Pullback exponential attractors for nonautonomous equations Part I: Semilinear parabolic problems[☆]

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Abstract

A family of compact and positively invariant sets with uniformly bounded fractal dimension which at a uniform exponential rate pullback attract bounded subsets of the phase space under the process is constructed. The existence of such a family, called a pullback exponential attractor, is proved for a nonautonomous semilinear abstract parabolic Cauchy problem. Specific examples will be presented in the forthcoming Part II of this work.

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1. Introduction

The study of the long-time behavior of infinite dimensional dynamical systems or semigroups generated by autonomous partial differential equations can be usually reduced to the description of the compact invariant set attracting all bounded subsets of the phase space called the *global attractor* (see [5], [11], [16], [17]). This uniquely determined object has frequently a finite (fractal) dimension, but the attraction to it may be arbitrarily slow. The need to overcome this drawback created the notion of the *exponential attractor* – a compact, *positively* invariant set of finite fractal dimension and exponentially attracting each bounded

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subset of the phase space at a uniform exponential rate. Though no longer uniquely determined, the exponential attractor still contains the global attractor. Its first construction in [8] worked only in Hilbert spaces, but was later completely modified in [9] to work also in Banach spaces.

In recent years more attention was paid to more general *nonautonomous* differential equations and the processes generated by them. Different approaches were made to find the counterpart of the global attractor in this case (see for example [2], [3]). One of the most suitable ones defines the notion of the *pullback global attractor* as a minimal family of compact invariant sets under the process and *pullback* attracting each bounded subset of the phase space.

In this paper we construct a *pullback exponential attractor* for an evolution process generated by a nonautonomous equation. This is a family of nonempty compact and positively invariant sets under the process that have finite fractal dimension uniformly bounded for all times and that pullback attract each bounded subset of the phase space at a uniform exponential rate.

Our main abstract result is given in Theorem 2.1. A result of this type, but for discrete pullback exponential attractors, was already presented in [10]. That paper also contains a construction of the continuous pullback exponential attractor in a specific case of nonautonomous reaction-diffusion equation with uniformly bounded nonautonomous term. Our abstract result in Theorem 2.1 also applies to this setting (see [7]). Our purpose in formulation of Theorem 2.1 was to follow the spirit of the results from [4, Section 2] for semigroups generated by autonomous equations. We formulate in Corollary 2.6 the counterpart of Theorem 2.1 in case the process is a semigroup.

During the final stage of preparation of this paper the authors have learned that similar questions concerning (continuous) pullback exponential attractors were also considered in [13]. Nevertheless, we present here a concurrent construction and establish a uniform setting for nonautonomous abstract semilinear parabolic equations in Section 3 (see Theorem 3.6). This result can be directly applied to various equations of mathematical physics. Some specific examples concerning reaction-diffusion systems will be presented in the forthcoming Part II of this work (see [7]).

2. Pullback exponential and global attractors

This section is devoted to the construction of (continuous) uniform pullback exponential attractors for evolution processes. We consider an evolution process $U(\tau,\sigma)\colon V \to V$, $\tau \geq \sigma$, $\tau,\sigma \in \mathbb{R}$, in a normed space $(V, \|\cdot\|_V)$, i.e.

$$U(\tau,\sigma)U(\sigma,\rho) = U(\tau,\rho),\ \tau \geq \sigma \geq \rho,\ \tau,\sigma,\rho \in \mathbb{R}, \quad U(\tau,\tau) = I,\ \tau \in \mathbb{R}, \tag{A1}$$

where I denotes an identity operator on V. Our aim is to construct a family $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ of precompact subsets of V, positively invariant under the process, with a uniform bound on their fractal dimension and which have the property of uniform pullback exponential attraction

$$\exists_{\varphi>0}\forall_{B_1\subset V,\text{ bounded}} \quad \lim_{t\to\infty} e^{\varphi t}\sup_{\tau\in\mathbb{R}} \operatorname{dist}_V(U(\tau,\tau-t)B_1,\mathcal{M}(\tau))=0, \tag{2.1}$$

where dist_V denotes the Hausdorff semidistance in V. Note that this property is equivalent to the uniform forwards exponential attraction

$$\exists_{\varphi>0}\forall_{B_1\subset V,\text{ bounded}} \quad \lim_{t\to\infty} e^{\varphi t}\sup_{\tau\in\mathbb{R}} \operatorname{dist}_V(U(t+\tau,\tau)B_1,\mathcal{M}(t+\tau))=0. \tag{2.2}$$

Uniform pullback exponential attractors are a direct generalization of exponential attractors for the semigroups (see Corollary 2.6). Moreover, using this notion a general abstract approach to nonautonomous semilinear parabolic equations can be given as in Section 3. Therefore the existence of uniform pullback exponential attractors can be proved for various models of mathematical physics. For the applications to time-perturbed reaction-diffusion systems we refer the reader to the Part II of this work ([7]). However, in some cases of nonautonomous equations expecting *uniform* pullback attraction may be too demanding (for examples we refer the reader to [14], [15]) and only pullback attraction is expected

$$\exists_{\varphi>0}\forall_{B_1\subset V,\text{ bounded}}\forall_{\tau\in\mathbb{R}} \quad \lim_{t\to\infty} e^{\varphi t} \operatorname{dist}_V(U(\tau,\tau-t)B_1,\mathcal{M}(\tau))=0. \tag{2.3}$$

Therefore we carry out the construction in such a way to capture not only the case of uniform pullback exponential attraction which, as observed above, implies forwards exponential attraction too, but also to emphasize the nonuniform pullback attraction by the sets without immediate forwards attraction (see Corollary 2.4). We also refer the reader to [13] for a concurrent construction.

To this end, in the theorem below, we fix a time $-\infty < \tau_0 \leq \infty$ and consider a time interval

$$\mathcal{T} = \{\tau \in \mathbb{R}\colon \tau \leq \tau_0\}.$$

Note that all the constants in the theorem may depend on τ_0. It will follow that if $\tau_0 = \infty$ and hence $\mathcal{T} = \mathbb{R}$, then the constructed family of sets will be the desired uniform pullback exponential attractor satisfying (2.1) and consequently (2.2).

Our construction will be carried out under the assumption of existence of a bounded set $B \subset V$ absorbing bounded subsets of V uniformly in time, i.e.

$$\exists_{B\subset V,\text{bounded}}\forall_{B_1\subset V,\text{ bounded}}\exists_{T_{B_1}>0}\forall_{t\geq T_{B_1}} \bigcup_{\tau\in\mathcal{T}} U(\tau,\tau-t)B_1 \subset B. \tag{A2}$$

Note again the duality with forwards absorbing property in case $\tau_0 = \infty$, since

$$\bigcup_{\tau\in\mathbb{R}} U(\tau, \tau - t)B_1 = \bigcup_{\tau\in\mathbb{R}} U(t + \tau, \tau)B_1.$$

Theorem 2.1. *Let $(W, \|\cdot\|_W)$ be an auxiliary normed space such that V is compactly embedded in W and assume that* (A1) *and* (A2) *hold. Let the process $\{U(\tau, \sigma)\colon \tau \geq \sigma\}$ satisfy the smoothing property with constant $\kappa > 0$*

$$\sup_{\tau\in\mathcal{T}} \|U(\tau, \tau - T_B)u_1 - U(\tau, \tau - T_B)u_2\|_V \leq \kappa \|u_1 - u_2\|_W\,,\ u_1, u_2 \in B, \tag{A3}$$

and the following continuity properties with exponents $0 < \xi_1, \xi_2 \leq 1$ and constants $\lambda_1, \lambda_2 > 0$

$$\sup_{\tau\in\mathcal{T}} \|U(\tau, \tau - T_B)u - U(\tau - t_1, \tau - t_1 - T_B)u\|_W \leq \lambda_1 {t_1}^{\xi_1},\ t_1 \in [0, T_B],\ u \in B, \tag{A4}$$

$$\sup_{\tau\in\mathcal{T}} \|U(\tau, \tau - t_1)u - U(\tau, \tau - t_2)u\|_W \leq \lambda_2 |t_1 - t_2|^{\xi_2}\,,\ t_1, t_2 \in [T_B, 2T_B],\ u \in B, \tag{A5}$$

where $T_B > 0$ is the time corresponding to the absorbing set B from (A2).

Then for any $\nu \in (0, \frac{1}{2})$ there exists a family $\{\mathcal{M}(\tau) = \mathcal{M}_\nu(\tau)\colon \tau \in \mathcal{T}\}$ of nonempty precompact in V subsets of B with the following properties:

(i) $\{\mathcal{M}(\tau)\colon \tau \in \mathcal{T}\}$ is positively invariant under the process $U(\tau, \sigma)$, i.e.

$$U(\tau, \sigma)\mathcal{M}(\sigma) \subset \mathcal{M}(\tau),\ \tau \geq \sigma,\ \tau \in \mathcal{T}, \tag{W1}$$

(ii) $\mathcal{M}_\nu(\tau)$ has a finite fractal dimension in V uniformly with respect to $\tau \in \mathcal{T}$, i.e.

$$\sup_{\tau\in\mathcal{T}} d_f^V(\mathcal{M}_\nu(\tau)) \leq \max\left\{\frac{1}{\xi_1}, \frac{1}{\xi_2}\right\} \Big(1 + \log_{\frac{1}{2\nu}}(1 + \mu\kappa)\Big) + \log_{\frac{1}{2\nu}} N^W_{\frac{\nu}{\kappa}}(B^V(0, 1)), \tag{W2}$$

where $\mu > 0$ is such that

$$\|u\|_W \leq \mu \|u\|_V\,,\ u \in V, \tag{2.4}$$

and $N^W_{\frac{\nu}{\kappa}}(B^V(0, 1))$ denotes the smallest number of balls in W of radius $\frac{\nu}{\kappa}$ necessary to cover the unit ball in V,

(iii) $\{\mathcal{M}(\tau)\colon \tau \in \mathcal{T}\}$ is pullback exponentially attracting bounded subsets of V, i.e.

$$\exists_{\varphi>0}\forall_{B_1\subset V,\ bounded} \quad \lim_{t\to\infty} e^{\varphi t} \sup_{\tau\in\mathcal{T}} \mathrm{dist}_V(U(\tau, \tau - t)B_1, \mathcal{M}(\tau)) = 0. \tag{W3}$$

If V is a Banach space and

$$U(\tau, \sigma)\colon\ \mathrm{cl}_V B \to V \text{ is continuous for } \tau \geq \sigma,\ \tau \in \mathcal{T}, \tag{A6}$$

then we can make $\mathcal{M}(\tau)$, $\tau \in \mathcal{T}$, *compact subsets of* $\mathrm{cl}_V B$.

Moreover, if the process is uniformly dissipative in V, *i.e. there exists a nondecreasing function* $Q\colon [0,\infty) \to [0,\infty)$ *and constants* $R_0, \omega > 0$ *such that*

$$\sup_{\tau\in\mathcal{T}} \|U(\tau,\tau-t)u\|_V \leq Q(\|u\|_V)e^{-\omega t} + R_0,\ t \geq 0,\ u \in V, \tag{A7}$$

(which in particular implies (A2)*), then* (W3) *can be improved to*

$$\exists_{\chi>0}\forall_{B_1\subset V,\ bounded}\ \exists_{c_{B_1}>0}\forall_{t\geq 0}\ \sup_{\tau\in\mathcal{T}} \mathrm{dist}_V(U(\tau,\tau-t)B_1, \mathcal{M}(\tau)) \leq c_{B_1}e^{-\chi t}. \tag{W3'}$$

Proof. **Step 1. Rescaling time and implications of** (A4) **and** (A5)**.** By scaling time we can assume that $T_B = 1$. Then the constant τ_0 turns into $\frac{\tau_0}{T_B}$, λ_1 into $\lambda_1 T_B^{\xi_1}$, λ_2 into $\lambda_2 T_B^{\xi_2}$ and the convergence rate ω in (A7) into ωT_B, but we keep the notation below.

Note that by (A2) in particular we have $U(\tau,\tau-n)B \subset B$ for $\tau \in \mathcal{T}$ and $n \in \mathbb{N}$. From (A3), (A4) and (2.4) it follows by induction that for $u \in B$ and $t_1 \in [0,1]$

$$\sup_{\tau\in\mathcal{T}} \|U(\tau,\tau-n)u - U(\tau-t_1,\tau-t_1-n)u\|_W \leq \sum_{j=0}^{n-1}(\mu\kappa)^j\lambda_1 t_1^{\xi_1},\ n \in \mathbb{N}. \tag{A8}$$

Observe also that (A3), (A5) and (2.4) imply that for any $n \in \mathbb{N}_0$, $t_1, t_2 \in [n+1, n+2]$ and $u \in B$

$$\sup_{\tau\in\mathcal{T}} \|U(\tau,\tau-t_1)u - U(\tau,\tau-t_2)u\|_W \leq (\mu\kappa)^n\lambda_2 |t_1-t_2|^{\xi_2}. \tag{A9}$$

Indeed, proceeding by induction we see that for $t_1, t_2 \in [n+2, n+3]$, $u \in B$ and $\tau \in \mathcal{T}$

$$\|U(\tau,\tau-t_1)u - U(\tau,\tau-t_2)u\|_W =$$

$$= \|U(\tau,\tau-1)U(\tau-1,\tau-1-(t_1-1))u - U(\tau,\tau-1)U(\tau-1,\tau-1-(t_2-1))u\|_W \leq$$

$$\leq \mu\kappa\, \|U(\tau-1,\tau-1-(t_1-1))u - U(\tau-1,\tau-1-(t_2-1))u\|_W \leq (\mu\kappa)^{n+1}\lambda_2 |t_1-t_2|^{\xi_2},$$

since $t_1 - 1, t_2 - 1 \in [n+1, n+2]$ and (A2) holds.

In the further part of the proof we simplify (A8) and (A9) by using the estimates

$$\sum_{j=0}^{n-1}(\mu\kappa)^j \leq (1+\mu\kappa)^n,\ n \in \mathbb{N}, \quad \text{and} \quad (\mu\kappa)^n \leq (1+\mu\kappa)^n,\ n \in \mathbb{N}_0.$$

Step 2. Construction of the sets $W_\tau^n \subset B$, $\tau \in \mathcal{T}$. Let $R > 0$ and $v_0 \in B$ be such that $B \subset B^W(v_0, R)$ and fix $\nu \in (0, \frac{1}{2})$. By compactness of the embedding $V \hookrightarrow W$ we define $N = N^W_{\frac{\nu}{\kappa}}(B^V(0,1))$ as the smallest number of balls in W with radius $\frac{\nu}{\kappa}$ needed to cover $B^V(0,1)$. We set $W^0_\tau = \{v_0\}$ for $\tau \in \mathcal{T}$ and construct the sets W^n_τ for $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$ by the following inductive procedure. Let the nonempty sets W^n_τ be already constructed in such a way that

(1) $W^n_\tau \subset U(\tau, \tau - n)B \subset B \quad$ and $\quad \# W^n_\tau \le N^n$,

(2) $U(\tau, \tau - n)B \subset \bigcup_{u \in W^n_\tau} B^W(u, (2\nu)^n R)$.

Fix $\tau \in \mathcal{T}$ and $u_0 \in W^n_{\tau-1} \subset B$. By (A3) we have

$$U(\tau, \tau-1)[U(\tau-1, \tau-1-n)B \cap B^W(u_0, (2\nu)^n R)] \subset B^V(U(\tau, \tau-1)u_0, \kappa(2\nu)^n R).$$

Thus if $v \in U(\tau, \tau-1)[U(\tau-1, \tau-1-n)B \cap B^W(u_0, (2\nu)^n R)]$ then

$$\frac{1}{\kappa(2\nu)^n R}(v - U(\tau, \tau-1)u_0) \in B^V(0,1) \subset \bigcup_{i=1}^N B^W\left(u_i, \frac{\nu}{\kappa}\right)$$

with some $u_i \in W$, $i = 1, \dots, N$, and consequently we have for some $\widetilde{u}_i \in W$, $i = 1, \dots, N$

$$U(\tau, \tau-1)[U(\tau-1, \tau-1-n)B \cap B^W(u_0, (2\nu)^n R)] \subset \bigcup_{i=1}^N B^W(\widetilde{u}_i, 2^n \nu^{n+1} R).$$

Increasing twice the radii, we cover $U(\tau, \tau-1)[U(\tau-1, \tau-1-n)B \cap B^W(u_0, (2\nu)^n R)]$ by at most N balls in W with centers from $U(\tau, \tau-1)[U(\tau-1, \tau-1-n)B \cap B^W(u_0, (2\nu)^n R)]$ and radius $(2\nu)^{n+1} R$.

We denote the set of all the centers of balls for all $u_0 \in W^n_{\tau-1}$ by W^{n+1}_τ. Thus we have

$$U(\tau, \tau-n-1)B = U(\tau, \tau-1)U(\tau-1, \tau-1-n)B =$$

$$= \bigcup_{u_0 \in W^n_{\tau-1}} U(\tau, \tau-1)[U(\tau-1, \tau-1-n)B \cap B^W(u_0, (2\nu)^n R)] \subset \bigcup_{u \in W^{n+1}_\tau} B^W(u, (2\nu)^{n+1} R).$$

Moreover, we know that $\# W^{n+1}_\tau \le \# W^n_{\tau-1} \cdot N \le N^{n+1}$ and

$$W^{n+1}_\tau \subset \bigcup_{u_0 \in W^n_{\tau-1}} U(\tau, \tau-1)[U(\tau-1, \tau-1-n)B \cap B^W(u_0, (2\nu)^n R)] = U(\tau, \tau-1-n)B.$$

This proves (1)–(2) for any $n \in \mathbb{N}_0$.

Step 3. Construction of the sets $\widetilde{W}_\tau^n \subset B$, $\tau \le \tau_0 - n$, and $\overline{W}_\tau^n \subset B$, $\tau \in \mathcal{T}$. We know that $W_\tau^n \subset U(\tau, \tau-n)B$, $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$. Therefore, there exists a set $\widetilde{W}_{\tau-n}^n \subset B$ such that $\#\widetilde{W}_{\tau-n}^n \le N^n$ and

$$W_\tau^n = U(\tau, \tau-n)\widetilde{W}_{\tau-n}^n, \; n \in \mathbb{N}_0, \; \tau \in \mathcal{T}.$$

Set $L = 1 + \mu\kappa > 1$. For each $n \in \mathbb{N}_0$ we define a number $0 < \delta(n) \le 1$ be the relation

$$L^n \delta(n)^{\xi_1} = (2\nu)^n. \tag{2.5}$$

We define the sets $\overline{W}_\tau^n$, $\tau \in \mathcal{T}$ by the formula

$$\overline{W}_\tau^n = U(\tau, \tau-n)\widetilde{W}_{k\delta(n)-n}^n, \; \tau \in \mathcal{T} \cap [k\delta(n), (k+1)\delta(n)), \; k \in \mathbb{Z}. \tag{2.6}$$

By (A8) we have for $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$, $0 \le s < \delta(n)$ and $u \in B$

$$\|U(\tau, \tau-n)u - U(\tau-s, \tau-s-n)u\|_W \le \lambda_1 L^n s^{\xi_1} < (2\nu)^n \lambda_1. \tag{2.7}$$

This implies that

$$U(\tau, \tau-n)B \subset \bigcup_{u \in \overline{W}_\tau^n} B^W(u, (2\nu)^n(2\lambda_1 + R)), \; n \in \mathbb{N}_0, \; \tau \in \mathcal{T}. \tag{2.8}$$

Indeed, fix $n \in \mathbb{N}_0$ and $\tau \in \mathcal{T}$. Let $k \in \mathbb{Z}$ be such that $\tau \in [k\delta(n), (k+1)\delta(n))$. Note that $k\delta(n) \in \mathcal{T}$ and take $x \in U(\tau, \tau-n)B$. By (2.7) there exists $z \in U(k\delta(n), k\delta(n)-n)B$ such that $\|x - z\|_W < (2\nu)^n \lambda_1$ and from (2) it follows the existence of

$$w \in W_{k\delta(n)}^n = U(k\delta(n), k\delta(n)-n)\widetilde{W}_{k\delta(n)-n}^n = \overline{W}_{k\delta(n)}^n$$

satisfying $\|z - w\|_W < (2\nu)^n R$. Again by (2.7) we deduce that there exists $v \in \overline{W}_\tau^n$ with $\|w - v\|_W < (2\nu)^n \lambda_1$. Thus the triangle inequality yields (2.8).

Therefore, we have

(I) $\overline{W}_\tau^n \subset U(\tau, \tau-n)B \subset B$ and $\#\overline{W}_\tau^n \le N^n$, $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$,

(II) $U(\tau, \tau-n)B \subset \bigcup_{u \in \overline{W}_\tau^n} B^W(u, (2\nu)^n(2\lambda_1 + R))$, $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$.

Step 4. Construction of the sets $E_\tau^n \subset B$, $\tau \in \mathcal{T}$. We now define the sets

$$E_\tau^0 = \overline{W}_\tau^0, \; \tau \in \mathcal{T}, \quad E_\tau^n = \overline{W}_\tau^n \cup U(\tau, \tau-1)E_{\tau-1}^{n-1}, \; n \in \mathbb{N}, \; \tau \in \mathcal{T}.$$

We show that the following properties hold:

(a) $E^n_\tau \subset U(\tau,\tau-n)B \subset B$, $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$,

(b) $U(\tau,\tau-n)B \subset \bigcup_{u\in E^n_\tau} B^W(u,(2\nu)^n(2\lambda_1+R))$, $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$,

(c) $U(\tau,\tau-1)E^{n-1}_{\tau-1} \subset E^n_\tau$, $n \in \mathbb{N}$, $\tau \in \mathcal{T}$,

(d) $E^n_\tau = \bigcup_{l=0}^n U(\tau,\tau-l)\overline{W}^{n-l}_{\tau-l}$, $n \in \mathbb{N}_0$, $\tau \in \mathcal{T}$.

To prove (a) observe that by (I) we have $E^0_\tau = \overline{W}^0_\tau \subset U(\tau,\tau)B = B$ and suppose that for some $n \in \mathbb{N}_0$ the property (a) holds. Then we have again by (I)

$$E^{n+1}_\tau = \overline{W}^{n+1}_\tau \cup U(\tau,\tau-1)E^n_{\tau-1} \subset$$

$$\subset U(\tau,\tau-n-1)B \cup U(\tau,\tau-1)U(\tau-1,\tau-1-n)B = U(\tau,\tau-n-1)B \subset B.$$

The assertion (b) follows from (II), since $\overline{W}^n_\tau \subset E^n_\tau$, whereas (c) is the consequence of the definition of E^n_τ. We prove (d) by induction. We have $E^0_\tau = U(\tau,\tau)\overline{W}^0_\tau = \overline{W}^0_\tau$, $\tau \in \mathcal{T}$. Assuming (d) for some $n \in \mathbb{N}_0$, we have

$$E^{n+1}_\tau = \overline{W}^{n+1}_\tau \cup U(\tau,\tau-1)E^n_{\tau-1} = \overline{W}^{n+1}_\tau \cup U(\tau,\tau-1)\bigcup_{l=0}^{n} U(\tau-1,\tau-1-l)\overline{W}^{n-l}_{\tau-1-l} =$$

$$= \overline{W}^{n+1}_\tau \cup \bigcup_{l=0}^{n} U(\tau,\tau-1-l)\overline{W}^{n-l}_{\tau-1-l} = \overline{W}^{n+1}_\tau \cup \bigcup_{j=1}^{n+1} U(\tau,\tau-j)\overline{W}^{n+1-j}_{\tau-j} = \bigcup_{l=0}^{n+1} U(\tau,\tau-l)\overline{W}^{n+1-l}_{\tau-l}.$$

Step 5. Construction of the sets $\widetilde{\mathcal{M}}(\tau)$, $\tau \in \mathcal{T}$. We define the following sets

$$\widetilde{\mathcal{M}}(\tau) = \bigcup_{s\in[0,1)} U(\tau,\tau-1-s)\bigcup_{n=0}^{\infty} E^n_{\tau-1-s},\ \tau \in \mathcal{T}.$$

We show that these nonempty subsets of B satisfy the following properties:

$$U(\tau,\sigma)\widetilde{\mathcal{M}}(\sigma) \subset \widetilde{\mathcal{M}}(\tau),\ \tau \ge \sigma,\ \tau \in \mathcal{T}, \tag{X1}$$

the sets $\widetilde{\mathcal{M}}(\tau)$, $\tau \in \mathcal{T}$, are precompact in W and

$$\sup_{\tau\in\mathcal{T}} d^W_f(\widetilde{\mathcal{M}}(\tau)) \le \max\left\{\frac{1}{\xi_1},\frac{1}{\xi_2}\right\}\left(1+\log_{\frac{1}{2\nu}}(1+\mu\kappa)\right) + \log_{\frac{1}{2\nu}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1)), \tag{X2}$$

$$\exists_{\chi>0}\forall_{B_1\subset V,\ \text{bounded}}\ \exists_{\widetilde{c}_{B_1}>0}\forall_{t\ge T_{B_1}+1}\ \sup_{\tau\in\mathcal{T}} \mathrm{dist}_W(U(\tau,\tau-t)B_1,\widetilde{\mathcal{M}}(\tau)) \le \widetilde{c}_{B_1}e^{-\chi t}. \tag{X3}$$

Indeed, fix $\tau \geq \sigma$, $\tau \in \mathcal{T}$, and $s \in [0,1)$. Let $\tau - \sigma + s = p + r$, where $p \in \mathbb{N}_0$ and $r \in [0,1)$. We have by (c)

$$U(\tau,\sigma)U(\sigma,\sigma-1-s)\bigcup_{n=0}^{\infty} E^n_{\sigma-1-s} = U(\tau,\tau-1-r)U(\tau-1-r,\sigma-1-s)\bigcup_{n=0}^{\infty} E^n_{\sigma-1-s} =$$

$$= U(\tau,\tau-1-r)U(\sigma-1-s+p,\sigma-1-s)\bigcup_{n=0}^{\infty} E^n_{\sigma-1-s} \subset U(\tau,\tau-1-r)\bigcup_{n=0}^{\infty} E^{n+p}_{\sigma-1-s+p} =$$

$$= U(\tau,\tau-1-r)\bigcup_{n=0}^{\infty} E^{n+p}_{\tau-1-r} \subset U(\tau,\tau-1-r)\bigcup_{n=0}^{\infty} E^n_{\tau-1-r}.$$

This implies (X1).

To prove (X2) we note that for any $\tau \in \mathcal{T}$ and $m \in \mathbb{N}_0$

$$\widetilde{\mathcal{M}}(\tau) \subset \bigcup_{s\in[0,1)}\bigcup_{n=0}^{m} U(\tau,\tau-1-s)E^n_{\tau-1-s} \cup U(\tau,\tau-m)B,$$

since by (a) and (A2) for any $n \geq m+1$ we have

$$U(\tau,\tau-1-s)E^n_{\tau-1-s} \subset U(\tau,\tau-1-s)U(\tau-1-s,\tau-1-s-n)B =$$

$$= U(\tau,\tau-m)U(\tau-m,\tau-1-s-n)B \subset U(\tau,\tau-m)B.$$

We fix now $0 < \varepsilon < \frac{2\lambda_1+R}{2\nu}$ and show below that $\widetilde{\mathcal{M}}(\tau)$ can be covered by an ε-net. Let $m \in \mathbb{N}_0$ be such that

$$(2\nu)^m(2\lambda_1+R) \leq \varepsilon < (2\nu)^{m-1}(2\lambda_1+R). \tag{2.9}$$

By (I) and (II) we have

$$U(\tau,\tau-m)B \subset \bigcup_{u\in\overline{W}^m_\tau} B^W(u,\varepsilon) \quad \text{and} \quad \#\overline{W}^m_\tau \leq N^m. \tag{2.10}$$

We define

$$\widetilde{\mathcal{M}}^m(\tau) = \bigcup_{s\in[0,1)}\bigcup_{n=0}^{m} U(\tau,\tau-1-s)E^n_{\tau-1-s},\ \tau \in \mathcal{T}.$$

We are going to construct a cover of this set by ε-balls. Note that by (d) we have

$$\widetilde{\mathcal{M}}^m(\tau) = \bigcup_{n=0}^{m}\bigcup_{l=0}^{n}\bigcup_{s\in[0,1)} U(\tau,\tau-1-s-l)\overline{W}^{n-l}_{\tau-1-s-l}.$$

From (2.6) it follows that

$$\widetilde{\mathcal{M}}^m(\tau) = \bigcup_{n=0}^{m}\bigcup_{l=0}^{n}\bigcup_{s\in[0,1)} U(\tau,\tau-1-s-n)\widetilde{W}^{n-l}_{k\delta(n-l)+l-n},\ \tau\in\mathcal{T},$$

where $k = k_{\tau,n,l,s}\in\mathbb{Z}$ is such that $-s\in[k\delta(n-l)+l+1-\tau,(k+1)\delta(n-l)+l+1-\tau)$.

We fix $\tau\in\mathcal{T}$, $0\le n\le m$ and $0\le l\le n$. We consider the function

$$[0,1)\ni s\mapsto \widetilde{W}^{n-l}_{k_{\tau,n,l,s}\delta(n-l)+l-n}.$$

This function is piecewise constant. Taking s_1, s_2 from a subinterval of $[0,1)$ such that $k = k_{\tau,n,l,s_1} = k_{\tau,n,l,s_2}$, we have by (A9)

$$\operatorname{dist}^{symm}_W(U(\tau,\tau-1-s_1-n)\widetilde{W}^{n-l}_{k\delta(n-l)+l-n}, U(\tau,\tau-1-s_2-n)\widetilde{W}^{n-l}_{k\delta(n-l)+l-n})\le$$

$$\le\sup_{u\in B}\|U(\tau,\tau-1-s_1-n)u-U(\tau,\tau-1-s_2-n)u\|_W\le\lambda_2L^n\left|s_1-s_2\right|^{\xi_2},$$

where $\operatorname{dist}^{symm}_W$ denotes the Hausdorff distance. This shows that the function

$$[0,1)\ni s\mapsto U(\tau,\tau-1-s-n)\widetilde{W}^{n-l}_{k_{\tau,n,l,s}\delta(n-l)+l-n}$$

is piecewise Hölder continuous (with the same exponent ξ_2 and constant λ_2L^n on the subintervals of length at most $\delta(n-l)$). Note that different numbers k may be attained on at most $\left[\frac{1}{\delta(n-l)}\right]+2$ intervals. Since we would like to have different numbers k on intervals of length not exceeding $d=\left(\frac{\varepsilon}{2\lambda_2L^n}\right)^{\frac{1}{\xi_2}}$, if necessary we divide each interval into subintervals of length d. Each interval gives rise to at most $\left[\frac{\delta(n-l)}{d}\right]+1$ subintervals. Let p_0 denote the total number of these subintervals of $[0,1)$ with length not exceeding d. We know that

$$p_0\le\left(\left[\frac{1}{\delta(n-l)}\right]+2\right)\left(\left[\frac{\delta(n-l)}{d}\right]+1\right).$$

From each of p_0 small intervals we choose a point s_p. Then for s from the small interval containing s_p we have

$$\operatorname{dist}^{symm}_W(U(\tau,\tau-1-s-n)\widetilde{W}^{n-l}_{k\delta(n-l)+l-n}, U(\tau,\tau-1-s_p-n)\widetilde{W}^{n-l}_{k\delta(n-l)+l-n})<\frac{\varepsilon}{2}.$$

This leads to

$$N^W_\varepsilon(\bigcup_{s\in[0,1)}U(\tau,\tau-1-s-n)\widetilde{W}^{n-l}_{k_{\tau,n,l,s}\delta(n-l)+l-n})\le\sum_{p=1}^{p_0}N^W_{\frac{\varepsilon}{2}}(U(\tau,\tau-1-s_p-n)\widetilde{W}^{n-l}_{k_{\tau,n,l,s_p}\delta(n-l)+l-n}).$$

Since for each p we have

$$\#U(\tau,\tau-1-s_p-n)\widetilde{W}^{n-l}_{k_{\tau,n,l,s_p}\delta(n-l)+l-n} \leq \#\widetilde{W}^{n-l}_{k_{\tau,n,l,s_p}\delta(n-l)+l-n} \leq N^{n-l},$$

it follows that

$$N^W_\varepsilon(\bigcup_{s\in[0,1)} U(\tau,\tau-1-s-n)\widetilde{W}^{n-l}_{k_{\tau,n,l,s}\delta(n-l)+l-n}) \leq p_0 N^{n-l} \leq$$

$$\leq \left(\left(\frac{\varepsilon}{2\lambda_2 L^n}\right)^{-\frac{1}{\xi_2}} + \frac{1}{\delta(n-l)} + 2\delta(n-l)\left(\frac{\varepsilon}{2\lambda_2 L^n}\right)^{-\frac{1}{\xi_2}} + 2\right) N^{n-l} \leq$$

$$\leq 3\left(\left(\frac{\varepsilon}{2\lambda_2 L^n}\right)^{-\frac{1}{\xi_2}} + \frac{1}{\delta(n-l)}\right) N^{n-l},$$

since $\delta(n-l) \leq 1$. Observe that by (2.9) we have

$$\left(\frac{\varepsilon}{2\lambda_2 L^n}\right)^{-\frac{1}{\xi_2}} \leq const(R,\xi_2,\lambda_1,\lambda_2)\left[\left(\frac{L}{2\nu}\right)^{\frac{1}{\xi_2}}\right]^m.$$

From (2.5) it follows that $\delta(j+1) \leq \delta(j)$, $j \in \mathbb{N}_0$, and thus

$$\frac{1}{\delta(n-l)} \leq \frac{1}{\delta(m)} = \left[\left(\frac{L}{2\nu}\right)^{\frac{1}{\xi_1}}\right]^m \leq \left[\left(\frac{L}{2\nu}\right)^{\max\{\frac{1}{\xi_1},\frac{1}{\xi_2}\}}\right]^m.$$

Therefore, we get

$$N^W_\varepsilon(\bigcup_{s\in[0,1)} U(\tau,\tau-1-s-n)\widetilde{W}^{n-l}_{k_{\tau,n,l,s}\delta(n-l)+l-n}) \leq const(R,\xi_2,\lambda_1,\lambda_2)\left[\left(\frac{L}{2\nu}\right)^{\max\left\{\frac{1}{\xi_1},\frac{1}{\xi_2}\right\}} N\right]^m.$$

Thus, we have

$$N^W_\varepsilon(\widetilde{\mathcal{M}}^m(\tau)) \leq C(m+1)^2\left[\left(\frac{L}{2\nu}\right)^{\max\left\{\frac{1}{\xi_1},\frac{1}{\xi_2}\right\}} N\right]^m,$$

where $C>0$ does not depend on m, N, L, ν and τ. This and (2.10) yield

$$N^W_\varepsilon(\widetilde{\mathcal{M}}(\tau)) \leq C(m+1)^2\left[\left(\frac{L}{2\nu}\right)^{\max\left\{\frac{1}{\xi_1},\frac{1}{\xi_2}\right\}} N\right]^m + N^m. \qquad (2.11)$$

This ensures that the set $\widetilde{\mathcal{M}}(\tau)$ is precompact in W. From (2.11) we derive the estimate of the fractal dimension of $\widetilde{\mathcal{M}}(\tau)$. Indeed, we have

$$\ln(N_\varepsilon^W(\widetilde{\mathcal{M}}(\tau))) \le \ln \widetilde{C} + 2\ln(m+1) + m \ln K$$

with $K = \left(\frac{L}{2\nu}\right)^{\max\left\{\frac{1}{\xi_1};\frac{1}{\xi_2}\right\}} N$. From (2.9) it follows that

$$m < 1 + \frac{\ln(2\lambda_1 + R)}{\ln \frac{1}{2\nu}} + \frac{\ln \frac{1}{\varepsilon}}{\ln \frac{1}{2\nu}}.$$

Hence we obtain

$$\ln(N_\varepsilon^W(\widetilde{\mathcal{M}}(\tau))) \le C_1 + \frac{\ln K}{\ln \frac{1}{2\nu}} \ln \frac{1}{\varepsilon} + 2\ln\left(C_2 + C_3 \ln \frac{1}{\varepsilon}\right)$$

with $C_1, C_2 \in \mathbb{R}$ and $C_3 > 0$ independent of ε and τ. This implies

$$\sup_{\tau\in\mathcal{T}} d_f^W(\widetilde{\mathcal{M}}(\tau)) = \sup_{\tau\in\mathcal{T}} \limsup_{\varepsilon\to 0} \frac{\ln(N_\varepsilon^W(\widetilde{\mathcal{M}}(\tau)))}{\ln \frac{1}{\varepsilon}} \le \frac{\ln K}{\ln \frac{1}{2\nu}},$$

and in consequence (X2).

It remains to prove (X3). Set $\chi = \ln \frac{1}{2\nu} > 0$ and fix a bounded subset B_1 of V. Define

$$\widetilde{c}_{B_1} = \frac{2\lambda_1 + R}{2\nu} \mu\kappa e^{\chi(T_{B_1}+1)} > 0,$$

where $T_{B_1} > 0$ is taken from (A2). Let $t \ge T_{B_1} + 1$ and $t = T_{B_1} + n_0 + s_0$ with $n_0 \in \mathbb{N}$ and $s_0 \in [0,1)$. Then we have for $\tau \in \mathcal{T}$

$$\mathrm{dist}_W(U(\tau, \tau-t)B_1, \widetilde{\mathcal{M}}(\tau)) =$$

$$= \mathrm{dist}_W(U(\tau,\tau-n_0)U(\tau-n_0,\tau-n_0-s_0-T_{B_1})B_1, \bigcup_{s\in[0,1)} U(\tau,\tau-1-s) \bigcup_{n=0}^{\infty} E_{\tau-1-s}^n) \le$$

$$\le \mathrm{dist}_W(U(\tau,\tau-1)U(\tau-1,\tau-n_0)B, U(\tau,\tau-1)E_{\tau-1}^{n_0-1}) \le$$

$$\le \mu\kappa\, \mathrm{dist}_W(U(\tau-1,\tau-1-(n_0-1))B, E_{\tau-1}^{n_0-1}) \le (2\lambda_1+R)\mu\kappa(2\nu)^{n_0-1} = \frac{2\lambda_1+R}{2\nu}\mu\kappa e^{-\chi n_0},$$

where we used (b), (A3) and (2.4). Thus, we have

$$\mathrm{dist}_W(U(\tau,\tau-t)B_1, \widetilde{\mathcal{M}}(\tau)) \le \frac{2\lambda_1+R}{2\nu}\mu\kappa e^{\chi(T_{B_1}+s_0)} e^{-\chi t} \le \widetilde{c}_{B_1} e^{-\chi t}.$$

Step 6. Pullback exponential attractor with precompact sections in V. We define

$$\mathcal{M}(\tau) = U(\tau, \tau - 1)\widetilde{\mathcal{M}}(\tau - 1),\ \tau \in \mathcal{T}.$$

Of course $\mathcal{M}(\tau)$ is a nonempty subset of B. Furthermore, we have by (X1) for $\tau \geq \sigma$, $\tau \in \mathcal{T}$

$$U(\tau, \sigma)\mathcal{M}(\sigma) = U(\tau, \sigma - 1)\widetilde{\mathcal{M}}(\sigma - 1) \subset U(\tau, \tau - 1)\widetilde{\mathcal{M}}(\tau - 1) = \mathcal{M}(\tau),$$

which proves (W1).

By (A3) the function $W \supset B \ni u \to U(\tau, \tau - 1)u \in V$ is Lipschitz continuous, so we know that $\mathcal{M}(\tau)$ is a precompact subset of V and

$$d_f^V(\mathcal{M}(\tau)) \leq d_f^W(\widetilde{\mathcal{M}}(\tau - 1)),$$

which together with (X2) implies (W2).

Let B_1 be a bounded subset of V. From (A3) and (X3) we obtain

$$\begin{aligned} &\sup_{\tau\in\mathcal{T}} \mathrm{dist}_V(U(\tau, \tau - t)B_1, \mathcal{M}(\tau)) \leq \\ \leq\ &\kappa \sup_{\tau\in\mathcal{T}} \mathrm{dist}_W(U(\tau - 1, \tau - 1 - (t - 1))B_1, \widetilde{\mathcal{M}}(\tau - 1)) \leq \kappa\widetilde{c}_{B_1} e^{\chi} e^{-\chi t},\ t \geq T_{B_1} + 2. \end{aligned} \tag{2.12}$$

This implies (W3).

Recall that $B \subset B^W(v_0, R)$ with $v_0 \in B$ and $R > 0$. If (A7) holds, we let $B_1 \subset B^V(0, R_{B_1})$ and set

$$c_{B_1} = \max\{(Q(R_{B_1}) + Q(\|v_0\|_V) + 2R_0)e^{\chi(T_{B_1}+2)}, \kappa\widetilde{c}_{B_1}e^{\chi}\} > 0.$$

It follows from (A7) that

$$\bigcup_{\tau\in\mathcal{T}} U(\tau, \tau - t)B_1 \subset B^V\left(0, Q(R_{B_1}) + R_0\right),\ t \geq 0.$$

Moreover, we know from the previous steps that for $\tau \in \mathcal{T}$

$$\mathcal{M}(\tau) = U(\tau, \tau - 1)\widetilde{\mathcal{M}}(\tau - 1) \supset U(\tau, \tau - 2)E^0_{\tau-2} =$$

$$= U(\tau, \tau - 2)\overline{W}^0_{\tau-2} = U(\tau, \tau - 2)\widetilde{W}^0_{k_{\tau-2}\delta(0)} = U(\tau, \tau - 2)W^0_{k_{\tau-2}\delta(0)} = \{U(\tau, \tau - 2)v_0\}.$$

Hence, for $t \in [0, T_{B_1} + 2]$ and $\tau \in \mathcal{T}$ we have

$$\mathrm{dist}_V(U(\tau, \tau - t)B_1, \mathcal{M}(\tau)) \leq \mathrm{dist}_V(B^V\left(0, Q(R_{B_1}) + R_0\right), \{U(\tau, \tau - 2)v_0\}) \leq$$

$$\leq Q(R_{B_1}) + R_0 + \|U(\tau, \tau - 2)v_0\|_V \leq (Q(R_{B_1}) + Q(\|v_0\|_V) + 2R_0)e^{\chi(T_{B_1}+2)}e^{-\chi t} \leq c_{B_1}e^{-\chi t}.$$

Thus, we get

$$\sup_{\tau\in\mathcal{T}} \operatorname{dist}_V(U(\tau,\tau-t)B_1, \widetilde{\mathcal{M}}(\tau)) \le c_{B_1} e^{-\chi t},\ t\in[0,T_{B_1}+2].$$

This and (2.12) imply (W3').

Step 7. Pullback exponential attractor with compact sections in V. Assume now that V is a Banach space and (A6) holds. We define

$$\widehat{\mathcal{M}}(\tau) = \operatorname{cl}_V \mathcal{M}(\tau),\ \tau\in\mathcal{T}.$$

These sets are nonempty subsets of $\operatorname{cl}_V B$. Moreover, they are compact. By (A6) and (W1) we have

$$U(\tau,\sigma)\widehat{\mathcal{M}}(\sigma) \subset \widehat{\mathcal{M}}(\tau),\ \tau\ge\sigma,\ \tau\in\mathcal{T}.$$

Since $d_f^V(\widehat{\mathcal{M}}(\tau)) = d_f^V(\operatorname{cl}_V(\mathcal{M}(\tau))) = d_f^V(\mathcal{M}(\tau))$, we obtain from (W2)

$$\sup_{\tau\in\mathcal{T}} d_f^V(\widehat{\mathcal{M}}(\tau)) \le \max\left\{\frac{1}{\xi_1},\frac{1}{\xi_2}\right\}\Big(1+\log_{\frac{1}{2\nu}}(1+\mu\kappa)\Big) + \log_{\frac{1}{2\nu}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1)).$$

Finally, for any bounded subset B_1 of V we have

$$\operatorname{dist}_V(U(\tau,\tau-t)B_1, \widehat{\mathcal{M}(\tau)}) \le \operatorname{dist}_V(U(\tau,\tau-t)B_1, \mathcal{M}(\tau)),\ t\ge 0,\ \tau\in\mathcal{T}.$$

Step 8. Rescaling time back. Observe that rescaling time back turns $\frac{\tau_0}{T_B}$ back into τ_0 and does not change the claim, but the rates φ and χ in (W3) and (W3') will change into φT_B^{-1} and χT_B^{-1}, respectively. This ends the proof. □

Remark 2.2. Observe that from the proof of the above theorem it follows that instead of a process $\{U(\tau,\sigma)\colon \tau\ge\sigma\}$ we could only consider a semiprocess $\{U(\tau,\sigma)\colon \sigma\le\tau,\ \tau\in\mathcal{T}\}$. If the process is a semigroup (see Corollary 2.6), the assumption (A4) is trivially satisfied.

Below we relate the sets constructed in the above theorem with a better known notion of a pullback global attractor. By the *pullback global attractor* we call a family $\{\mathcal{A}(\tau)\colon \tau\in\mathbb{R}\}$ of nonempty compact subsets of V, invariant under the process, i.e. $U(\tau,\sigma)\mathcal{A}(\sigma) = \mathcal{A}(\tau)$, $\tau\ge\sigma$, pullback attracting all bounded subsets B_1 of V

$$\lim_{t\to\infty} \operatorname{dist}_V(U(\tau,\tau-t)B_1, \mathcal{A}(\tau)) = 0,\ \tau\in\mathbb{R},$$

and minimal in the sense that if $\{\widetilde{\mathcal{A}}(\tau)\colon \tau\in\mathbb{R}\}$ is a family of closed sets in V pullback attracting all bounded subsets of V, then $\mathcal{A}(\tau)\subset\widetilde{\mathcal{A}}(\tau)$, $\tau\in\mathbb{R}$.

The proposition presented below gives the existence of a finite dimensional pullback global attractor in V if $\tau_0=\infty$. For the proof we refer the reader to [6, Theorem 1.1] (see also [1] for the discussion on the existence of pullback global attractors).

Proposition 2.3. *Let V be a Banach space compactly embedded in a normed space W and assume that* (A1)-(A6) *are satisfied with some $\tau_0 \leq \infty$. Then for any bounded subset B_1 of V and any $\tau \in \mathcal{T}$ the ω-limit set of B_1 at time τ, i.e.*

$$\omega(B_1,\tau) = \bigcap_{s\geq 0} \mathrm{cl}_V \bigcup_{t\geq s} U(\tau,\tau-t)B_1,$$

is a nonempty compact subset of $\mathcal{M}(\tau)$ from Theorem 2.1. Moreover, we have

$$\omega(B_1,\tau) = \{u \in V \colon \exists_{u_n\in B_1} \exists_{t_n\geq 0}\, t_n \to \infty \textit{ and } U(\tau,\tau-t_n)u_n \to u\}.$$

Furthermore, (W3) *implies*

$$\forall_{\tau\in\mathcal{T}} \lim_{t\to\infty} \mathrm{dist}_V(U(\tau,\tau-t)B_1, \omega(B_1,\tau)) = 0.$$

We also have

$$U(\tau,\sigma)\omega(B_1,\sigma) = \omega(B_1,\tau),\ \tau \geq \sigma,\ \tau \in \mathcal{T}.$$

Setting

$$\mathcal{A}(\tau) = \mathrm{cl}_V \bigcup_{B_1\subset V,\ bounded} \omega(B_1,\tau),\ \tau \in \mathcal{T},$$

we see that $\{\mathcal{A}(\tau)\colon \tau \in \mathcal{T}\}$ is a family of nonempty compact subsets of V,

$$U(\tau,\sigma)\mathcal{A}(\sigma) = \mathcal{A}(\tau),\ \tau \geq \sigma,\ \tau \in \mathcal{T},$$

and the family pullback attracts all bounded sets in V for any $\tau \in \mathcal{T}$, i.e.

$$\forall_{B_1\subset V,\ bounded} \forall_{\tau\in\mathcal{T}} \lim_{t\to\infty} \mathrm{dist}_V(U(\tau,\tau-t)B_1, \mathcal{A}(\tau)) = 0.$$

Moreover, if $\{\widetilde{\mathcal{A}}(\tau)\colon \tau \in \mathcal{T}\}$ is a family of closed sets in V pullback attracting all bounded subsets of V for any $\tau \in \mathcal{T}$, then $\mathcal{A}(\tau) \subset \widetilde{\mathcal{A}}(\tau)$, $\tau \in \mathcal{T}$. In particular, we have $\mathcal{A}(\tau) \subset \mathcal{M}(\tau)$, $\tau \in \mathcal{T}$.

Observe that if $\tau_0 = \infty$, then Theorem 2.1 gives a construction of the uniform pullback exponential attractor, satisfying (2.1) and, in consequence, also (2.2). Then the union of precompact (compact) sets $\mathcal{M}(\tau)$, $\tau \in \mathbb{R}$, is a subset of the bounded absorbing set B ($\mathrm{cl}_V\, B$). However, if $\tau_0 < \infty$ it can be proved, following [13, Theorem 2.3], that under an additional assumption we can still expect pullback exponential attraction as in (2.3). As we show below, in that case the union of sets $\mathcal{M}(\tau)$, $\tau \in \mathbb{R}$, may not be bounded (in the future).

Corollary 2.4. *Let V be a normed space compactly embedded in a normed space W and assume that* (A1)-(A5) *are satisfied with some $\tau_0 < \infty$. If the process $\{U(\tau,\sigma)\colon \tau \geq \sigma\}$ satisfies the additional assumption on Lipschitz continuity*

$$\forall_{t>0}\exists_{k(t)>0}\forall_{u_1,u_2\in B}\,\|U(t+\tau_0,\tau_0)u_1 - U(t+\tau_0,\tau_0)u_2\|_V \leq k(t)\,\|u_1-u_2\|_V\,, \tag{A10}$$

then for any $\nu \in (0,\frac{1}{2})$ there exists a family $\{\mathcal{M}(\tau) = \mathcal{M}_\nu(\tau)\colon \tau \in \mathbb{R}\}$ of nonempty precompact subsets of V with the following properties:

(i) $\{\mathcal{M}(\tau)\colon \tau\in\mathbb{R}\}$ is positively invariant under the process $U(\tau,\sigma)$, i.e.

$$U(\tau,\sigma)\mathcal{M}(\sigma) \subset \mathcal{M}(\tau),\ \tau \geq \sigma, \tag{Z1}$$

(ii) $\mathcal{M}_\nu(\tau)$ has a finite fractal dimension in V uniformly with respect to $\tau\in\mathbb{R}$, i.e.

$$\sup_{\tau\in\mathbb{R}} d_f^V(\mathcal{M}_\nu(\tau)) \leq \max\left\{\frac{1}{\xi_1},\frac{1}{\xi_2}\right\}\Big(1+\log_{\frac{1}{2\nu}}(1+\mu\kappa)\Big) + \log_{\frac{1}{2\nu}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1)), \tag{Z2}$$

where $\mu > 0$ is given in (2.4),

(iii) $\{\mathcal{M}(\tau)\colon \tau\in\mathbb{R}\}$ is pullback exponentially attracting bounded subsets of V, i.e.

$$\exists_{\varphi>0}\forall_{B_1\subset V,\ bounded}\,\forall_{\tau\in\mathbb{R}}\quad \lim_{t\to\infty} e^{\varphi t}\operatorname{dist}_V(U(\tau,\tau-t)B_1,\mathcal{M}(\tau)) = 0. \tag{Z3}$$

Moreover, if V is a Banach space and the process $\{U(\tau,\sigma)\colon \tau\geq\sigma\}$ is continuous on $\operatorname{cl}_V B$, i.e. the mapping $U(\tau,\sigma)\colon \operatorname{cl}_V B \to V$ is continuous for $\tau \geq \sigma$, then we can make $\mathcal{M}(\tau)$, $\tau\in\mathbb{R}$, compact subsets of V.

Proof. For $\tau \leq \tau_0$ the sets $\mathcal{M}(\tau)$ have already been constructed in Theorem 2.1. Therefore, we only set

$$\mathcal{M}(\tau) = U(\tau,\tau_0)\mathcal{M}(\tau_0),\ \tau > \tau_0.$$

To show (Z1) it remains to consider only two cases. If $\sigma \leq \tau_0 < \tau$, then we have

$$U(\tau,\sigma)\mathcal{M}(\sigma) = U(\tau,\tau_0)U(\tau_0,\sigma)\mathcal{M}(\sigma) \subset U(\tau,\tau_0)\mathcal{M}(\tau_0) = \mathcal{M}(\tau),$$

while if $\tau_0 < \sigma < \tau$ we have

$$U(\tau,\sigma)\mathcal{M}(\sigma) = U(\tau,\sigma)U(\sigma,\tau_0)\mathcal{M}(\tau_0) = U(\tau,\tau_0)\mathcal{M}(\tau_0) = \mathcal{M}(\tau).$$

If $\tau > \tau_0$, then by (A10), $\mathcal{M}(\tau)$ is a precompact subset of V and $d_f^V(\mathcal{M}(\tau)) \leq d_f^V(\mathcal{M}(\tau_0))$. Moreover, we have for $t > \tau - \tau_0$ and a bounded subset B_1 of V

$$e^{\varphi t}\operatorname{dist}_V(U(\tau,\tau-t)B_1,\mathcal{M}(\tau)) = e^{\varphi t}\operatorname{dist}_V(U(\tau,\tau_0)U(\tau_0,\tau-t)B_1, U(\tau,\tau_0)\mathcal{M}(\tau_0)) \leq$$

$$\leq k(\tau-\tau_0)e^{\varphi(\tau-\tau_0)}e^{\varphi(t+\tau_0-\tau)}\operatorname{dist}_V(U(\tau_0,\tau_0-(t+\tau_0-\tau)),\mathcal{M}(\tau_0))$$

and the right-hand side tends to 0 as $t\to\infty$ by (W3). □

If $\tau_0 < \infty$ it can be proved that under assumption (A10) we can still construct a (finite dimensional) pullback global attractor contained in the pullback exponential attractor from Corollary 2.4.

Proposition 2.5. *Let V be a Banach space compactly embedded in a normed space W and assume that* (A1)-(A5) *and* (A10) *are satisfied with some $\tau_0 < \infty$ and the process $\{U(\tau,\sigma)\colon \tau \geq \sigma\}$ is continuous on $\mathrm{cl}_V B$. Then there exists a family $\{\mathcal{A}(\tau)\colon \tau \in \mathbb{R}\}$ of nonempty compact subsets of V, invariant under the process*

$$U(\tau,\sigma)\mathcal{A}(\sigma) = \mathcal{A}(\tau),\ \tau \geq \sigma,$$

and pullback attracting all bounded sets in V

$$\forall_{B_1 \subset V,\ bounded}\ \forall_{\tau\in\mathbb{R}}\ \lim_{t\to\infty} \mathrm{dist}_V(U(\tau,\tau-t)B_1, \mathcal{A}(\tau)) = 0.$$

We also know that $\{\mathcal{A}(\tau)\colon \tau \in \mathbb{R}\}$ is minimal among the families of closed sets in V that pullback attract all bounded subsets of V at any time $\tau \in \mathbb{R}$. In particular, we have $\mathcal{A}(\tau) \subset \mathcal{M}(\tau)$, $\tau \in \mathbb{R}$, where the pullback exponential attractor $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ is taken from Corollary 2.4. Moreover, the pullback global attractor is given as

$$\mathcal{A}(\tau) = \mathrm{cl}_V \bigcup_{B_1\subset V,\ bounded} \omega(B_1,\tau),\ \tau \in \mathbb{R},$$

where $\omega(B_1,\tau)$ is the ω-limit set of B_1 at time $\tau \in \mathbb{R}$ defined as

$$\omega(B_1,\tau) = \bigcap_{s\geq 0} \mathrm{cl}_V \bigcup_{t\geq s} U(\tau,\tau-t)B_1.$$

Proof. For $\tau > \tau_0$ we set $\mathcal{A}(\tau) = U(\tau,\tau_0)\mathcal{A}(\tau_0)$. The rest of the proof is similar to the proof of Corollary 2.4. □

Theorem 2.1 and Proposition 2.3 also give a method for a construction of exponential and global attractors for semigroups. The corollary that we present below is in the vein of results from [4, Section 2].

Corollary 2.6. *Assume that $S(t)\colon V \to V$, $t \geq 0$, is a semigroup in a normed space V, i.e.*

$$S(t)S(s) = S(t+s),\ t,s \geq 0, \quad S(0) = I, \tag{SA1}$$

where I denotes an identity operator on V. Moreover, assume that there exists a bounded set B in V absorbing bounded subsets of V, i.e. for any bounded set B_1 in V there exists a time $T_{B_1} > 0$ such that

$$S(t)B_1 \subset B,\ t \geq T_{B_1}. \tag{SA2}$$

Let $(W, \|\cdot\|_W)$ be an auxiliary normed space such that V is compactly embedded in W and the semigroup $\{S(t)\colon t \geq 0\}$ satisfies the following properties with $\kappa > 0$, $0 < \theta \leq 1$ and $\lambda > 0$

$$\|S(T_B)u_1 - S(T_B)u_2\|_V \leq \kappa \|u_1 - u_2\|_W, \; u_1, u_2 \in B, \tag{SA3}$$

$$\|S(t_1)u - S(t_2)u\|_W \leq \lambda |t_1 - t_2|^\theta, \; t_1, t_2 \in [T_B, 2T_B], \; u \in B. \tag{SA4}$$

Then for any $\nu \in (0, \frac{1}{2})$ there exists a precompact in V subset $\mathcal{M} = \mathcal{M}_\nu$ of B with the following properties:

(i) $\mathcal{M}$ is positively invariant under the semigroup $\{S(t)\colon t \geq 0\}$, i.e.

$$S(t)\mathcal{M} \subset \mathcal{M}, \; t \geq 0, \tag{SW1}$$

(ii) $\mathcal{M}_\nu$ has a finite fractal dimension in V, i.e.

$$d_f^V(\mathcal{M}_\nu) \leq \frac{1}{\theta}\Big(1 + \log_{\frac{1}{2\nu}}(1 + \mu\kappa)\Big) + \log_{\frac{1}{2\nu}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1)), \tag{SW2}$$

where $\mu > 0$ is taken from (2.4) and

(iii) $\mathcal{M}$ has the property of exponential attraction, i.e.

$$\exists_{\varphi > 0} \forall_{B_1 \subset V,\ bounded} \; \lim_{t \to \infty} e^{\varphi t} \operatorname{dist}_V(S(t)B_1, \mathcal{M}) = 0. \tag{SW3}$$

If V is a Banach space and the semigroup $\{S(t)\colon t \geq 0\}$ is continuous on $\operatorname{cl}_V B$:

$$S(t)\colon \operatorname{cl}_V B \to V \text{ is continuous for } t \geq 0, \tag{SA5}$$

then we can make $\mathcal{M}$ a compact subset of $\operatorname{cl}_V B$. Moreover, $\mathcal{M}$ contains the (finite dimensional) global attractor $\mathcal{A}$, that is a nonempty compact set $\mathcal{A}$, invariant under the semigroup and attracting each bounded subset of V

$$\forall_{B_1 \subset V,\ bounded} \; \lim_{t \to \infty} \operatorname{dist}_V(S(t)B_1, \mathcal{A}) = 0.$$

Moreover, if the semigroup is dissipative in V, i.e. there exists a nondecreasing function $Q\colon [0, \infty) \to [0, \infty)$ and constants $R_0, \omega > 0$ such that

$$\|S(t)u\|_V \leq Q(\|u\|_V)e^{-\omega t} + R_0, \; t \geq 0, \; u \in V, \tag{SA6}$$

(which implies in particular the existence of the absorbing set B), then (SW3) *can be improved to*

$$\exists_{\chi > 0} \forall_{B_1 \subset V,\ bounded} \exists_{c_{B_1} > 0} \forall_{t \geq 0} \; \operatorname{dist}_V(S(t)B_1, \mathcal{M}) \leq c_{B_1} e^{-\chi t}. \tag{SW3'}$$

Proof. We define

$$U(\tau,\sigma) = S(\tau-\sigma),\ \tau \geq \sigma,\ \tau,\sigma \in \mathbb{R}.$$

Then (A1)–(A5) are satisfied with $\tau_0 = \infty$, $\lambda_1 = \lambda_2 = \lambda$ and $\xi_1 = \xi_2 = \theta$. We apply the above theorem, but note that the construction in the proof of the sets W^n_τ, $\widetilde{W}^n_\tau$, $\overline{W}^n_\tau$, E^n_τ, $\widetilde{\mathcal{M}}(\tau)$, $\widetilde{\mathcal{M}}^m(\tau)$ and $\mathcal{M}(\tau)$ is independent of $\tau \in \mathbb{R}$. Therefore, for any $\nu \in (0, \frac{1}{2})$ there exists a precompact in V subset $\mathcal{M} = \mathcal{M}_\nu$ of B satisfying (SW1)–(SW3). Moreover, (SA5) implies (A6) and in consequence the possibility of making $\mathcal{M}$ a compact subset of $\mathrm{cl}_V B$. Also (SA6) yields (A7) and in this case implies (SW3'). □

3. Abstract nonautonomous semilinear parabolic problems

In this section we present general conditions on nonautonomous semilinear parabolic problems that guarantee existence of pullback global and exponential attractors.

Let X denote a Banach space and let $A\colon X \supset D(A) \to X$ be a positive sectorial operator in X and X^γ, $\gamma \geq 0$, be the associated fractional power spaces (see [12]). It is known that $-A$ generates in X a strongly continuous analytic semigroup $\{e^{-At}\}$ and

$$\left\|e^{-At}\right\|_{\mathcal{L}(X,X^\gamma)} \leq c_\gamma \frac{e^{-at}}{t^\gamma},\ \gamma \geq 0,\ t > 0, \tag{3.1}$$

where $a > 0$ is such that $\mathrm{Re}\,\sigma(A) > a$ and c_γ are certain positive constants. Suppose also that A has a compact resolvent. This yields the compactness of the embedding X^{γ_2} into X^{γ_1} for $\gamma_1 < \gamma_2$.

We fix $\alpha \in [0,1)$ and assume that $F\colon \mathbb{R} \times X^\alpha \to X$ satisfies the following assumption

$$\begin{gathered} \forall_{G \subset X^\alpha,\ \text{bounded}} \exists_{0<\theta=\theta(G)<1} \forall_{T_1,T_2 \in \mathbb{R}, T_1<T_2} \exists_{L=L(T_2-T_1,G)>0} \forall_{\tau_1,\tau_2 \in [T_1,T_2]} \forall_{u_1,u_2 \in G} \\ \left\|F(\tau_1,u_1) - F(\tau_2,u_2)\right\|_X \leq L(|\tau_1-\tau_2|^\theta + \|u_1-u_2\|_{X^\alpha}). \end{gathered} \tag{F1}$$

Note that L depends only on the difference $T_2 - T_1$ and on G. Moreover, if F is Lipschitz continuous with respect to time on $[T_1,T_2] \times G$, then of course it is also Hölder continuous and thus satisfies the condition (F1) with any $0 < \theta(G) < 1$. Under this assumption for any $\sigma \in \mathbb{R}$ and $u_0 \in X^\alpha$ there exists a unique (forward) X^α solution to the problem

$$\begin{cases} u_\tau + Au = F(\tau,u),\ \tau > \sigma, \\ u(\sigma) = u_0, \end{cases} \tag{3.2}$$

defined on the maximal interval of existence $[\sigma, \tau_{max})$, i.e. a function

$$u \in C([\sigma,\tau_{max}), X^\alpha) \cap C((\sigma,\tau_{max}), X^1) \cap C^1((\sigma,\tau_{max}), X)$$

satisfying (3.2) in X and such that either $\tau_{max} = \infty$ or $\tau_{max} < \infty$ and in the latter case

$$\limsup_{\tau \to \tau_{max}} \|u(\tau)\|_{X^\alpha} = \infty.$$

Moreover, the solution u satisfies the variation of constants formula

$$u(\tau) = e^{-A(\tau-\sigma)}u_0 + \int_\sigma^\tau e^{-A(\tau-s)}F(s,u(s))ds,\ \tau \in [\sigma, \tau_{max}). \tag{3.3}$$

For the purpose of considerations in this section we define

$$\mathcal{T} = \{\tau \in \mathbb{R} \colon \tau \le \tau_0\}$$

with $\tau_0 \le \infty$ fixed from now on and we further assume that for some $M > 0$

$$\sup_{\tau \in \mathcal{T}} \|F(\tau, 0)\|_X \le M. \tag{F2}$$

In order to prove that the local solutions can be extended globally (forward) in time and obtain the existence of a bounded absorbing set in X^α in applications we verify an appropriate a priori condition. Here we will assume that

$$\text{each local solution can be extended globally (forward) in time, i.e. } \tau_{max} = \infty, \tag{F3a}$$

there exists a constant $\omega > 0$ and a nondecreasing function $Q \colon [0,\infty) \to [0,\infty)$ (both independent of σ) such that

$$\|u(\tau)\|_{X^\alpha} \le Q(\|u_0\|_{X^\alpha})e^{-\omega(\tau-\sigma)} + R_0,\ \sigma \le \tau,\ \tau \in \mathcal{T}, \tag{F3b}$$

holds with a constant $R_0 = R_0(\tau_0) > 0$ independent of σ, τ and u_0 and (in case $\tau_0 < \infty$) for any $T > 0$ there exists $R_{T,\sigma} > 0$ and a nondecreasing function $\widetilde{Q}_{T,\sigma} \colon [0,\infty) \to [0,\infty)$ such that

$$\|u(\tau)\|_{X^\alpha} \le \widetilde{Q}_{T,\sigma}(\|u_0\|_{X^\alpha}) + R_{T,\sigma},\ \tau \in [\sigma, \sigma + T]. \tag{F3c}$$

Note that assumptions (F3a)–(F3c) can be replaced by a single stronger requirement that (3.2) admits the following dissipativity condition in X^α

$$\|u(\tau)\|_{X^\alpha} \le Q(\|u_0\|_{X^\alpha})e^{-\omega(\tau-\sigma)} + R(\tau),\ \tau \in [\sigma, \tau_{max}), \tag{F3}$$

where $\omega > 0$ is a constant, $Q \colon [0,\infty) \to [0,\infty)$ is nondecreasing and $R \colon \mathbb{R} \to [0,\infty)$ is a continuous function such that for some positive constant R_0 (independent of u_0, σ, τ)

$$\sup_{\tau \in \mathcal{T}} R(\tau) \le R_0.$$

On account of (F3a) we define the evolution process $\{U(\tau,\sigma)\colon \tau \geq \sigma\}$ on X^α by

$$U(\tau,\sigma)u_0 := u(\tau),\ \tau \geq \sigma,\ u_0 \in X^\alpha, \tag{3.4}$$

where $u(\tau)$ is the value at time τ of the X^α solution of (3.2) starting at time σ from u_0.

Moreover, assumption (F3b) implies that $B_0 = B^{X^\alpha}(0, 2R_0)$ pullback absorbs bounded subsets of X^α uniformly in time, i.e. for any bounded set B_1 in X^α there exists $\widetilde{T}_{B_1} > 0$ such that for $t \geq \widetilde{T}_{B_1}$

$$\bigcup_{\tau\in\mathcal{T}} U(\tau,\tau-t)B_1 \subset B_0.$$

Let $\beta \in (\alpha, 1)$. Then we have continuous and compact embedding of X^β in X^α and

$$\|u\|_{X^\alpha} \leq c_{\alpha,\beta}\, \|u\|_{X^\beta}\,,\ u \in X^\beta. \tag{3.5}$$

This guarantees that (F1) holds with α replaced by β. Thus (3.2) has a unique local (forward) in time X^β solution for any $u_0 \in X^\beta$ and $\sigma \in \mathbb{R}$. Since these solutions are X^α solutions, they are also global (forward) in time. Hence we can consider our process $\{U(\tau,\sigma)\colon \tau \geq \sigma\}$ on X^β. We define

$$B = \bigcup_{\tau\in\mathcal{T}} U(\tau,\tau-\widetilde{T}_{B_0})B_0$$

and see that B pullback absorbs bounded subsets of X^β uniformly in time, since

$$\forall_{B_1\subset X^\beta,\text{bounded}}\exists_{T_{B_1}>0}\forall_{t\geq T_{B_1}} \bigcup_{\tau\in\mathcal{T}} U(\tau,\tau-t)B_1 \subset B. \tag{3.6}$$

Furthermore, from (3.1) and (3.3) it follows that B is a bounded subset of X^β. Indeed, for $\tau \in \mathcal{T}$ and $u \in B_0$ we have

$$\left\| U(\tau,\tau-\widetilde{T}_{B_0})u \right\|_{X^\beta} \leq \frac{c_{\beta-\alpha}}{\widetilde{T}_{B_0}^{\beta-\alpha}}\, \|u\|_{X^\alpha} +$$

$$+\int_0^{\widetilde{T}_{B_0}} \frac{c_\beta}{(\widetilde{T}_{B_0}-s)^\beta} \left\| F(s+\tau-\widetilde{T}_{B_0}, U(s+\tau-\widetilde{T}_{B_0}, \tau-\widetilde{T}_{B_0})u) \right\|_X ds.$$

Note that by (F1), (F2) and (F3b) for $s \in [0, \widetilde{T}_{B_0}]$ we have

$$\left\| F(s+\tau-\widetilde{T}_{B_0}, U(s+\tau-\widetilde{T}_{B_0}, \tau-\widetilde{T}_{B_0})u) \right\|_X \leq L\left(Q(2R_0)+R_0\right)+M,$$

where $L = L(\widetilde{T}_{B_0}, B^{X^\alpha}(0, Q(2R_0) + R_0))$, since $s + \tau - \widetilde{T}_{B_0} \in [\tau - \widetilde{T}_{B_0}, \tau]$ and

$$\left\| U(s + \tau - \widetilde{T}_{B_0}, \tau - \widetilde{T}_{B_0})u \right\|_{X^\alpha} \leq Q(2R_0) + R_0.$$

Summarizing, we obtain

$$\left\| U(\tau, \tau - \widetilde{T}_{B_0})u \right\|_{X^\beta} \leq \frac{2c_{\beta-\alpha}R_0}{\widetilde{T}_{B_0}^{\beta-\alpha}} + c_\beta(L(Q(2R_0) + R_0) + M)\frac{\widetilde{T}_{B_0}^{1-\beta}}{1-\beta},$$

which proves the boundedness of B in X^β. Let $R_B > 0$ be such that $B \subset B^{X^\beta}(0, R_B)$.

Therefore, we have verified assumptions (A1)–(A2) with $V = X^\beta$. In a series of lemmas we show that (F1)–(F3) guarantee that the process is continuous on $\mathrm{cl}_{X^\beta} B$ and (A3)–(A5) and (A10) hold automatically with $V = X^\beta$, $W = X^\alpha$.

First we verify that the smoothing property (A3) holds.

Lemma 3.1. *Under the above assumptions we have*

$$\exists_{\kappa_{\alpha,\beta}>0}\forall_{u_1,u_2\in\mathrm{cl}_{X^\beta} B} \quad \sup_{\tau\in\mathcal{T}} \|U(\tau, \tau - T_B)u_1 - U(\tau, \tau - T_B)u_2\|_{X^\beta} \leq \kappa_{\alpha,\beta} \|u_1 - u_2\|_{X^\alpha}, \quad (3.7)$$

where the constant $T_B > 0$ comes from (3.6).

Proof. Fix $t \in (0, T_B]$ and let $u_1, u_2 \in \mathrm{cl}_{X^\beta} B$ and $\tau \in \mathcal{T}$. We have by (3.3) and (F1)

$$\|U(t + \tau - T_B, \tau - T_B)u_1 - U(t + \tau - T_B, \tau - T_B)u_2\|_{X^\beta} \leq \left\|A^{\beta-\alpha}e^{-At}A^\alpha(u_1 - u_2)\right\|_X +$$

$$+ \int_0^t \left\| e^{-A(t-s)}(F(s + \tau - T_B, U(s + \tau - T_B, \tau - T_B)u_1) + \right.$$

$$\left. -F(s + \tau - T_B, U(s + \tau - T_B, \tau - T_B)u_2)) \right\|_{X^\beta} ds \leq \frac{c_{\beta-\alpha}}{t^{\beta-\alpha}} \|u_1 - u_2\|_{X^\alpha} +$$

$$+c_\beta c_{\alpha,\beta} L \int_0^t \frac{1}{(t-s)^\beta} \|U(s + \tau - T_B, \tau - T_B)u_1 - U(s + \tau - T_B, \tau - T_B)u_2\|_{X^\beta} ds$$

with L independent of t and τ, since $s + \tau - T_B \in [\tau - T_B, \tau]$ for $s \in [0, t]$ and by (F3b)

$$\|U(s + \tau - T_B, \tau - T_B)u_i\|_{X^\alpha} \leq Q(c_{\alpha,\beta}R_B) + R_0, \ i = 1, 2.$$

From the Volterra type inequality it follows that

$$\|U(t + \tau - T_B, \tau - T_B)u_1 - U(t + \tau - T_B, \tau - T_B)u_2\|_{X^\beta} \leq \frac{c_{T_B}}{t^{\beta-\alpha}} \|u_1 - u_2\|_{X^\alpha}, \ t \in (0, T_B].$$

Taking $t = T_B$ we obtain (3.7). □

Now we check that the process is continuous on $\mathrm{cl}_{X^\beta} B$ and (A10) holds if $\tau_0 < \infty$.

Lemma 3.2. *Under the above assumptions for any $\sigma \in \mathbb{R}$ and $T > 0$ there exists $c_{T,\sigma} > 0$ such that*

$$\|U(t+\sigma,\sigma)u_1 - U(t+\sigma,\sigma)u_2\|_{X^\beta} \leq \frac{c_{T,\sigma}}{t^{\beta-\alpha}} \|u_1 - u_2\|_{X^\alpha}, \ u_1, u_2 \in \mathrm{cl}_{X^\beta} B, \ t \in (0, T]. \quad (3.8)$$

In particular, the process $\{U(\tau,\sigma) \colon \tau \geq \sigma\}$ is continuous on $\mathrm{cl}_{X^\beta} B$ in the space X^β. Moreover, if $\tau_0 < \infty$ then we also have

$$\forall_{t>0} \exists_{k(t)>0} \forall_{u_1,u_2 \in B} \ \|U(t+\tau_0,\tau_0)u_1 - U(t+\tau_0,\tau_0)u_2\|_{X^\beta} \leq k(t) \|u_1 - u_2\|_{X^\beta}. \quad (3.9)$$

Proof. Let $\sigma \in \mathbb{R}$, $T > 0$ and fix $t \in (0, T]$ and $u_1, u_2 \in \mathrm{cl}_{X^\beta} B$. Following the argument as in Lemma 3.1 we see that

$$\|U(t+\sigma,\sigma)u_1 - U(t+\sigma,\sigma)u_2\|_{X^\beta} \leq \frac{c_{\beta-\alpha}}{t^{\beta-\alpha}} \|u_1 - u_2\|_{X^\alpha} +$$

$$+ c_\beta c_{\alpha,\beta} L \int_0^t \frac{1}{(t-s)^\beta} \|U(s+\sigma,\sigma)u_1 - U(s+\sigma,\sigma)u_2\|_{X^\beta} \, ds$$

with L depending now on T and σ, since for $s \in [0,t]$ we have $s + \sigma \in [\sigma, \sigma + T]$ and by (F3c)

$$\|U(s+\sigma,\sigma)u_i\|_{X^\alpha} \leq \widetilde{Q}_{T,\sigma}(c_{\alpha,\beta} R_B) + R_{T,\sigma}, \ i = 1, 2.$$

From the Volterra type inequality it follows that

$$\|U(t+\sigma,\sigma)u_1 - U(t+\sigma,\sigma)u_2\|_{X^\beta} \leq \frac{c_{T,\sigma}}{t^{\beta-\alpha}} \|u_1 - u_2\|_{X^\alpha}, \ t \in (0, T]$$

with $c_{T,\sigma} > 0$ depending only on B, T, σ, R_B, α and β. □

The lemma below shows that assumption (A4) is also satisfied.

Lemma 3.3. *Under the above assumptions there exist $0 < \theta < 1$ and $\lambda_1 > 0$ such that*

$$\sup_{\tau \in \mathcal{T}} \|U(\tau, \tau - T_B)u - U(\tau - t_1, \tau - t_1 - T_B)u\|_{X^\alpha} \leq \lambda_1 t_1^\theta, \ t_1 \in [0, T_B], \ u \in B, \quad (3.10)$$

where $T_B > 0$ is defined in (3.6) and θ comes from (F1) and is specified in (3.15).

Proof. Fix $\tau \in \mathcal{T}$, $t_1 \in [0, T_B]$, $u \in B$ and let $t \in (0, T_B]$. We want to estimate

$$\|U(\tau, \tau - t)u - U(\tau - t_1, \tau - t_1 - t)u\|_{X^\alpha}.$$

To this end, note that $\widetilde{u}(\rho) = U(\rho - t, \tau - t)u$, $\rho \geq \tau$, is the solution of

$$\begin{cases} \widetilde{u}_\rho + A\widetilde{u} = F(\rho - t, \widetilde{u}(\rho)), \ \rho > \tau, \\ \widetilde{u}(\tau) = u. \end{cases} \tag{3.11}$$

Denoting the evolution process corresponding to the equation in (3.11), we obtain

$$\widetilde{U}(\rho, \tau)u = e^{-A(\rho - \tau)}u + \int_\tau^\rho e^{-A(\rho - s)} F(s - t, \widetilde{U}(s, \tau)u)ds, \ \rho \geq \tau. \tag{3.12}$$

Since $\widetilde{U}(\rho, \tau)u = U(\rho - t, \tau - t)u$, we would like to set $\rho = \tau + t$.

Similarly, we note that $\overline{u}(\rho) = U(\rho - t_1 - t, \tau - t_1 - t)u$, $\rho \geq \tau$, is the solution of

$$\begin{cases} \overline{u}_\rho + A\overline{u} = F(\rho - t_1 - t, \overline{u}(\rho)), \ \rho > \tau, \\ \overline{u}(\tau) = u. \end{cases} \tag{3.13}$$

Thus denoting by $\overline{U}$ the evolution process corresponding to the equation in (3.13), we get

$$\overline{U}(\rho, \tau)u = e^{-A(\rho - \tau)}u + \int_\tau^\rho e^{-A(\rho - s)} F(s - t_1 - t, \overline{U}(s, \tau)u)ds, \ \rho \geq \tau. \tag{3.14}$$

Since $\overline{U}(\rho, \tau)u = U(\rho - t_1 - t, \tau - t_1 - t)u$, we would like to set $\rho = \tau + t$.

Using (F1) we estimate

$$\begin{aligned} \|U(\tau, \tau - t)u - U(\tau - t_1, \tau - t_1 - t)u\|_{X^\alpha} &= \left\|\widetilde{U}(t + \tau, \tau)u - \overline{U}(t + \tau, \tau)u\right\|_{X^\alpha} \leq \\ &\leq \int_0^t \left\|e^{-A(t-s)}[F(s + \tau - t, \widetilde{U}(s, \tau)u) - F(s - t_1 - t, \overline{U}(s, \tau)u)]\right\|_{X^\alpha} ds \leq \\ &\leq \int_0^t \frac{c_\alpha L}{(t-s)^\alpha} \left(t_1^\theta + \left\|\widetilde{U}(s + \tau, \tau)u - \overline{U}(s + \tau, \tau)u\right\|_{X^\alpha}\right) ds \leq \\ &\leq \frac{c_\alpha L}{1 - \alpha} T_B^{1-\alpha} t_1^\theta + \int_0^t \frac{c_\alpha L}{(t-s)^\alpha} \left\|\widetilde{U}(s + \tau, \tau)u - \overline{U}(s + \tau, \tau)u\right\|_{X^\alpha} ds, \end{aligned}$$

where

$$\theta = \theta(B^{X^\alpha}(0, Q(c_{\alpha,\beta} R_B) + R_0)) \ \text{ and } \ L = L(2T_B, B^{X^\alpha}(0, Q(c_{\alpha,\beta} R_B) + R_0)), \tag{3.15}$$

since $s+\tau-t\in[\tau-t,\tau]\subset[\tau-2T_B,\tau]$ and $s+\tau-t_1-t\in[\tau-t_1-t,\tau-t_1]\subset[\tau-2T_B,\tau]$ for $s\in[0,t]$ and by (F3b)

$$\left\|\widetilde{U}(s+\tau,\tau)u\right\|_{X^\alpha}=\|U(s+\tau-t,\tau-t)u\|_{X^\alpha}\leq Q(c_{\alpha,\beta}R_B)+R_0$$

and

$$\left\|\overline{U}(s+\tau,\tau)u\right\|_{X^\alpha}=\|U(s+\tau-t_1-t,\tau-t_1-t)u\|_{X^\alpha}\leq Q(c_{\alpha,\beta}R_B)+R_0.$$

By the Volterra type inequality it follows that

$$\left\|\widetilde{U}(t+\tau,\tau)u-\overline{U}(t+\tau,\tau)\right\|_{X^\alpha}\leq\lambda_1 t_1^\theta,\ t\in(0,T_B],$$

where $\lambda_1>0$ depends only on α, T_B, L and is independent of τ and t. Taking $t=T_B$ in the above inequality, we obtain (3.10). □

Let us now recall an auxiliary result from [5, the formula (2.2.2)].

Lemma 3.4. *For any $\gamma\in(0,1]$ we have*

$$\left\|(e^{-At}-I)v\right\|_X\leq\frac{c_{1-\gamma}}{\gamma}t^\gamma\|v\|_{X^\gamma},\ v\in X^\gamma,\ t\geq 0. \tag{3.16}$$

Proof. Observe that

$$(e^{-At}-I)v=\int_0^t\frac{d}{ds}(e^{-As}v)ds=-\int_0^t Ae^{-As}vds=-\int_0^t A^{1-\gamma}A^\gamma e^{-As}vds,\ v\in X.$$

If $v\in X^\gamma$ we get

$$\left\|(e^{-At}-I)v\right\|_X\leq\int_0^t\left\|A^{1-\gamma}e^{-As}\right\|_{\mathcal{L}(X,X)}\|A^\gamma v\|_X ds\leq c_{1-\gamma}\|v\|_{X^\gamma}\int_0^t\frac{ds}{s^{1-\gamma}}=c_{1-\gamma}\|v\|_{X^\gamma}\frac{t^\gamma}{\gamma}.$$

This proves the claim. □

We are now in a position to prove that (A5) also holds in our setting.

Lemma 3.5. *Under the above assumptions for any $\gamma\in(0,1)$ there exists $\lambda_2>0$ such that*

$$\sup_{\tau\in\mathcal{T}}\|U(\tau,\tau-t_1)u-U(\tau,\tau-t_2)u\|_{X^\alpha}\leq\lambda_2|t_1-t_2|^\gamma,\ t_1,t_2\in[T_B,2T_B],\ u\in B, \tag{3.17}$$

where $T_B>0$ is defined in (3.6).

Proof. Let $t_1, t_2 \in [T_B, 2T_B]$. It is enough to consider $t_1 \neq t_2$ and by symmetry it suffices to suppose that $t_1 < t_2$. Thus we have $t_2 = t_1 + h$ with $h > 0$. Note that $0 < h < t_1 + h = t_2 \leq 2T_B$. We fix $u \in B$ and $\tau \in \mathcal{T}$. We consider $t \in (0, 2T_B - h]$ and want to estimate $\|U(\tau, \tau - t - h)u - U(\tau, \tau - t)u\|_{X^\alpha}$. To this end, note that $\widetilde{u}(\rho) = U(\rho - t, \tau - t)u$, $r \geq \tau$, is the solution of (3.11) and the evolution process $\widetilde{U}$ corresponding to the equation in (3.11) satisfies (3.12). Since $\widetilde{U}(\rho, \tau)u = U(\rho - t, \tau - t)u$, we would like to set $\rho = \tau + t$.

Note also that $\widehat{u}(\rho) = U(\rho - t - h, \tau - t - h)u$, $\rho \geq \tau$, is the solution of

$$\begin{cases} \widehat{u}_\rho + A\widehat{u} = F(\rho - t - h, \widehat{u}(\rho)), \ \rho > \tau, \\ \widehat{u}(\tau) = u. \end{cases} \tag{3.18}$$

Thus denoting by $\widehat{U}$ the evolution process corresponding to the equation in (3.18), we obtain

$$\widehat{U}(\rho, \tau)u = e^{-A(\rho - \tau)}u + \int_\tau^\rho e^{-A(\rho - s)} F(s - t - h, \widehat{U}(s, \tau)u) ds, \ \rho \geq \tau. \tag{3.19}$$

Since $\widehat{U}(\rho, \tau)u = U(\rho - t - h, \tau - t - h)u$, we would like to set here $\rho = \tau + t + h$.

Concluding, we have

$$\|U(\tau, \tau - t - h)u - U(\tau, \tau - t)u\|_{X^\alpha} \leq \left\|e^{-A(t+h)}u - e^{-At}u\right\|_{X^\alpha} +$$
$$+ \left\| \int_\tau^{\tau+t+h} e^{-A(\tau+t+h-s)} F(s - t - h, \widehat{U}(s, \tau)u) ds - \int_\tau^{\tau+t} e^{-A(\tau+t-s)} F(s - t, \widetilde{U}(s, \tau)u) ds \right\|_{X^\alpha}.$$

Changing the variables we get

$$\|U(\tau, \tau - t - h)u - U(\tau, \tau - t)u\|_{X^\alpha} \leq \left\|(e^{-Ah} - I)e^{-At}u\right\|_{X^\alpha} +$$
$$+ \int_0^h \left\|e^{-A(t+h-s)} F(s + \tau - t - h, \widehat{U}(s + \tau, \tau)u)\right\|_{X^\alpha} ds +$$
$$+ \int_0^t \left\|e^{-A(t-s)} (F(s + \tau - t, \widehat{U}(s + h + \tau, \tau)u) - F(s + \tau - t, \widetilde{U}(s + \tau, \tau)u))\right\|_{X^\alpha} ds.$$

By Lemma 3.4 we have for any $\gamma \in (0, 1)$

$$\left\|(e^{-Ah} - I)e^{-At}u\right\|_{X^\alpha} \leq \frac{c_{1-\gamma} c_\gamma}{\gamma} \frac{h^\gamma}{t^\gamma} \|u\|_{X^\alpha} \leq \frac{c_{1-\gamma} c_\gamma c_{\alpha,\beta} R_B (2T_B)^{\delta-\gamma}}{\gamma} \frac{h^\gamma}{t^\delta}, \tag{3.20}$$

where $\delta = \max\{\alpha, \gamma\} < 1$. Note that for $s \in [0, h]$ we have by (F3b)

$$\left\|\widehat{U}(s + \tau, \tau)u\right\|_{X^\alpha} = \|U(s + \tau - t - h, \tau - t - h)u\|_{X^\alpha} \leq Q(c_{\alpha,\beta} R_B) + R_0.$$

Moreover, we know that $s+\tau-t-h \in [\tau-t-h, \tau-t] \subset [\tau-t-2T_B, \tau-t]$. Thus we get by (F1) and (F2)

$$\left\| F(s+\tau-t-h, \widehat{U}(s+\tau,\tau)u) \right\|_X \leq L\left(Q(c_{\alpha,\beta}R_B)+R_0\right)+M=\widetilde{M}$$

with $L = L(2T_B, B^{X^\alpha}(0, Q(c_{\alpha,\beta}R_B)+R_0))$ and in consequence

$$\begin{aligned}\int_0^h \left\| e^{-A(t+h-s)} F(s+\tau-t-h, \widehat{U}(s+\tau,\tau)u) \right\|_{X^\alpha} ds \leq c_\alpha \widetilde{M} \int_0^h \frac{ds}{(t+h-s)^\alpha} \leq \\ \leq c_\alpha \widetilde{M} \frac{h}{t^\alpha} \leq c_\alpha \widetilde{M} (2T_B)^{1-\gamma} (2T_B)^{\delta-\alpha} \frac{h^\gamma}{t^\delta}.\end{aligned} \tag{3.21}$$

Finally, using (F1) we estimate

$$\int_0^t \left\| e^{-A(t-s)} (F(s+\tau-t, \widehat{U}(s+h+\tau,\tau)u) - F(s+\tau-t, \widetilde{U}(s+\tau,\tau)u)) \right\|_{X^\alpha} ds \leq$$

$$\leq c_\alpha L \int_0^t \frac{1}{(t-s)^\alpha} \left\| \widehat{U}(s+h+\tau,\tau)u - \widetilde{U}(s+\tau,\tau)u \right\|_{X^\alpha} ds,$$

since for $s \in [0,t]$ we have $s+\tau-t \in [\tau-t, \tau] \subset [\tau-2T_B, \tau]$ and by (F3b)

$$\left\| \widehat{U}(s+h+\tau,\tau)u \right\|_{X^\alpha} = \|U(s+\tau-t, \tau-t-h)u\|_{X^\alpha} \leq Q(c_{\alpha,\beta}R_B)+R_0$$

and

$$\left\| \widetilde{U}(s+\tau,\tau)u \right\|_{X^\alpha} = \|U(s+\tau-t, \tau-t)u\|_{X^\alpha} \leq Q(c_{\alpha,\beta}R_B)+R_0.$$

Combining the above estimate with (3.20) and (3.21) we get for any $\gamma \in (0,1)$

$$\begin{aligned}\left\| \widehat{U}(t+\tau+h,\tau)u - \widetilde{U}(t+\tau,\tau)u \right\|_{X^\alpha} \leq \\ \leq \tilde{c}\frac{h^\gamma}{t^\delta} + c_\alpha L \int_0^t \frac{1}{(t-s)^\alpha} \left\| \widehat{U}(s+\tau+h,\tau)u - \widetilde{U}(s+\tau,\tau)u \right\|_{X^\alpha} ds,\ t \in (0, 2T_B-h].\end{aligned} \tag{3.22}$$

Considering the function

$$\begin{aligned} y(t) = \left\| \widehat{U}(t+\tau+h,\tau)u - \widetilde{U}(t+\tau,\tau)u \right\|_{X^\alpha} = \\ = \|U(\tau, \tau-t-h)u - U(\tau,\tau-t)u\|_{X^\alpha},\ t \in [0, 2T_B-h],\end{aligned}$$

we see that

$$y(t) \leq \widetilde{c}\frac{h^\gamma}{t^\delta} + c_\alpha L \int_0^t \frac{1}{(t-s)^\alpha} y(s) ds, \ t \in (0, 2T_B - h].$$

By the Volterra type inequality we get

$$y(t) \leq \widehat{c}\frac{h^\gamma}{t^\delta}, \ t \in (0, 2T_B - h],$$

where $\widehat{c} > 0$ depends on γ, but is independent of t and h. We conclude that

$$\|U(\tau, \tau - t - h)u - U(\tau, \tau - t)u\|_{X^\alpha} \leq \frac{\widehat{c}}{(T_B)^\delta} h^\gamma, \ t \in [T_B, 2T_B - h].$$

In particular, we have for any $\gamma \in (0, 1)$

$$\|U(\tau, \tau - t_2)u - U(\tau, \tau - t_1)u\|_{X^\alpha} \leq \frac{\widehat{c}}{(T_B)^\delta} (t_2 - t_1)^\gamma.$$

This ends the proof. □

Using Lemmas 3.1, 3.2, 3.3 and 3.5 we apply Theorem 2.1, Corollary 2.4 and Propositions 2.3, 2.5 and infer the following result.

Theorem 3.6. *Let $A\colon X \supset D(A) \to X$ be a positive sectorial operator in a Banach space X with compact resolvent. Given $\alpha \in [0, 1)$ let the function $F\colon \mathbb{R} \times X^\alpha \to X$ satisfy assumption* (F1)*. Moreover, assume that for some $\tau_0 \leq \infty$ the function F satisfies* (F2) *and conditions* (F3a)–(F3c) *hold giving rise to the dissipative evolution process $\{U(\tau, \sigma)\colon \tau \geq \sigma,\ \tau, \sigma \in \mathbb{R}\}$ in X^α for the problem* (3.2).

Then for any $\beta \in (\alpha, 1)$ there exists a family $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ of nonempty compact subsets of X^β with the following properties:

(i) $\{\mathcal{M}(\tau)\colon \tau \in \mathbb{R}\}$ is positively invariant under the process $U(\tau, \sigma)$, i.e.

$$U(\tau, \sigma)\mathcal{M}(\sigma) \subset \mathcal{M}(\tau), \ \tau \geq \sigma,$$

(ii) $\mathcal{M}(\tau)$ has a finite fractal dimension in X^β uniformly with respect to $\tau \in \mathbb{R}$, i.e.

$$d_f^{X^\alpha}(\mathcal{M}(\tau)) \leq d_f^{X^\beta}(\mathcal{M}(\tau)) \leq \frac{1}{\theta}\left(1 + \log_2(1 + c_{\alpha,\beta}\kappa_{\alpha,\beta})\right) + \log_2 N^{X^\alpha}_{\frac{1}{4\kappa_{\alpha,\beta}}}(B^{X^\beta}(0, 1)), \ \tau \in \mathbb{R},$$

where $0 < \theta < 1$ is specified in Lemma 3.3, $c_{\alpha,\beta} > 0$ is the embedding constant from (3.5) *and $\kappa_{\alpha,\beta} > 0$ is taken from* (3.7), *(iii)* $\{\mathcal{M}(\tau)\colon \tau\in\mathbb{R}\}$ *has the property of pullback exponential attraction, i.e.*

$$\exists_{\varphi>0}\forall_{B_1\subset X^\beta,\ bounded}\forall_{\tau\in\mathbb{R}}\quad \lim_{t\to\infty} e^{\varphi t}\,\mathrm{dist}_{X^\beta}(U(\tau,\tau-t)B_1,\mathcal{M}(\tau))=0$$

and if $\tau_0=\infty$*, the pullback attraction is uniform with respect to* τ

$$\exists_{\varphi>0}\forall_{B_1\subset X^\beta,\ bounded}\quad \lim_{t\to\infty} e^{\varphi t}\sup_{\tau\in\mathbb{R}}\mathrm{dist}_{X^\beta}(U(\tau,\tau-t)B_1,\mathcal{M}(\tau))=0.$$

Furthermore, the pullback exponential attractor $\{\mathcal{M}(\tau)\colon \tau\in\mathbb{R}\}$ *contains a (finite dimensional) pullback global attractor* $\{\mathcal{A}(\tau)\colon \tau\in\mathbb{R}\}$*, i.e. a family of nonempty compact subsets of* X^β*, invariant under the process* $\{U(\tau,\sigma)\colon \tau\geq\sigma\}$

$$U(\tau,\sigma)\mathcal{A}(\sigma)=\mathcal{A}(\tau),\ \tau\geq\sigma,$$

and pullback attracting all bounded subsets of X^β

$$\forall_{B_1\subset X^\beta,\ bounded}\forall_{\tau\in\mathbb{R}}\quad \lim_{t\to\infty}\mathrm{dist}_{X^\beta}(U(\tau,\tau-t)B_1,\mathcal{A}(\tau))=0.$$

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