|^2_{[L^2(\Omega)]^k}\,. \tag{3.33}$$

By the Poincaré inequality we obtain

$$\partial_t \|u(t)\|^2_{[L^2(\Omega)]^k} + 2\lambda_1\nu \|u(t)\|^2_{[L^2(\Omega)]^k} \le \varepsilon \|u(t)\|^2_{[L^2(\Omega)]^k} + \frac{1}{\varepsilon} \|F(t,u)\|^2_{[L^2(\Omega)]^k}$$

Taking $\varepsilon = \frac{7}{4}\lambda_1\nu$, multiplying by $e^{\frac{\lambda_1\nu}{4}t}$ and integrating over $[\sigma,\tau]$ gives

$$\|u(\tau)\|^2_{[L^2(\Omega)]^k} \le \|u(\sigma)\|^2_{[L^2(\Omega)]^k} e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)} + \frac{4}{7\lambda_1\nu}\int_\sigma^\tau \|F(t,u)\|^2_{[L^2(\Omega)]^k} e^{-\frac{\lambda_1\nu}{4}(\tau-t)}dt.$$

Let $\tau \in J$ and E be the corresponding set from (3.28). We apply (3.29) and obtain

$$\|u(\tau)\|^2_{[L^2(\Omega)]^k} \le \big(\|u(\sigma)\|^2_{[L^2(\Omega)]^k} + c_5\sum_{l=1}^k \|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}\big)e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)} + c_6 P(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k})$$

with $c_5 = c_5(h) > 0$, $c_6 = c_6(h) > 0$. This yields (3.31), since $\|u_l\|^2_{L^2(\Omega)} \le \|u_l\|^{2+q_l}_{L^{2+q_l}(\Omega)} + |\Omega|$.

Assume now that $\tau \in J_h$. Integrating (3.33) with $\varepsilon = 1$ over $[\tau-h,\tau]$ we get

$$\|u(\tau)\|^2_{[L^2(\Omega)]^k} + 2\nu\int_{\tau-h}^\tau \sum_{l=1}^k \||\nabla u_l(t)|\|^2_{L^2(\Omega)}\,dt \le$$

$$\le \|u(\tau-h)\|^2_{[L^2(\Omega)]^k} + \int_{\tau-h}^\tau \left(\|u(t)\|^2_{[L^2(\Omega)]^k} + \|F(t,u)\|^2_{[L^2(\Omega)]^k}\right)dt.$$

Using (3.31) and (3.29) we obtain for $\tau \in J_h$

$$\int_{\tau-h}^\tau \sum_{l=1}^k \||\nabla u_l(t)|\|^2_{L^2(\Omega)}\,dt \le c_7\sum_{l=1}^k \|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}\left(e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)} + \int_{\tau-h}^\tau e^{-\frac{\lambda_1\nu}{4}(t-\sigma)}dt\right) +$$

$$+\widehat{Q}\left(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\right) \le c_8\sum_{l=1}^k \|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)} e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)} + \widehat{Q}\left(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\right),$$

where $c_7 = c_7(h) > 0$, $c_8 = c_8(h) > 0$ and $\widehat{Q} = \widehat{Q}(h)$ is a nondecreasing positive function. This gives (3.32). □

Proposition 3.8. *Let $p_l \le q_l \le 4$, $l = 1,\dots,k$ and $u = (u_1,\dots,u_k)$ be an $X^{\frac{1}{2}}$ solution of* (3.1) *on $[\sigma,\tau_{max})$. Then we have for $\tau \in J$*

$$\sum_{l=1}^k \||\nabla u_l(\tau)|\|^2_{L^2(\Omega)} \le R_1\big(\sum_{l=1}^k \||\nabla u_l(\sigma)|\|^2_{L^2(\Omega)}\big)e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)} + R_2\big(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\big), \quad (3.34)$$

where $R_1 = R_1(h), R_2 = R_2(h)$ are both nondecreasing positive functions and h, J and E come from each of the three cases in (3.28).

Proof. For each $l = 1, \dots, k$ we multiply the l-th equation in (3.1) by $A_l u_l$, integrate over Ω and add the obtained equations. Since a_{ij}^l do not depend on time and $a_{ij}^l = a_{ji}^l$, integration by parts and the Schwarz inequality imply

$$-\frac{1}{2}\sum_{l=1}^{k}\sum_{i,j=1}^{3}\int_\Omega \partial_t(a_{ij}^l \partial_{x_i} u_l \partial_{x_j} u_l)dx + \sum_{l=1}^{k}\|A_l u_l\|_{L^2(\Omega)}^2 \le \frac{1}{2}\|F(t,u)\|_{[L^2(\Omega)]^k}^2 + \frac{1}{2}\sum_{l=1}^{k}\|A_l u_l\|_{L^2(\Omega)}^2 .$$

We add to both sides a term with

$$z_u(t) = -\sum_{l=1}^{k}\sum_{i,j=1}^{3}\int_\Omega a_{ij}^l \partial_{x_i} u_l \partial_{x_j} u_l dx$$

and obtain

$$\partial_t z_u(t) + \lambda_1 \nu z_u(t) + \sum_{l=1}^{k}\|A_l u_l\|_{L^2(\Omega)}^2 \le \lambda_1 \nu z_u(t) + \|F(t,u)\|_{[L^2(\Omega)]^k}^2 . \tag{3.35}$$

Since the functions a_{ij}^l are continuous on $\overline{\Omega}$, we know that

$$\left|a_{ij}^l(x)\right| \le \max_{\substack{i,j=1,\dots,3 \\ l=1,\dots,k}} \sup_{x\in\overline{\Omega}} \left|a_{ij}^l(x)\right| = C_a.$$

Therefore, it follows from (3.3) that

$$\nu \sum_{l=1}^{k} \||\nabla u_l|\|_{L^2(\Omega)}^2 \le z_u(t) \le 3C_a \sum_{l=1}^{k} \||\nabla u_l|\|_{L^2(\Omega)}^2 . \tag{3.36}$$

Applying (3.36) to (3.35), multiplying by $e^{\lambda_1 \nu t}$ and integrating over $[\sigma, \tau]$, we obtain

$$z_u(\tau) \le z_u(\sigma)e^{-\lambda_1\nu(\tau-\sigma)} + 3\lambda_1\nu C_a \int_\sigma^\tau \sum_{l=1}^{k} \||\nabla u_l(t)|\|_{L^2(\Omega)}^2 e^{-\lambda_1\nu(\tau-t)}dt +$$

$$+ \int_\sigma^\tau \|F(t,u)\|_{[L^2(\Omega)]^k}^2 e^{-\lambda_1\nu(\tau-t)}dt.$$

Let $\tau \in J$ and E be the corresponding set from (3.28). We apply (3.29), (3.36) and get

$$\sum_{l=1}^{k} \||\nabla u_l(\tau)|\|_{L^2(\Omega)}^2 \le \frac{3C_a}{\nu}\sum_{l=1}^{k} \||\nabla u_l(\sigma)|\|_{L^2(\Omega)}^2 e^{-\lambda_1\nu(\tau-\sigma)} + \frac{1}{\lambda_1 \nu^2} P\big(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\big) +$$

$$+c_4\frac{2}{\lambda_1\nu^2}\sum_{l=1}^{k}\|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}e^{-\frac{\lambda_1\nu}{2}(\tau-\sigma)}+3\lambda_1 C_a\int_\sigma^\tau\sum_{l=1}^{k}|||\nabla u_l(t)|||^2_{L^2(\Omega)}e^{-\lambda_1\nu(\tau-t)}dt.$$

We consider now two cases. In the first case we assume that τ belongs to J_h corresponding to the appropriate case in (3.28). We use Lemma 3.3 and (3.32) to estimate

$$\int_\sigma^\tau\sum_{l=1}^{k}|||\nabla u_l(t)|||^2_{L^2(\Omega)}e^{-\lambda_1\nu(\tau-t)}dt\le c_{10}\sup_{t\in[\sigma+h,\tau]}\left(e^{-\frac{\lambda_1\nu}{2}(\tau-t)}\int_{t-h}^{t}\sum_{l=1}^{k}|||\nabla u_l(s)|||^2_{L^2(\Omega)}ds\right)\le$$

$$\le c_{11}\sum_{l=1}^{k}\|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)}+c_{10}Q\big(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\big),\ \tau\in J_h,$$

where $c_{10}=c_{10}(h)>0$ and $c_{11}=c_{11}(h)>0$.

In the second case when $\sigma\le\tau\le\sigma+h$, $\tau\in J$, we have by (3.32)

$$\int_\sigma^\tau\sum_{l=1}^{k}|||\nabla u_l(t)|||^2_{L^2(\Omega)}e^{-\lambda_1\nu(\tau-t)}dt\le c_9\sum_{l=1}^{k}\|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)}+Q\big(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\big).$$

Combining the two cases we obtain

$$\sum_{l=1}^{k}|||\nabla u_l(\tau)|||^2_{L^2(\Omega)}\le c_{12}\sum_{l=1}^{k}\Big(|||\nabla u_l(\sigma)|||^2_{L^2(\Omega)}+\|u_l(\sigma)\|^{2+q_l}_{L^{2+q_l}(\Omega)}\Big)e^{-\frac{\lambda_1\nu}{4}(\tau-\sigma)}+$$

$$+R\big(\sup_{s\in E}\|g(s)\|_{[L^2(\Omega)]^k}\big),\ \tau\in J,$$

where $c_{12}=c_{12}(h)$ is a positive constant and $R=R(h)$ is a nondecreasing positive function. Since $q_l\le 4$, $l=1,\dots,k$ and $u_l\in H^1_0(\Omega)$, it follows from the Sobolev embedding and the Poincaré inequality that

$$\|u_l\|_{L^{2+q_l}(\Omega)}\le D\,|||\nabla u_l|||_{L^2(\Omega)}.$$

This ends the proof of (3.34). □

Propositions 3.7 and 3.8 imply global (forward) in time solvability of (3.1), thus (F3a) holds. Moreover, assumptions (F3b) and (F3c) are also satisfied.

Corollary 3.9. *If $p_l\le q_l\le 4$, $l=1,\dots,k$ and $u=(u_1,\dots,u_k)$ is an $X^{\frac12}$ solution of (3.1) on $[\sigma,\tau_{max})$, then $\tau_{max}=\infty$ and for $\sigma\le\tau$, $\tau\in\mathcal{T}$ we have*

$$\|u(\tau)\|_{[H^1_0(\Omega)]^k}\le Q_1\big(\|u(\sigma)\|_{[H^1_0(\Omega)]^k}\big)e^{-\frac{\lambda_1\nu}{8}(\tau-\sigma)}+Q_2\big(\sup_{s\in(-\infty,\tau_0+2)}\|g(s)\|_{[L^2(\Omega)]^k}\big),\qquad(3.37)$$

where Q_1, Q_2 are both nondecreasing positive functions and for any $T > 0$ there exist nondecreasing positive functions $\widetilde{Q}_1 = \widetilde{Q}_1(T)$, $\widetilde{Q}_2 = \widetilde{Q}_2(T)$ such that for $\sigma \le \tau \le \sigma + T$

$$\|u(\tau)\|_{[H_0^1(\Omega)]^k} \le \widetilde{Q}_1\big(\|u(\sigma)\|_{[H_0^1(\Omega)]^k} \big) e^{-\frac{\lambda_1 \nu}{8}(\tau-\sigma)} + \widetilde{Q}_2\big(\sup_{s\in[\sigma,\sigma+T]} \|g(s)\|_{[L^2(\Omega)]^k} \big). \tag{3.38}$$

Proof. The fact that $X^{\frac{1}{2}}$ solutions of (3.1) exist globally (forward) in time follows from (3.13) and Propositions 3.7 and 3.8 in the context of the first case in (3.28), while (3.37) and (3.38) correspond to the second and the third case in (3.28), respectively. □

Therefore, we can apply Theorem 1.1 and obtain

Theorem 3.10. *Under assumptions* (3.4), (3.5) *with* $0 \le p_l \le q_l \le 4$, $l = 1, \ldots, k$ *and assumptions* (3.6), (3.7) *the problem* (3.1) *generates an evolution process* $\{U(\tau, \sigma) \colon \tau \ge \sigma\}$ *in* $[H_0^1(\Omega)]^k$ *and for any* $\beta \in (\frac{1}{2}, 1)$ *there exists a family* $\{\mathcal{M}(\tau) \colon \tau \in \mathbb{R}\}$ *of nonempty compact subsets of* $[H_0^{2\beta}(\Omega)]^k$ *with the following properties*

(i) $\{\mathcal{M}(\tau) \colon \tau \in \mathbb{R}\}$ *is positively invariant under the process* $U(\tau, \sigma)$, *i.e.*

$$U(\tau, \sigma)\mathcal{M}(\sigma) \subset \mathcal{M}(\tau), \ \tau \ge \sigma,$$

(ii) $\mathcal{M}(\tau)$ *has a finite fractal dimension in* $[H_0^{2\beta}(\Omega)]^k$ *uniformly w.r.t.* $\tau \in \mathbb{R}$, *i.e.*

$$d_f^{[H_0^1(\Omega)]^k}(\mathcal{M}(\tau)) \le d_f^{[H_0^{2\beta}(\Omega)]^k}(\mathcal{M}(\tau)) \le d < \infty, \ \tau \in \mathbb{R},$$

(iii) $\{\mathcal{M}(\tau) \colon \tau \in \mathbb{R}\}$ *has the property of pullback exponential attraction, i.e.*

$$\exists_{\varphi>0} \forall_{B_1 \subset [H_0^{2\beta}(\Omega)]^k, \ bounded} \forall_{\tau\in\mathbb{R}} \ \lim_{t\to\infty} e^{\varphi t} \operatorname{dist}_{[H_0^{2\beta}(\Omega)]^k}(U(\tau, \tau - t)B_1, \mathcal{M}(\tau)) = 0$$

and if $\tau_0 = \infty$, *the pullback attraction is uniform with respect to* τ

$$\exists_{\varphi>0} \forall_{B_1 \subset [H_0^{2\beta}(\Omega)]^k, \ bounded} \ \lim_{t\to\infty} e^{\varphi t} \sup_{\tau\in\mathbb{R}} \operatorname{dist}_{[H_0^{2\beta}(\Omega)]^k}(U(\tau, \tau - t)B_1, \mathcal{M}(\tau)) = 0.$$

Furthermore, the pullback exponential attractor $\{\mathcal{M}(\tau) \colon \tau \in \mathbb{R}\}$ *contains a (finite dimensional) pullback global attractor* $\{\mathcal{A}(\tau) \colon \tau \in \mathbb{R}\}$, *i.e. a family of nonempty compact subsets of* $[H_0^{2\beta}(\Omega)]^k$, *invariant under the process* $\{U(\tau, \sigma) \colon \tau \ge \sigma\}$

$$U(\tau, \sigma)\mathcal{A}(\sigma) = \mathcal{A}(\tau), \ \tau \ge \sigma,$$

and pullback attracting all bounded subsets of $[H_0^{2\beta}(\Omega)]^k$

$$\forall_{B_1 \subset [H_0^{2\beta}(\Omega)]^k, \ bounded} \forall_{\tau\in\mathbb{R}} \ \lim_{t\to\infty} \operatorname{dist}_{[H_0^{2\beta}(\Omega)]^k}(U(\tau, \tau - t)B_1, \mathcal{A}(\tau)) = 0.$$

Remark 3.11. Observe that the assumptions of the above theorem are satisfied in case of the FitzHugh-Nagumo system and the chemical reaction system considered in Remark 3.1.

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