TRANSVERSALITY IN SCALAR REACTION-DIFFUSION EQUATIONS ON A CIRCLE

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Abstract. We prove that stable and unstable manifolds of hyperbolic periodic orbits for general scalar reaction-diffusion equations on a circle always intersect transversally. The argument also shows that for a periodic orbit there are no homoclinic connections. The main tool used in the proofs is Matano's zero number theory dealing with the Sturm nodal properties of the solutions.

1. Introduction

We consider the scalar reaction-diffusion equation of the form

$$u_t = u_{xx} + f(x, u, u_x) \tag{1.1}$$

for one real variable $u = u(t, x)$ on a circle $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In other words, we consider (1.1) together with periodic boundary conditions

$$u(t, 0) = u(t, 2\pi), \quad u_x(t, 0) = u_x(t, 2\pi)$$

and discuss (1.1) with initial condition

$$u(0, x) = u_0(x),\ x \in S^1. \tag{1.2}$$

Below we use suitable assumptions on f so that the problem (1.1), (1.2) defines a global semiflow in $X^\alpha = H^{2\alpha}(S^1)$, $\frac{3}{4} < \alpha < 1$, for which there exists a global attractor, i.e. a nonempty compact invariant set attracting every bounded subset of X^α. The existence of global attractors and other qualitative properties of the dynamical systems generated by reaction-diffusion equations under various boundary conditions have been extensively considered in the literature. For the interested reader we mention the following excellent surveys [11, 25, 26].

It has been shown in [3, 19, 20] that time periodic solutions may appear in the description of dynamics of (1.1). In case the function f does not explicitly depend on the x variable, i.e. $f = f(u, u_x)$, it was proved (see [3, 8] for details) that the global attractor consists exclusively of equilibria, orbits of periodic solutions of the form

$$u(t, x) = v(x - ct),\ t \in \mathbb{R},\ x \in S^1 \text{ with some } c \neq 0, \tag{1.3}$$

(called *rotating waves*) and heteroclinic orbits connecting the above-mentioned critical elements, when all are assumed hyperbolic. Moreover, necessary and sufficient

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conditions for the existence of heteroclinic orbits between critical elements were established in [8]. However, as it follows from [29], in case of general x-dependent nonlinearities homoclinic orbits may belong to the attractor as well and the periodic solutions do not have to be, in general, of the form (1.3). This happens due to the lack of S^1-equivariance, which was a crucial property used in [8] to exclude homoclinic connections.

One of the most important results concerning (1.1) is the Poincaré-Bendixson type theorem proved by Fiedler and Mallet-Paret in [7, Theorem 1] (see also [21]). It states that if $u_0 \in X^\alpha$, $\frac{3}{4} < \alpha < 1$, then either its ω-limit set $\omega(u_0)$ consists in precisely one periodic orbit or $\alpha(v_0)$ and $\omega(v_0)$ are subsets of the set of all equilibria for any $v_0 \in \omega(u_0)$.

In this paper we investigate closely the situation when a bounded orbit from the global attractor connects two hyperbolic periodic orbits. First, we exclude the existence of a homoclinic connection for a hyperbolic periodic orbit (cf. [22]) in order to finally prove the main result of this paper stating that the intersection of the global unstable manifold of a hyperbolic periodic orbit Π^- with the local stable manifold of another hyperbolic periodic orbit Π^+ is always transversal, i.e.

$$W^u(\Pi^-) \pitchfork W^s_{loc}(\Pi^+).$$

The paper is organized as follows. In Section 2 we formulate the abstract Cauchy problem for (1.1)-(1.2) and using the theory from [13] we solve the problem locally. Further we obtain a priori and subordination estimates, which ensure that the solutions exist globally in time. The semiflow of global solutions constructed in this way is point dissipative and compact, thus has a compact global attractor. In Section 3 we examine the properties of the semiflow and the evolution system for the linearization around a given solution. Moreover, we recall the properties of the *zero number* of solutions of linear parabolic equations. Section 4 is devoted to the operator called a *period map* for a periodic orbit. We describe its spectrum and decompose the phase space according to the spectrum. We also recall the notions of local stable and unstable manifolds of a hyperbolic periodic orbit and list their properties. In Section 5 we analyze the local stable manifold of a hyperbolic periodic orbit Π and show that for any $u_0 \notin \Pi$ from the local stable manifold of Π there exists $a \in \Pi$ such that $u(t; u_0) - p(t; a)$ tends exponentially to 0 as $t \to \infty$ and

$$z(u_0 - a) \geq i(\Pi) + 1 + \frac{1 + (-1)^{i(\Pi)}}{2}, \tag{1.4}$$

where $i(\Pi)$ denotes the total algebraic multiplicity of eigenvalues of the period map for Π outside the closed unit ball. Similarly, in Section 6 we investigate the global unstable manifold of a hyperbolic periodic orbit Π. We prove that for any $u_0 \notin \Pi$ from the global unstable manifold there exists $a \in \Pi$ such that $u(t; u_0) - p(t; a)$ tends exponentially to 0 as $t \to -\infty$ and

$$z(u_0 - a) \leq i(\Pi) - 1 + \frac{1 + (-1)^{i(\Pi)}}{2}. \tag{1.5}$$

In Section 7 we combine the estimates (1.4) and (1.5) and find, in particular, that there is no homoclinic connection for a hyperbolic periodic orbit. Finally, in Section 8 we follow the ideas from [5] and introduce filtrations of the phase space with respect to the asymptotic behavior of solutions for the linearized equation around an orbit connecting two hyperbolic periodic orbits. A proper choice of the spaces from the filtrations carefully combined with the corresponding zero number estimates for the functions from these spaces yields the transversality of the intersection of the stable and unstable manifolds of two hyperbolic periodic orbits. The transversal intersection of invariant manifolds of critical elements is one of the ingredients for genericity results (cf. e.g. Kupka-Smale theorem) or structural stability theorems (cf. [12, Chapter 10], [23]) in the theory of dynamical systems. In Section 9 we make some concluding remarks about structural stability for the semiflow generated by (1.1).

Under different boundary conditions many authors have considered problems of the same type as discussed here. For separated boundary conditions, the results of Henry [14] and Angenent [1] on the transversality of the stable and unstable manifolds of stationary solutions constitute obligatory references. A problem of this type has also been considered by Chen, Chen and Hale in [5] for nonautonomous time periodic equations with $f = f(t, x, u)$ under Dirichlet boundary conditions. The effect of radial symmetry on the transversality of stable and unstable manifolds of equilibria for problems defined on symmetric domains in $\mathbb{R}^n$ has been studied by Poláčik in [24]. For special classes of ordinary differential equations on $\mathbb{R}^n$, Fusco and Oliva have considered the transversality between stable and unstable manifolds of equilibria and periodic orbits (see [9, 10]). Here we extend the results of [10] realizing the plans sketched by these authors for further possible extensions.

2. Abstract setting of the problem and existence of the global attractor

Assume that $f\colon S^1\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying the following conditions

$$\text{there exist } 0 \le \gamma < 2 \text{ and a continuous function } k\colon [0,\infty) \to [0,\infty) \text{ such that } |f(x,y,z)| \le k(r)(1+|z|^\gamma),\ (x,y,z) \in S^1 \times [-r,r] \times \mathbb{R} \text{ for each } r > 0, \tag{2.1}$$

$$yf(x,y,0) < 0,\ (x,y) \in S^1 \times \mathbb{R},\ |y| \ge K \text{ for some } K > 0. \tag{2.2}$$

In this paper we are going to use fractional Sobolev spaces of 2π-periodic functions $H^s(S^1)$, $s > 0$, and their properties (cf. [27, Appendix A]). Among others we will frequently use the Sobolev embedding

$$H^s(S^1) \hookrightarrow C^1(S^1) \text{ if } s > \frac{3}{2}.$$

We consider the operator $A\colon L^2(S^1) \supset H^2(S^1) \to L^2(S^1)$ given by

$$Au = -u_{xx} + u,\ u \in H^2(S^1).$$

Since A is a positive definite selfadjoint operator, it is a positive sectorial operator. Henceforth we consider fractional power spaces

$$X^\alpha = D(A^\alpha),\ \alpha \ge 0,$$

with norms $\|u\|_{X^\alpha} = \|A^\alpha u\|_{L^2(S^1)}$, $u \in X^\alpha$ (cf. [13, Section 1.4]). Note that $X^0 = L^2(S^1)$, $X^1 = H^2(S^1)$ and

$$X^\alpha = [L^2(S^1), H^2(S^1)]_\alpha = H^{2\alpha}(S^1),\ \alpha \in (0,1)$$

(see [33, Section 1.18.10] and [30, Section 3.6.1]). Since $H^2(S^1)$ is compactly embedded in $L^2(S^1)$, it follows that A has a compact resolvent.

We rewrite (1.1), (1.2) as an abstract Cauchy problem in X^0

$$\begin{cases} u_t + Au = F(u), \\ u(0) = u_0, \end{cases} \tag{2.3}$$

where F is the Nemycki operator corresponding to

$$F(u)(x) = f(x, u(x), u_x(x)) + u(x),\ x \in S^1.$$

For a fixed $\alpha \in (\frac{3}{4}, 1)$, F takes X^α into X^0 and is Lipschitz continuous on bounded subsets of X^α.

By the theory presented in [13] it follows that for each $u_0 \in X^\alpha$ there exists a unique local forward X^α solution defined on a maximal interval of existence, i.e.

$$u \in C([0, \tau_{u_0}), X^\alpha) \cap C^1((0, \tau_{u_0}), X^0) \cap C((0, \tau_{u_0}), X^1)$$

and satisfies (2.3) on $[0, \tau_{u_0})$ in X^0. Moreover, either $\tau_{u_0} = \infty$ or

$$\tau_{u_0} < \infty \text{ and } \limsup_{t \to \tau_{u_0}} \|u(t; u_0)\|_{X^\alpha} = \infty.$$

Using assumption (2.2) and the maximum principle it follows that if for some $R \geq 0$ we have $\|u_0\|_{L^\infty(S^1)} \leq K+R$, then there exists a positive constant $\delta = \delta(K, R)$ such that

$$\|u(t; u_0)\|_{L^\infty(S^1)} \leq K + Re^{-\delta t},\ t \in [0, \tau_{u_0}). \tag{2.4}$$

This implies that each forward X^α solution is bounded in $L^\infty(S^1)$.

Observe that by using Young's inequality we can assume without loss of generality that $1 < \gamma < 2$ in (2.1). Applying (2.4) to (2.1), we obtain

$$\|F(u(t; u_0))\|_{X^0} \leq c(\|u_0\|_{L^\infty(S^1)})(1 + \|u(t; u_0)\|^\gamma_{W^{1,2\gamma}(S^1)}),\ t \in (0, \tau_{u_0}). \tag{2.5}$$

Fix $r > \max\{\frac{\gamma-1}{2-\gamma}, 2\}$ and let $\beta \geq \alpha$ be such that

$$\frac{1}{2}\left(\gamma + \frac{1}{r}(\gamma - 1)\right) < \beta < 1.$$

Then for a chosen

$$\theta \in \left(\frac{1 + \frac{1}{r} - \frac{1}{2\gamma}}{2\beta + \frac{1}{r} - \frac{1}{2}}, \frac{1}{\gamma} \right)$$

such that $\theta \geq \frac{r-2\gamma}{\gamma(r-2)}$ the following interpolation inequality is satisfied

$$\|u(t; u_0)\|_{W^{1,2\gamma}(S^1)} \leq c_\theta \|u(t; u_0)\|^\theta_{H^{2\beta}(S^1)} \|u(t; u_0)\|^{1-\theta}_{L^r(S^1)},\ t \in (0, \tau_{u_0}), \tag{2.6}$$

due to the embedding (cf. [30, Section 3.6.1])

$$[L^r(S^1), H^{2\beta}(S^1)]_\theta \hookrightarrow W^{1,2\gamma}(S^1).$$

Since $X^\beta = H^{2\beta}(S^1)$ and $L^\infty(S^1)$ is continuously embedded in $L^r(S^1)$, we obtain from (2.5), (2.6) and again (2.4) the following subordination condition

$$\|F(u(t;u_0))\|_{X^0} \le \widetilde{c}(\|u_0\|_{L^\infty(S^1)})(1+\|u(t;u_0)\|_{X^\beta}^{\theta\gamma}),\ t\in(0,\tau_{u_0}), \tag{2.7}$$

with $\theta\gamma<1$.

By [6, Theorem 3.1.1] it follows that each forward X^β solution of (2.3) exists globally in time ($\tau_{u_0}=\infty$) and denoting by $u(\cdot;u_0)$ this solution,

$$S(t)u_0 = u(t;u_0),\ t\ge 0,$$

defines a C^0 semiflow of global forward X^β solutions having positive semiorbits of bounded sets bounded. In fact the above statement holds for X^α solutions, since we have proved the existence of local forward X^α solutions and we know that, by definition, they enter X^1 immediately and for $t>0$ we may consider them as X^β solutions that exist globally in time.

Note that now (2.4) implies that the estimate of solutions in $L^\infty(S^1)$ is asymptotically independent of initial conditions

$$\limsup_{t\to\infty}\|u(t;u_0)\|_{L^\infty(S^1)}\le K.$$

Then by [6, Theorem 4.1.1] there exists a constant $K_1>0$ such that

$$\limsup_{t\to\infty}\|u(t;u_0)\|_{X^\alpha}\le K_1. \tag{2.8}$$

Therefore the semiflow $\{S(t)\colon t\ge 0\}$ is point dissipative in X^α. Note also that $S(t)$ is a compact map on X^α for each $t>0$ by [6, Theorem 3.3.1], since A has a compact resolvent. Thus the semiflow $\{S(t)\colon t\ge 0\}$ has a global attractor $\mathcal{A}$ in X^α. We recall that $\mathcal{A}$ is then the union of all bounded orbits.

3. Properties of the semiflow

Fix $s\in\mathbb{R}$. Let $u(\cdot;s,\xi)$ be the global forward X^α solution of the problem

$$\begin{cases} u_t + Au = F(u),\ t>s,\\ u(s)=\xi. \end{cases} \tag{3.1}$$

Since f is C^2, it follows from [13, Theorem 3.4.4, Corollary 3.4.6] that the function

$$(s,\infty)\times X^\alpha \ni (t,\xi)\mapsto u(t;s,\xi)\in X^\alpha$$

is continuously differentiable. Moreover, for each fixed $t\ge s$ the function

$$X^\alpha\ni\xi\mapsto u(t;s,\xi)\in X^\alpha$$

is also continuously differentiable and, for each $\zeta\in X^\alpha$, its derivative in the ζ-direction given by

$$w(t;s,\zeta) = D_\xi u(t;s,\xi)\zeta\in X^\alpha,\ t\ge s,$$

is a unique global forward X^α solution of the linear variational problem

$$\begin{cases} w_t + Aw = D_uF(u(t;s,\xi))w,\ t>s,\\ w(s)=\zeta. \end{cases} \tag{3.2}$$

Taking into account the regularity of X^α solutions we see that (3.2) is the abstract equivalent of

$$\begin{cases} w_t = w_{xx} + b(t,x)w + d(t,x)w_x, \ t > s, \ x \in S^1, \\ w(s,x) = \zeta(x), \ x \in S^1, \end{cases} \tag{3.3}$$

where

$$b(t,x) = f_y(x, u(t;s,\xi)(x), u_x(t;s,\xi)(x)), \ t > s, \ x \in S^1,$$
$$d(t,x) = f_z(x, u(t;s,\xi)(x), u_x(t;s,\xi)(x)), \ t > s, \ x \in S^1.$$

We define the evolution system

$$T(t,s)\zeta = w(t;s,\zeta), \ t \geq s, \ \zeta \in X^\alpha, \tag{3.4}$$

where $w(t;s,\zeta)$ is a unique global forward X^α solution of (3.2). Note that we have $T(t,s)\zeta = D_\xi u(t;s,\xi)\zeta$, so it follows that $T(t,0)\zeta = (D_{u_0} S(t)u_0)\zeta$. Moreover, for $t > s$ the operator $T(t,s) \in \mathcal{L}(X^\alpha, X^\alpha)$ is compact in the Hilbert space X^α (see [13, Section 7.1]).

Below we prove the injectivity of the semiflow $\{S(t) \colon t \geq 0\}$ and the injectivity of $\{T(t,s) \colon t \geq s\}$.

To show that the semiflow is injective in X^α suppose that for some $u_1, u_2 \in X^\alpha$ and some $t_0 > 0$ we have

$$S(t_0)u_1 = S(t_0)u_2.$$

Define $v(t) = S(t)u_1 - S(t)u_2$, $t \geq 0$. Then we have

$$\begin{cases} v_t + Av = F(S(t)u_1) - F(S(t)u_2), \ t > 0, \\ v(0) = u_1 - u_2. \end{cases} \tag{3.5}$$

Moreover, we know that $v(t_0) = 0$. Note that A is a positive definite selfadjoint operator in the Hilbert space $X^0 = L^2(S^1)$ and $X^\alpha \hookrightarrow X^{\frac{1}{2}}$. Furthermore,

$$v \in C([0,\infty), X^\alpha) \cap C^1((0,\infty), X^0) \cap C((0,\infty), X^1)$$

and

$$\|F(S(t)u_1) - F(S(t)u_2)\|_{X^0} \leq L \|S(t)u_1 - S(t)u_2\|_{X^{\frac{1}{2}}} = L \|v(t)\|_{X^{\frac{1}{2}}}, \ t \in [0,\infty),$$

where L is a constant depending on

$$\sup_{t \in [0,\infty)} \|S(t)u_i\|_{X^\alpha} < \infty, \ i = 1,2.$$

By [6, Proposition 7.1.1] (see also [32, Lemmas 6.1,6.2]) we get

$$v(t) = 0, \ t \in [0, t_0].$$

In particular, we obtain $u_1 = u_2$. This proves the injectivity of the semiflow.

Suppose now that

$$T(t_0, s_0)\zeta = 0$$

for some $t_0 > s_0$ and $\zeta \in X^\alpha$. Define $w(t) = T(t + s_0, s_0)\zeta$, $t \geq 0$, and choose any $T_0 > t_0 - s_0$. Then we have

$$\begin{cases} w_t + Aw = D_u F(u(t + s_0; s_0, \xi))w, \ 0 < t \leq T_0, \\ w(0) = \zeta. \end{cases} \tag{3.6}$$

Moreover, we know that $w(t_0 - s_0) = 0$. For $t \in [0, T_0]$ we estimate

$$\|D_u F(u(t+s_0; s_0, \xi))w(t)\|_{X^0} \le C_1 \|w(t)\|_{X^0} + C_2 \|w(t)\|_{X^{\frac{1}{2}}} \le M \|w(t)\|_{X^{\frac{1}{2}}},$$

where C_1 and C_2 depend on

$$\sup_{(t,x)\in[0,T_0]\times S^1} |f_y(x, u(t+s_0; s_0, \xi)(x), u_x(t+s_0; s_0, \xi)(x))|,$$

$$\sup_{(t,x)\in[0,T_0]\times S^1} |f_z(x, u(t+s_0; s_0, \xi)(x), u_x(t+s_0; s_0, \xi)(x))|,$$

respectively. Thus the assumptions of [6, Proposition 7.1.1] are fulfilled again and

$$w(t) = 0, \ t \in [0, t_0 - s_0].$$

In particular, we obtain $\zeta = 0$. This proves the injectivity of $T(t,s)$, $t \ge s$.

Observe also that by the backward uniqueness of the adjoint equation of (3.3) (see [16]) the adjoint operator $T(t,s)^*$ is injective and by [13, Theorem 7.3.3] each operator $T(t,s)$, $t \ge s$, has a dense range.

In what follows we are going to use frequently the properties of the *zero number* of a C^1 function referring to the Sturm nodal properties of the solutions of (3.7) (see [31]) so successfully reintroduced by Matano (cf. [18]) as an essential tool for the description of the dynamics of scalar semilinear parabolic equations. We denote by $z(\varphi)$ the number of strict sign changes of a C^1 function $\varphi\colon S^1 \to \mathbb{R}$. Then, as a consequence of the maximum principle, the zero number has the following properties.

Lemma 3.1. *(*[20, Lemma 3.2],[2]*) Let $J \subset \mathbb{R}$ be an open interval and v be a nontrivial classical solution of the linear parabolic equation*

$$v_t = v_{xx} + b(t,x)v + d(t,x)v_x, \ t \in J, \ x \in S^1, \tag{3.7}$$

where b, b_x, b_t and d are bounded on any compact subset of $J \times S^1$, then the zero number of $v(t)$ has the following properties:

(i) *$z(v(t))$ is finite for any $t \in J$,*
(ii) *$z(v(t))$ is nonincreasing in t on J,*
(iii) *$z(v(t))$ drops strictly at $t = t_0$ if and only if*

$$S^1 \ni x \mapsto v(t_0)(x) \in \mathbb{R},$$

has a multiple zero.

Observe that the assertions of this lemma hold for the zero number of the difference of two different solutions for a scalar semilinear parabolic equation.

Lemma 3.2. *(*[20, Lemma 3.4]*) If u_1 and u_2 are two different X^α solutions of* (1.1) *defined on an open interval J, then $v(t) = u_1(t) - u_2(t)$, $t \in J$, satisfies the linear parabolic equation* (3.7) *with*

$$b(t,x) = \int_0^1 f_y(x, \theta u_1 + (1-\theta)u_2, \theta(u_1)_x + (1-\theta)(u_2)_x) d\theta,$$

$$d(t,x) = \int_0^1 f_z(x, \theta u_1 + (1-\theta)u_2, \theta(u_1)_x + (1-\theta)(u_2)_x) d\theta,$$

and the assertions of Lemma 3.1 hold.

4. The period map

Consider a periodic orbit Π with period $\omega > 0$ and choose a periodic point $a \in \Pi$. Thus

$$\Pi = \{p(t) \colon t \in [0, \omega)\},$$

where $p \colon \mathbb{R} \to X^1$ is a periodic solution of (2.3) with $p(0) = a$. We consider the linear variational problem (3.2) around p and the corresponding evolution operators $T(t, s)$, $t \geq s$. In particular, the operator $T_\omega = T(\omega, 0) = D_{u_0} S(\omega) a$ is called a *period map* (cf. [13, Definition 7.2.1]) and the function $w(t) = T(t, 0)\zeta$ satisfies the linear nonautonomous equation

$$w_t = w_{xx} + b(t, x)w + d(t, x)w_x, \ t > 0, \ x \in S^1, \tag{4.1}$$

with

$$b(t, x) = f_y(x, p(t)(x), p_x(t)(x)), \quad d(t, x) = f_z(x, p(t)(x), p_x(t)(x)).$$

Since T_ω is a bounded compact operator in the Hilbert infinite-dimensional space X^α, the spectrum $\sigma(T_\omega)$ of T_ω consists of 0 and a countable number of eigenvalues converging to 0. Each of these eigenvalues is called a *characteristic multiplier* and has a finite algebraic multiplicity.

Moreover, if we choose $p(\theta) \in \Pi$ instead of a and linearize around the periodic solution $p(\cdot + \theta)$, then the evolution operators are $T(\theta + t, \theta + s)$, so the period map is equal to $T(\theta + \omega, \theta) = D_{u_0} S(\omega) p(\theta)$. Thus, by [13, Lemma 7.2.2], the spectrum of $T_\omega = D_{u_0} S(\omega) a$ does not depend on the choice of the periodic point $a \in \Pi$, but the eigenfunctions do depend on a. Observe also that 1 is always a characteristic multiplier with the corresponding eigenfunction $p_t(0) \in X^1$ (cf. [13, Lemma 8.2.2]). If 1 is a simple eigenvalue of T_ω unique on the unit circle, we say that Π is a *hyperbolic* periodic orbit.

We put the multipliers in a sequence $\{\lambda_j\}_{j \geq 0}$ such that they appear according to their algebraic multiplicity and are ordered by $|\lambda_{j+1}| \leq |\lambda_j|$. It was shown in [3] that for all $j \geq 0$ we have $|\lambda_{2j+1}| < |\lambda_{2j}|$. In other words, denoting by $E_j(\Pi)$ the real generalized eigenspace of $\{\lambda_{2j-1}, \lambda_{2j}\}$ for $j \geq 1$ and by $E_0(\Pi)$ the real eigenspace corresponding to the isolated eigenvalue λ_0, we know that $\dim E_0(\Pi) = 1$ and $\dim E_j(\Pi) = 2$, $j \geq 1$. Moreover, [3, Theorem 2.2] yields that any nonzero $\phi \in E_j(\Pi)$, $j \geq 0$, has only simple zeros and $z(\phi) = 2j$.

Now we consider three projections connected with the decomposition of the spectrum of T_ω

$$P_\ell = \frac{1}{2\pi i} \int_{\gamma_\ell} (\mu I - T_\omega)^{-1} d\mu, \ \ell \in \{s, c, u\}$$

where γ_ℓ, $\ell \in \{s, c, u\}$, is a closed regular curve surrounding in mathematically positive sense and separating from the rest of the spectrum of T_ω the following subsets of the spectrum of T_ω

$$\sigma_s = \{\lambda \in \sigma(T_\omega) \colon \ |\lambda| < 1\}, \sigma_c = \{\lambda \in \sigma(T_\omega) \colon \ |\lambda| = 1\}, \sigma_u = \{\lambda \in \sigma(T_\omega) \colon \ |\lambda| > 1\},$$

respectively. Note that $\dim P_u X^\alpha$, called the *Morse index* $i(\Pi)$, is finite and equals the total algebraic multiplicity of multipliers outside the closed unit ball. Similarly

$\dim P_c X^\alpha$ is finite and equals the total algebraic multiplicity of multipliers on the unit circle.

Observe that $X^\alpha = P_u X^\alpha \oplus P_c X^\alpha \oplus P_s X^\alpha$ and $P_\ell X^\alpha$, $\ell \in \{s, c, u\}$ are positively invariant subspaces of T_ω and

$$\sigma(T_\omega|_{P_\ell X^\alpha}) = \sigma_\ell.$$

Furthermore, the eigenvectors of T_ω belong to X^1, so $P_\ell X^\alpha \subset X^1$, $\ell \in \{c, u\}$. Moreover, T_ω maps bijectively $P_u X^\alpha$ onto $P_u X^\alpha$ and $P_c X^\alpha$ onto $P_c X^\alpha$.

Assume that Π is a hyperbolic periodic orbit. Consequently, we have $P_c X^\alpha = \operatorname{span}\{p_t(0)\}$. We consider the *Poincaré map* $\mathcal{P}_a$ for the semiflow $\{S(t)\}$ corresponding to the cross section $a + P_u X^\alpha + P_s X^\alpha$ (see [13, Section 8.4],[28, Section 4.1]). Then the spectrum of the tangent map to $\mathcal{P}_a$ at a is equal to $\sigma(T_\omega) \setminus \{1\}$ and hence a is a hyperbolic fixed point of $\mathcal{P}_a$. Therefore Π is hyperbolic in the sense of [28].

Since a hyperbolic periodic orbit Π is a normally hyperbolic manifold for $\{S(t)\}$ (see [28, Remark 14.3 (c)]), it follows from [28, Theorem 14.2, Remark 14.3] (see also [28, Section 6.3]) that the *local stable manifold of* Π in a small neighborhood U of Π defined by

$$W^s_{loc}(\Pi) = \{u_0 \in X^\alpha \colon S(t)u_0 \in U,\ t \geq 0\}$$

is a C^1 submanifold of X^α with $\operatorname{codim} W^s_{loc}(\Pi) = i(\Pi)$, whereas the *local unstable manifold of* Π in U defined by

$$W^u_{loc}(\Pi) = \{u_0 \in X^\alpha \colon \exists_{\{u_{-s}\}_{s\geq 0}} S(t)u_{-s} = u_{t-s},\ 0 \leq t \leq s \text{ and } u_{-s} \in U,\ s \geq 0\}$$

is a C^1 submanifold of X^α with $\dim W^u_{loc}(\Pi) = i(\Pi) + 1$.

Moreover, $W^s_{loc}(\Pi)$ is fibrated by *local strong stable manifolds at each* $a \in \Pi$

$$W^s_{loc}(\Pi) = \bigcup_{a \in \Pi} W^{ss}_{loc}(a)$$

and $W^u_{loc}(\Pi)$ by *local strong unstable manifolds at each* $a \in \Pi$

$$W^u_{loc}(\Pi) = \bigcup_{a \in \Pi} W^{su}_{loc}(a),$$

where, for sufficiently small $\rho > 0$, we have the following characterizations with certain $\kappa, \kappa' > 0$

$$\begin{aligned} W^{ss}_{loc}(a) = \{u_0 \in X^\alpha \colon & \|S(t)u_0 - S(t)a\|_{X^\alpha} < \rho \text{ for } t \geq 0 \\ & \text{and } \lim_{t\to\infty} e^{\kappa t} \|S(t)u_0 - S(t)a\|_{X^\alpha} = 0\}, \end{aligned}$$

$$\begin{aligned} W^{su}_{loc}(a) = \{u_0 \in X^\alpha \colon & \exists_{\{u_{-t}\}_{t\geq 0}} \left\|u_{-t} - S(t)^{-1}a\right\|_{X^\alpha} < \rho \text{ for } t \geq 0, \\ & S(r)u_{-s} = u_{r-s} \text{ for } 0 \leq r \leq s \text{ and } \lim_{t\to\infty} e^{\kappa' t} \left\|u_{-t} - S(t)^{-1}a\right\|_{X^\alpha} = 0\}. \end{aligned}$$

From [28, Section 15.2] it follows that for each $a \in \Pi$, $W^{ss}_{loc}(a)$ is a C^1 submanifold of X^α tangent at a to $P_s X^\alpha$ and $W^{su}_{loc}(a)$ is a C^1 submanifold of X^α tangent at a to $P_u X^\alpha$.

5. Local stable manifold of a hyperbolic periodic orbit

In this section we consider a hyperbolic parabolic orbit Π and show that for any $u_0 \in W^s_{loc}(\Pi) \setminus \Pi$ there exists $a \in \Pi$ such that $u(t;u_0) - p(t;a)$ tends exponentially to 0 as $t \to \infty$ and

$$z(u_0 - a) \geq i(\Pi) + 1 + \frac{1 + (-1)^{i(\Pi)}}{2}. \tag{5.1}$$

Choose $u_0 \in W^s_{loc}(\Pi) \setminus \Pi$ and let $a \in \Pi$ be such that $u_0 \in W^{ss}_{loc}(a)$. We consider the corresponding solutions $u(t) = S(t)u_0$ and $p(t) = S(t)a$ of (2.3). Let $v(t) = u(t) - p(t)$, $t \geq 0$, and note that v satisfies the nonautonomous linear equation[1]

$$v_t = v_{xx} + \hat{b}(t,x)v + \hat{d}(t,x)v_x,\ t > 0,\ x \in S^1, \tag{5.2}$$

where

$$\hat{b}(t,x) = \int_0^1 f_y(x, \theta u(t)(x) + (1-\theta)p(t)(x), \theta u_x(t)(x) + (1-\theta)p_x(t)(x))d\theta,$$

$$\hat{d}(t,x) = \int_0^1 f_z(x, \theta u(t)(x) + (1-\theta)p(t)(x), \theta u_x(t)(x) + (1-\theta)p_x(t)(x))d\theta.$$

We also have

$$\lim_{t\to\infty} e^{\kappa t} \|v(t)\|_{X^\alpha} = 0. \tag{5.3}$$

We consider the sequence $v(n\omega) = u(n\omega) - a$, $n \in \mathbb{N}$. Note that $u(n\omega) \in W^{ss}_{loc}(a)$ for all $n \in \mathbb{N}$. Changing the norms to the equivalent ones, if necessary, but keeping the notation, we observe that

$$W^{ss}_{loc}(a) = \{u = a + h(P_s(u-a)) + P_s(u-a) \colon u \in B_{X^\alpha}(a,\rho)\}, \tag{5.4}$$

where $h \colon B_{P_sX^\alpha}(0,\rho) \to B_{P_uX^\alpha \oplus P_cX^\alpha}(0,\rho)$ is a C^1 function such that $h(0) = 0$ and $h'(0) = 0$. Let

$$\gamma = \max\{|\lambda_j| :\ |\lambda_j| < 1\}.$$

Then $\{\lambda_j :\ |\lambda_j| = \gamma\}$ is a spectral set for T_ω and we denote the corresponding projection in X^α by P. If $i(\Pi) = 2N-1$, then $\lambda_{2N-1} = 1$ and λ_{2N} form a spectral set and thus PX^α is the one-dimensional space spanned by the eigenfunction corresponding to λ_{2N}, so $PX^\alpha \subset E_N(\Pi)$ and

$$z(\phi) = 2N = i(\Pi) + 1 \text{ for } \phi \in PX^\alpha \setminus \{0\}. \tag{5.5}$$

If $i(\Pi) = 2N$, then $\lambda_{2N} = 1$ and PX^α is either $E_{N+1}(\Pi)$ or the one-dimensional space spanned by the eigenfunction corresponding to λ_{2N+1}. In both cases we have $PX^\alpha \subset E_{N+1}(\Pi)$ and

$$z(\phi) = 2N + 2 = i(\Pi) + 2 \text{ for } \phi \in PX^\alpha \setminus \{0\}. \tag{5.6}$$

It can be shown that for each $a \in \Pi$ the set

$$\begin{aligned} W^{fs}_{loc}(a) = \{u_0 \in X^\alpha \colon\ &\|S(t)u_0 - S(t)a\|_{X^\alpha} < \rho \text{ for } t \geq 0 \\ &\text{and } \lim_{t\to\infty} e^{\tilde{\kappa}t} \|S(t)u_0 - S(t)a\|_{X^\alpha} = 0\} \end{aligned}$$

[1] $v(t) \neq 0$ for all $t \geq 0$, since $u_0 \notin \Pi$.

for a certain $\tilde{\kappa} = \tilde{\kappa}(\gamma) > \kappa$ is a C^1 submanifold of X^α, tangent at a to $(P_s - P)X^\alpha$. We call $W^{fs}_{loc}(a)$ the *local fast stable manifold*.

We are going to show that if $u_0 \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$, then there exists a sequence $t_k \to \infty$ such that the normalized vectors $u(t_k; u_0) - p(t_k; a)$ tend to some $\varphi \in PX^\alpha \setminus \{0\}$. Consequently, the zero number estimates for elements from $PX^\alpha \setminus \{0\}$ given in (5.5) and (5.6) will lead to (5.1) for $u_0 \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$. We will also show that (5.1) for $u_0 \in W^{fs}_{loc}(a) \setminus \{a\}$ follows from the previous case and the fact that $W^{fs}_{loc}(a)$ is a submanifold of $W^{ss}_{loc}(a)$ with codimension 1 or 2 within $W^{ss}_{loc}(a)$.

Following [4, Lemma 2.2], we begin by proving that for $u_0 \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$ the $(P_s - P)X^\alpha$-coordinate of $v(n\omega)$ tends faster to zero than its PX^α-coordinate.

Lemma 5.1. *For $u_0 \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$ we have*

$$\frac{\|(P_s - P)v(n\omega)\|_{X^\alpha}}{\|Pv(n\omega)\|_{X^\alpha}} \to 0 \text{ as } n \to \infty. \tag{5.7}$$

Proof. Note that

$$W^{fs}_{loc}(a) = \{u = a + g((P_s - P)(u - a)) + (P_s - P)(u - a),\ u \in B_{X^\alpha}(a, \rho)\},$$

where $g\colon B_{(P_s - P)X^\alpha}(0, \rho) \to B_{(P_u + P + P_c)X^\alpha}(0, \rho)$ is C^1 and $g(0) = 0$, $g'(0) = 0$, is a subset of $W^{ss}_{loc}(a)$. Taking into account (5.4) and setting $y = P(u - a)$ and $z = (P_s - P)(u - a)$ for $u \in W^{ss}_{loc}(a)$, we see that

$$W^{fs}_{loc}(a) = \{u = a + h(y + z) + y + z \in W^{ss}_{loc}(a)\colon y = Pg(z), z \in B_{(P_s - P)X^\alpha}(0, \rho)\}.$$

This means that in the coordinates (y, z) for $W^{ss}_{loc}(a)$ the manifold $W^{fs}_{loc}(a)$ is a graph of the function $y = Pg(z)$.

Consider first the behavior of the sequence $\{v(n\omega)\}$ for $u_0 \in W^{ss}_{loc}(a)$. Denote by $\hat{T}(t, s)\colon X^\alpha \to X^\alpha$, $t \geq s \geq 0$, the linear evolution operator corresponding to (5.2). We know that

$$U_n = \hat{T}((n + 1)\omega, n\omega) - T_\omega \to 0 \text{ as } n \to \infty \tag{5.8}$$

in the operator norm of $\mathcal{L}(X^\alpha, X^\alpha)$ (see (4.1), (5.2)). Indeed, $v(t) = \hat{T}(t, n\omega)\xi$, $t \in [n\omega, (n + 1)\omega]$, with $\xi \in X^\alpha$, satisfies

$$v_t = v_{xx} + \hat{b}(t, x)v + \hat{d}(t, x)v_x,\ t \in (n\omega, (n + 1)\omega],\ x \in S^1, \quad v(n\omega) = \xi.$$

We change the variables $\tilde{v}^n(s) = v(s + n\omega)$, $s \in [0, \omega]$. Then $\tilde{v}^n$ satisfies

$$\begin{cases} \tilde{v}^n_s = \tilde{v}^n_{xx} + \hat{b}(s + n\omega, x)\tilde{v}^n + \hat{d}(s + n\omega, x)\tilde{v}^n_x,\ s \in (0, \omega],\ x \in S^1, \\ \tilde{v}^n(0) = \xi. \end{cases} \tag{5.9}$$

Moreover, for $w(s) = T(s, 0)\xi$, $s \in [0, \omega]$, from (4.1) we have

$$w_s = w_{xx} + b(s, x)w + d(s, x)w_x,\ s \in (0, \omega],\ x \in S^1, \quad w(0) = \xi.$$

Define $z^n(s) = \tilde{v}^n(s) - w(s)$, $s \in [0, \omega]$, and note that it satisfies

$$z^n_s = z^n_{xx} + b(s, x)z^n + d(s, x)z^n_x + (\hat{b}(s + n\omega, x) - b(s, x))\tilde{v}^n + (\hat{d}(s + n\omega, x) - d(s, x))\tilde{v}^n_x,$$

with $z^n(0)=0$. If we denote by $G(t,\sigma)$, $0\le\sigma\le t\le\omega$, the evolution operator in X^0 for $z_s=z_{xx}+b(s,x)z+d(s,x)z_x$, then we obtain (see [17, (6.1.18),(6.1.19)])

$$\|G(t,\sigma)\zeta\|_{X^\alpha}\le C\,\|\zeta\|_{X^\alpha}\,,\ 0\le\sigma\le t\le\omega,\ \zeta\in X^\alpha,$$

$$\|G(t,\sigma)\zeta\|_{X^\alpha}\le \frac{C}{(t-\sigma)^\alpha}\,\|\zeta\|_{X^0}\,,\ 0\le\sigma< t\le\omega,\ \zeta\in X^0,$$

and

$$z^n(s)=\int_0^s G(s,\sigma)h^n(\sigma)d\sigma,\ 0\le s\le\omega,$$

where $h^n(\sigma)=(\hat b(\sigma+n\omega)-b(\sigma))\tilde v^n+(\hat d(\sigma+n\omega)-d(\sigma))\tilde v^n_x$. Thus we get

$$\|z^n(s)\|_{X^\alpha}\le C\int_0^s\frac{1}{(s-\sigma)^\alpha}\,\|h^n(\sigma)\|_{X^0}\,d\sigma. \tag{5.10}$$

Moreover, from the variation of constants formula and (5.9) we obtain

$$\begin{aligned}\|\tilde v^n(s)\|_{X^\alpha}\le c\,\|\xi\|_{X^\alpha}+c\int_0^s\frac{1}{(s-\sigma)^\alpha}\|b(\sigma)\tilde v^n+d(\sigma)\tilde v^n_x\\ +(\hat b(\sigma+n\omega)-b(\sigma))\tilde v^n+(\hat d(\sigma+n\omega)-d(\sigma))\tilde v^n_x\|_{X^0}d\sigma.\end{aligned}$$

Note that by the regularity of f we have

$$|b(\sigma,x)|\le M,\ |d(\sigma,x)|\le M,\ \sigma\in[0,\omega],\ x\in S^1,$$

and

$$\left|\hat b(\sigma+n\omega,x)-b(\sigma,x)\right|\le M,\ \left|\hat d(\sigma+n\omega,x)-d(\sigma,x)\right|\le M,$$

for any $\sigma\in[0,\omega]$, $x\in S^1$ and $n\in\mathbb{N}$. Therefore

$$\|\tilde v^n(s)\|_{X^\alpha}\le c\,\|\xi\|_{X^\alpha}+2Mc\tilde c\int_0^s\frac{1}{(s-\sigma)^\alpha}\,\|\tilde v^n(\sigma)\|_{X^\alpha}\,d\sigma,\ s\in[0,\omega].$$

From a Volterra type inequality we obtain for $L=L(c,\tilde c,M,\omega)>0$

$$\|\tilde v^n(s)\|_{X^\alpha}\le L\,\|\xi\|_{X^\alpha}\,,\ s\in[0,\omega]. \tag{5.11}$$

Fix $\varepsilon>0$ and let $n_0\in\mathbb{N}$ be such that for $n\ge n_0$ we have for any $\sigma\in[0,\omega]$, $x\in S^1$

$$\left|\hat b(\sigma+n\omega,x)-b(\sigma,x)\right|<\frac{(1-\alpha)\varepsilon}{C\tilde cL\omega^{1-\alpha}},\ \left|\hat d(\sigma+n\omega,x)-d(\sigma,x)\right|<\frac{(1-\alpha)\varepsilon}{C\tilde cL\omega^{1-\alpha}}.$$

Hence from (5.10) and (5.11) we get

$$\|z^n(\omega)\|_{X^\alpha}\le\frac{(1-\alpha)\varepsilon}{L\omega^{1-\alpha}}\int_0^\omega\frac{1}{(\omega-\sigma)^\alpha}\,\|\tilde v^n(\sigma)\|_{X^\alpha}\,d\sigma\le\varepsilon\,\|\xi\|_{X^\alpha}\,.$$

This shows that $U_n\to 0$ in $\mathcal{L}(X^\alpha,X^\alpha)$.

Note that by definition $v(n\omega)=u(n\omega)-a$ with $u(n\omega)\in W^{ss}_{loc}(a)$ and

$$v((n+1)\omega)=\hat T((n+1)\omega,n\omega)v(n\omega).$$

Rewriting this equation in coordinates (y,z), we obtain with $y_n=Pv(n\omega)$ and $z_n=(P_s-P)v(n\omega)$

$$\begin{cases}y_{n+1}=P\hat T((n+1)\omega,n\omega)(h(y_n+z_n)+y_n+z_n),\\ z_{n+1}=(P_s-P)\hat T((n+1)\omega,n\omega)(h(y_n+z_n)+y_n+z_n).\end{cases}$$

Using the definition of U_n in (5.8) and the properties of projections, our system can be written as

$$\begin{cases} y_{n+1} = PT_\omega P y_n + PU_n(h(y_n+z_n)+y_n+z_n), \\ z_{n+1} = (P_s-P)T_\omega(P_s-P)z_n + (P_s-P)U_n(h(y_n+z_n)+y_n+z_n). \end{cases} \tag{5.12}$$

Finally, we make the change of coordinates $\tilde{y} = y - Pg(z)$, $\tilde{z} = z$, use the fact that in the new coordinates $W^{fs}_{loc}(a)$ is described by the equation $\tilde{y} = 0$ and get

$$\begin{cases} \tilde{y}_{n+1} = PT_\omega P\tilde{y}_n + G_n(\tilde{y}_n, \tilde{z}_n), \\ \tilde{z}_{n+1} = (P_s-P)T_\omega(P_s-P)\tilde{z}_n + H_n(\tilde{y}_n,\tilde{z}_n), \end{cases} \tag{5.13}$$

where

$$G_n(\tilde{y}_n,\tilde{z}_n) = PU_n(h(\tilde{y}_n + Pg(\tilde{z}_n)+\tilde{z}_n) + \tilde{y}_n - h(Pg(\tilde{z}_n)+\tilde{z}_n)),$$
$$H_n(\tilde{y}_n,\tilde{z}_n) = (P_s-P)U_n(h(\tilde{y}_n + Pg(\tilde{z}_n)+\tilde{z}_n) + \tilde{y}_n + Pg(\tilde{z}_n)+\tilde{z}_n).$$

Considering the spectra $\sigma(PT_\omega P)$ and $\sigma((P_s-P)T_\omega(P_s-P))$, we see that there exist $0<r<\gamma$ and $0<\mu<\frac{\gamma-r}{2}$ and norms equivalent to the original ones in the spaces PX^α and $(P_s-P)X^\alpha$ such that

$$|\!|\!|(P_s-P)T_\omega(P_s-P)\tilde{z}|\!|\!|_{(P_s-P)X^\alpha} \le (r+\mu)\,|\!|\!|\tilde{z}|\!|\!|_{(P_s-P)X^\alpha}\,,\ \tilde{z}\in(P_s-P)X^\alpha. \tag{5.14}$$

$$|\!|\!|PT_\omega P\tilde{y}|\!|\!|_{PX^\alpha} \ge (\gamma-\mu)\,|\!|\!|\tilde{y}|\!|\!|_{PX^\alpha}\,,\ \tilde{y}\in PX^\alpha, \tag{5.15}$$

Indeed, choose $0<r<\gamma$ so that

$$\lim_{n\to\infty}\|[(P_s-P)T_\omega(P_s-P)]^n\|^{\frac{1}{n}}_{\mathcal{L}((P_s-P)X^\alpha,(P_s-P)X^\alpha)} < r.$$

Then we set

$$|\!|\!|\tilde{z}|\!|\!|_{(P_s-P)X^\alpha} = \sum_{n=0}^{\infty} r^{-n}\,\|[(P_s-P)T_\omega(P_s-P)]^n\tilde{z}\|_{(P_s-P)X^\alpha}\,,\ \tilde{z}\in(P_s-P)X^\alpha.$$

The estimate (5.14) follows easily. Since

$$\lim_{n\to\infty}\left\|(PT_\omega P)^{-n}\right\|^{\frac{1}{n}}_{\mathcal{L}(PX^\alpha,PX^\alpha)} = \frac{1}{\gamma},$$

we can define

$$|\!|\!|\tilde{y}|\!|\!|_{PX^\alpha} = \sum_{n=0}^{\infty}(\gamma-\mu)^n\left\|(PT_\omega P)^{-n}\tilde{y}\right\|_{PX^\alpha}\,,\ \tilde{y}\in PX^\alpha.$$

Then the estimate (5.15) is straightforward.

Choose now $\beta\in\left(\frac{r+\mu}{\gamma-\mu},1\right)$. Then there exists $\varepsilon_0=\varepsilon_0(r,\gamma,\mu,\beta)>0$ such that for every $0<\varepsilon<\varepsilon_0$ we have

$$\frac{r+\mu+\varepsilon}{\gamma-\mu-\varepsilon}<\beta.$$

Observe that (5.8) and the properties of h and g as well as the fact that

$$|\!|\!|\tilde{y}_n|\!|\!|_{PX^\alpha} + |\!|\!|\tilde{z}_n|\!|\!|_{(P_s-P)X^\alpha} \to 0 \text{ as } n\to\infty$$

imply that for every $0 < \varepsilon < \varepsilon_0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for any $n \geq n_0$ we have

$$\|G_n(\tilde{y}_n, \tilde{z}_n)\|_{PX^\alpha} \leq \varepsilon \, \|\tilde{y}_n\|_{PX^\alpha} \tag{5.16}$$

and

$$\|H_n(\tilde{y}_n, \tilde{z}_n)\|_{(P_s-P)X^\alpha} \leq \varepsilon(\|\tilde{y}_n\|_{PX^\alpha} + \|\tilde{z}_n\|_{(P_s-P)X^\alpha}). \tag{5.17}$$

Assume now that $u_0 \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$. Then $\tilde{y}_n \neq 0$ for all $n \in \mathbb{N}$. Applying (5.15), (5.14), (5.16) and (5.17) to (5.13), we estimate for $n \geq n_0$

$$\frac{\|\tilde{z}_{n+1}\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_{n+1}\|_{PX^\alpha}} \leq \frac{r+\mu+\varepsilon}{\gamma-\mu-\varepsilon} \frac{\|\tilde{z}_n\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_n\|_{PX^\alpha}} + \frac{\varepsilon}{\gamma-\mu-\varepsilon}. \tag{5.18}$$

Choose $\delta \in (0, 1-\beta)$. Therefore, from (5.18) we infer that

$$\text{if } \frac{\|\tilde{z}_n\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_n\|_{PX^\alpha}} \geq \frac{\varepsilon}{\delta(\gamma-\mu-\varepsilon)}, \text{ then } \frac{\|\tilde{z}_{n+1}\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_{n+1}\|_{PX^\alpha}} \leq (\beta+\delta)\frac{\|\tilde{z}_n\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_n\|_{PX^\alpha}}. \tag{5.19}$$

Fix any $\eta > 0$. Choose $0 < \varepsilon < \varepsilon_0$ such that $\frac{\varepsilon}{\delta(\gamma-\mu-\varepsilon)} < \eta$. Suppose that for every $n \geq n_0 = n_0(\varepsilon)$ we have $\|\tilde{z}_n\|_{(P_s-P)X^\alpha} \geq \eta \, \|\tilde{y}_n\|_{PX^\alpha}$. Then by (5.19) we get

$$\frac{\varepsilon}{\delta(\gamma-\mu-\varepsilon)} < \eta \leq \frac{\|\tilde{z}_n\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_n\|_{PX^\alpha}} \leq (\beta+\delta)^{n-n_0} \frac{\|\tilde{z}_{n_0}\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_{n_0}\|_{PX^\alpha}}, \ n \geq n_0,$$

which is a contradiction. Therefore, there exists $n_1 \geq n_0$ such that $\|\tilde{z}_{n_1}\|_{(P_s-P)X^\alpha} < \eta \, \|\tilde{y}_{n_1}\|_{PX^\alpha}$. Hence from (5.18) it follows that for every $n \geq n_1$ we have

$$\frac{\|\tilde{z}_n\|_{(P_s-P)X^\alpha}}{\|\tilde{y}_n\|_{PX^\alpha}} < \eta.$$

Since $\eta > 0$ was chosen arbitrarily, this shows that

$$\frac{\|z_n\|_{X^\alpha}}{\|y_n - Pg(z_n)\|_{X^\alpha}} \to 0 \text{ as } n \to \infty. \tag{5.20}$$

Since $g(0) = 0$ and $g'(0) = 0$, we know that $y_n \neq 0$ for all sufficiently large n and

$$\frac{\|z_n\|_{X^\alpha}}{\|y_n\|_{X^\alpha}} \leq \frac{\|z_n\|_{X^\alpha}}{\|y_n - Pg(z_n)\|_{X^\alpha}} \left(1 + \frac{\|Pg(z_n)\|_{X^\alpha}}{\|y_n\|_{X^\alpha}}\right),$$

i.e. we have

$$\frac{\|z_n\|_{X^\alpha}}{\|y_n\|_{X^\alpha}} \left(1 - \frac{\|Pg(z_n)\|_{X^\alpha}}{\|y_n - Pg(z_n)\|_{X^\alpha}}\right) \leq \frac{\|z_n\|_{X^\alpha}}{\|y_n - Pg(z_n)\|_{X^\alpha}}.$$

This shows that

$$\frac{\|z_n\|_{X^\alpha}}{\|y_n\|_{X^\alpha}} \to 0 \text{ as } n \to \infty,$$

or in other words (5.7), which completes the proof. $\square$

We compute

$$\frac{v(n\omega)}{\|v(n\omega)\|_{X^\alpha}} = \frac{h(P_s v(n\omega))}{\|v(n\omega)\|_{X^\alpha}} + \frac{P_s v(n\omega)}{\|v(n\omega)\|_{X^\alpha}}. \tag{5.21}$$

Observe that[2]

$$\lim_{n\to\infty} \frac{h(P_s v(n\omega))}{\|v(n\omega)\|_{X^\alpha}} = \lim_{n\to\infty} \frac{h(P_s v(n\omega))}{\|P_s v(n\omega)\|_{X^\alpha}} \frac{\|P_s v(n\omega)\|_{X^\alpha}}{\|v(n\omega)\|_{X^\alpha}} = 0, \tag{5.22}$$

since $v(n\omega) \to 0$ as $n \to \infty$ and $h(0) = 0$, $h'(0) = 0$.

Let $u_0 \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$ and note that for n large enough

$$\frac{P_s v(n\omega)}{\|v(n\omega)\|_{X^\alpha}} = \frac{(P_s - P)v(n\omega)}{\|Pv(n\omega)\|_{X^\alpha}} \frac{\|Pv(n\omega)\|_{X^\alpha}}{\|v(n\omega)\|_{X^\alpha}} + \frac{Pv(n\omega)}{\|v(n\omega)\|_{X^\alpha}}. \tag{5.23}$$

Since PX^α is finite-dimensional and the sequence $\frac{Pv(n\omega)}{\|v(n\omega)\|_{X^\alpha}}$ is bounded there, we can find a subsequence $\{t_{n_k}\} \subset \{n\omega \colon n \in \mathbb{N}\}$ and $\varphi \in PX^\alpha \setminus \{0\}$ such that

$$\lim_{k\to\infty} \frac{Pv(t_{n_k})}{\|v(t_{n_k})\|_{X^\alpha}} = \varphi \tag{5.24}$$

and, by (5.21),(5.22),(5.7) and (5.23), we obtain

$$\lim_{k\to\infty} \frac{v(t_{n_k})}{\|v(t_{n_k})\|_{X^\alpha}} = \lim_{k\to\infty} \frac{S(t_{n_k})u_0 - a}{\|S(t_{n_k})u_0 - a\|_{X^\alpha}} = \varphi. \tag{5.25}$$

Since Lemma 3.2 applies to (5.2), we have for $u_0 \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$ and k large enough

$$z(u_0 - a) = z(v(0)) \geq z(v(t_{n_k})) = z(\varphi). \tag{5.26}$$

Let now $u_0 \in W^{fs}_{loc}(a) \setminus \{a\}$. Since Lemma 3.2 applies to (5.2) and the zero number is bounded from below, there exists $n \in \mathbb{N}$ large enough so that $v(n\omega) = u(n\omega) - a$ has only simple zeros. Note that $u(n\omega) \in W^{fs}_{loc}(a)$ and choose $\tilde{u} \in W^{ss}_{loc}(a) \setminus W^{fs}_{loc}(a)$ such that

$$z(v(n\omega)) = z(u(n\omega) - a) = z(\tilde{u} - a).$$

Therefore, by the above considerations there exists $\psi \in PX^\alpha \setminus \{0\}$ such that

$$z(u_0 - a) \geq z(v(n\omega)) = z(\tilde{u} - a) \geq z(\psi).$$

Recalling (5.5) and (5.6), we summarize our considerations in the following

Theorem 5.2. *For any $u_0 \in W^s_{loc}(\Pi) \setminus \Pi$ there exist $a \in \Pi$ and $\kappa > 0$ such that*

$$\lim_{t\to\infty} e^{\kappa t} \|S(t)u_0 - S(t)a\|_{X^\alpha} = 0$$

and, for $2N = z(p_t(0; a))$,

$$z(u_0 - a) \geq \begin{cases} i(\Pi) + 1 = 2N & \text{if } i(\Pi) = 2N - 1, \\ i(\Pi) + 2 = 2N + 2 & \text{if } i(\Pi) = 2N. \end{cases} \tag{5.27}$$

[2] $P_s v(n\omega) \neq 0$ for any $n \in \mathbb{N}$, because otherwise $v(n\omega) = h(P_s v(n\omega)) + P_s v(n\omega)$ would be 0.

6. Global unstable manifold of a hyperbolic periodic orbit

Following [13, Theorem 6.1.9], we prove a general result concerning the extension of submanifolds.

Lemma 6.1. *Let $S(t)\colon X^\alpha \to X^\alpha$, $t \geq 0$, be a semiflow, which admits a compact global attractor $\mathcal{A}$ in X^α. Assume that Σ is a bounded subset of $\mathcal{A}$, V is an open subset of an m-dimensional closed linear subspace E of X^α and $k\colon \Sigma \to V$ is a homeomorphism (with Σ endowed with the induced topology from X^α) and its inverse $h = k^{-1}\colon V \to \Sigma$ belongs to $C^1(V, X^\alpha)$ with $D_v h(v) \in \mathcal{L}(E, X^\alpha)$ injective for any $v \in V$. Moreover, let the semiflow $S(t)\colon X^\alpha \to X^\alpha$, $t \geq 0$, be injective, belonging to $C^1(X^\alpha, X^\alpha)$ and let $D_w S(t)(w) \in \mathcal{L}(X^\alpha, X^\alpha)$ be injective for any $t \geq 0$ and $w \in \Sigma$. Then each set $S(t)\Sigma$ is a C^1 submanifold of X^α with dimension m.*

Proof. Define $f^t\colon V \to X^\alpha$ by

$$f^t(v) = S(t)h(v),\ v \in V.$$

Since $S(t)|_{\mathcal{A}}$ is a homeomorphism of $\mathcal{A}$ onto $\mathcal{A}$, we infer that $S(t)|_\Sigma$ is a homeomorphism of Σ onto $S(t)\Sigma$ (both equipped with the induced topology from X^α). Thus f^t is a homeomorphism of V onto $S(t)\Sigma$, $f^t \in C^1(V, X^\alpha)$ and for any $v \in V$ we have

$$D_v f^t(v) = D_w S(t)(h(v)) \circ D_v h(v) \in \mathcal{L}(E, X^\alpha) \text{ is injective.}$$

Moreover, $(D_v f^t(v))E$ is an m-dimensional closed linear subspace of X^α, so it has a closed complement in X^α. Thus by [28, Corollary B.3.4] f^t is an injective C^1 immersion (at any point $v \in V$). Since f^t is a homeomorphism of V onto $S(t)\Sigma$ with the induced topology from X^α, it follows from [28, Proposition B.4.3] that $S(t)\Sigma$ is a C^1 submanifold of X^α with dimension m. □

In our problem we define the *global unstable manifold of a hyperbolic periodic orbit* Π by

$$W^u(\Pi) = \bigcup_{t\geq 0} S(t) W^u_{loc}(\Pi).$$

Using Lemma 6.1 we infer that this invariant subset of the global attractor $\mathcal{A}$ is the union of C^1 submanifolds of X^α. Moreover, we have

$$W^u(\Pi) = \bigcup_{t\geq 0} \bigcup_{a\in\Pi} S(t) W^{su}_{loc}(a),$$

where again by Lemma 6.1 each $S(t)W^{su}_{loc}(a)$ is a C^1 submanifold of X^α.

Below we examine the unstable manifold and, for simplicity, we keep the same notation as in the stable manifold case. The aim here is to show that for any $u_0 \in W^u(\Pi) \setminus \Pi$ there exists $a \in \Pi$ such that $u(t; u_0) - p(t; a)$ tends exponentially to 0 as $t \to -\infty$ and

$$z(u_0 - a) \leq i(\Pi) - 1 + \frac{1 + (-1)^{i(\Pi)}}{2}.$$

The crucial observation is, similarly to the case of $W^s_{loc}(\Pi)$, the existence of a sequence $t_k \to -\infty$ such that the normalized vectors $u(t_k; u_0) - p(t_k; a)$ tend to some $\varphi \in P_u X^\alpha \setminus \{0\}$. However, the proof is now easier, since $W^u_{loc}(\Pi)$ is a finite-dimensional submanifold of X^α.

Choose $u_0 \in W^u(\Pi) \setminus \Pi$ and note that then there exist $\tau \geq 0$ and $\tilde{a} \in \Pi$ such that $u_0 \in S(\tau) W^{su}_{loc}(\tilde{a})$. Let $\tilde{u}_0 \in W^{su}_{loc}(\tilde{a})$ and $a \in \Pi$ satisfy $u_0 = S(\tau)\tilde{u}_0$ and $a = S(\tau)\tilde{a}$. We consider the corresponding backward solutions $u(t) = u(t; \tilde{u}_0)$ and $\tilde{p}(t) = \tilde{p}(t; \tilde{a}) = p(t-\tau; a)$ of (2.3). Let $v(t) = u(t) - \tilde{p}(t)$, $t \leq 0$, and note that v satisfies the nonautonomous linear equation[3]

$$v_t = v_{xx} + \hat{b}(t,x)v + \hat{d}(t,x)v_x,\ t < 0,\ x \in S^1, \tag{6.1}$$

where

$$\hat{b}(t,x) = \int_0^1 f_y(x, \theta u(t)(x) + (1-\theta)\tilde{p}(t)(x), \theta u_x(t)(x) + (1-\theta)\tilde{p}_x(t)(x))d\theta,$$

$$\hat{d}(t,x) = \int_0^1 f_z(x, \theta u(t)(x) + (1-\theta)\tilde{p}(t)(x), \theta u_x(t)(x) + (1-\theta)\tilde{p}_x(t)(x))d\theta.$$

We also have

$$\lim_{t\to-\infty} e^{-\kappa' t} \|v(t)\|_{X^\alpha} = 0. \tag{6.2}$$

We consider the sequence $v(-n\omega) = u(-n\omega) - \tilde{a}$, $n \in \mathbb{N}$. Note that $u(-n\omega) \in W^{su}_{loc}(\tilde{a})$ for all $n \in \mathbb{N}$. Changing the norms to the equivalent ones, if necessary, but keeping the notation, we see that

$$W^{su}_{loc}(\tilde{a}) = \{u = \tilde{a} + h(P_u(u-\tilde{a})) + P_u(u-\tilde{a})\colon u \in B_{X^\alpha}(\tilde{a}, \rho)\}, \tag{6.3}$$

where $h\colon B_{P_u X^\alpha}(0,\rho) \to B_{P_s X^\alpha \oplus P_c X^\alpha}(0,\rho)$ is a C^1 function such that $h(0) = 0$ and $h'(0) = 0$. We compute

$$\frac{v(-n\omega)}{\|v(-n\omega)\|_{X^\alpha}} = \frac{h(P_u v(-n\omega))}{\|v(-n\omega)\|_{X^\alpha}} + \frac{P_u v(-n\omega)}{\|v(-n\omega)\|_{X^\alpha}}. \tag{6.4}$$

Again observe that[4]

$$\lim_{n\to\infty} \frac{h(P_u v(-n\omega))}{\|v(-n\omega)\|_{X^\alpha}} = \lim_{n\to\infty} \frac{h(P_u v(-n\omega))}{\|P_u v(-n\omega)\|_{X^\alpha}} \frac{\|P_u v(-n\omega)\|_{X^\alpha}}{\|v(-n\omega)\|_{X^\alpha}} = 0. \tag{6.5}$$

Since

$$\frac{P_u v(-n\omega)}{\|v(-n\omega)\|_{X^\alpha}},\ n \in \mathbb{N},$$

is a bounded sequence in a finite-dimensional subspace of X^α, there exist a subsequence $\{t_{n_k}\} \subset \{-n\omega\colon n \in \mathbb{N}\}$ and $\varphi \in P_u X^\alpha \setminus \{0\}$ such that

$$\lim_{k\to\infty} \frac{P_u v(t_{n_k})}{\|v(t_{n_k})\|_{X^\alpha}} = \varphi \tag{6.6}$$

and by (6.4) and (6.5), we obtain

$$\lim_{k\to\infty} \frac{v(t_{n_k})}{\|v(t_{n_k})\|_{X^\alpha}} = \varphi. \tag{6.7}$$

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[3] $v(t) \neq 0$ for all $t \leq 0$, since $\tilde{u}_0 \notin \Pi$.

[4] $P_u v(-n\omega) \neq 0$ for any $n \in \mathbb{N}$, because otherwise $v(-n\omega) = h(P_u v(-n\omega)) + P_u v(-n\omega)$ would be 0.

If $i(\Pi) = 2N-1$, then $\lambda_{2N-1} = 1$, $2N = z(\tilde{p}_t(0;\tilde{a})) = z(p_t(0;a))$ and

$$z(\phi) \leq 2N-2 = i(\Pi) - 1 \text{ for } \phi \in P_u X^\alpha \setminus \{0\}, \tag{6.8}$$

whereas if $i(\Pi) = 2N$, then $\lambda_{2N} = 1$, $2N = z(\tilde{p}_t(0;\tilde{a})) = z(p_t(0;a))$ and

$$z(\phi) \leq 2N = i(\Pi) \text{ for } \phi \in P_u X^\alpha \setminus \{0\}. \tag{6.9}$$

Using Lemma 3.2, we have for $u_0 \in W^u(\Pi) \setminus \Pi$ and k large enough

$$z(u_0 - a) \leq z(\tilde{u}_0 - \tilde{a}) = z(v(0)) \leq z(v(t_{n_k})) = z(\varphi). \tag{6.10}$$

Summarizing the above considerations we obtain

Theorem 6.2. *For any $u_0 \in W^u(\Pi) \setminus \Pi$ there exist $a \in \Pi$ and $\kappa' > 0$ such that*

$$\lim_{t\to\infty} e^{\kappa' t} \left\| S(t)^{-1} u_0 - S(t)^{-1} a \right\|_{X^\alpha} = 0$$

and, for $2N = z(p_t(0;a))$,

$$z(u_0 - a) \leq \begin{cases} i(\Pi) - 1 = 2N-2 & \textit{if } i(\Pi) = 2N-1, \\ i(\Pi) = 2N & \textit{if } i(\Pi) = 2N. \end{cases} \tag{6.11}$$

7. Exclusion of a homoclinic connection for a hyperbolic periodic orbit

In this section we will consider two (not necessarily distinct) hyperbolic periodic orbits Π^- and Π^+ with periods $\omega^- > 0$ and $\omega^+ > 0$, respectively. We also assume that there exists a point

$$u_0 \in (W^u(\Pi^-) \cap W^s_{loc}(\Pi^+)) \setminus (\Pi^- \cup \Pi^+).$$

Note that if $\Pi^- = \Pi^+$, then u_0 is a homoclinic point and the corresponding orbit is a homoclinic connection for the periodic orbit.

Consequently, to $u_0 \notin \Pi^- \cup \Pi^+$ there corresponds an X^α solution $u(\cdot; u_0)$ of (2.3), which is defined for all $t \in \mathbb{R}$, its orbit is bounded in X^α (thus belongs to the global attractor $\mathcal{A}$) and there exist initial data $a^\pm \in \Pi^\pm$ together with periodic solutions $p^\pm(\cdot; a^\pm)$ of (2.3) such that

$$\lim_{t\to\pm\infty} u(t;u_0) - p^\pm(t;a^\pm) = 0 \tag{7.1}$$

in X^α (cf. [13, Theorem 8.2.3]). We also define $N^\pm \in \mathbb{N}$ so that $2N^\pm = z(p_t^\pm(0;a^\pm))$.

In order to combine the estimates (5.27) and (6.11) we observe in the following two lemmas that there exists some neighborhood of $\{p^+(t;a^+) - p^-(t;a^-) \colon t \in \mathbb{R}\}$ in X^α consisting of a finite number of balls such that in a sufficiently bigger neighborhood the zero number of functions is constant.

Lemma 7.1. *If $\Pi^- = \Pi^+$ and $a^- \neq a^+$, then*

$$z(p^+(t;a^+) - p^-(t;a^-)) = const., \ t \in \mathbb{R}.$$

Therefore there exists a finite cover $\bigcup_{i=1}^n B_{X^\alpha}(p^+(t_i;a^+) - p^-(t_i;a^-), \varepsilon_i)$ of the set $\{p^+(t;a^+) - p^-(t;a^-) \colon t \in \mathbb{R}\}$ in X^α such that the zero number is constant in $\bigcup_{i=1}^n B_{X^\alpha}(p^+(t_i;a^+) - p^-(t_i;a^-), 2\varepsilon_i)$.

Proof. Suppose that $v(t) = p^+(t; a^+) - p^-(t; a^-)$, $t \in \mathbb{R}$, has a multiple zero at $t = t_0$. Since v is periodic with period $\omega = \omega^+ = \omega^-$, then by Lemma 3.2 we have

$$z(v(t_0)) = z(v(t_0 - \omega)) > z(v(t_0)),$$

which is a contradiction.

The second claim follows from the compactness of $\{p^+(t; a^+) - p^-(t; a^-) : t \in \mathbb{R}\}$ in X^α (as a continuous image of a compact interval), since to each point v of this set there corresponds a ball $B_{X^\alpha}(v, 2\varepsilon_v)$ in which the zero number is constant and we can choose a finite subcover from $\bigcup_v B_{X^\alpha}(v, \varepsilon_v)$. $\square$

Lemma 7.2. *If* $\Pi^- \neq \Pi^+$*, then we have*

$$z(p^+(t; a^+) - p^-(s; a^-)) = const., \ s, t \in \mathbb{R}.$$

Therefore there exists a finite cover $\bigcup_{i=1}^n B_{X^\alpha}(p^+(t_i; a^+) - p^-(s_i; a^-), \varepsilon_i)$ *of the set* $\{p^+(t; a^+) - p^-(s; a^-) : s, t \in \mathbb{R}\}$ *in* X^α *such that the zero number is constant in* $\bigcup_{i=1}^n B_{X^\alpha}(p^+(t_i; a^+) - p^-(s_i; a^-), 2\varepsilon_i)$.

Proof. Fix $\tau \in \mathbb{R}$ and suppose that $v(t) = p^+(t; a^+) - p^-(t + \tau; a^-)$, $t \in \mathbb{R}$, has a multiple zero at $t = t_0$. We consider two cases. If $\frac{\omega^+}{\omega^-}$ is rational, say $\frac{\omega^+}{\omega^-} = \frac{k}{m}$ for some $k, m \in \mathbb{N}$, then with $\omega = m\omega^+ = k\omega^-$ we have for any $t \in \mathbb{R}$

$$v(t+\omega) = p^+(t+m\omega^+; a^+) - p^-(t+k\omega^- + \tau; a^-) = p^+(t; a^+) - p^-(t+\tau; a^-) = v(t),$$

i.e. v is periodic with period ω. Then we have

$$z(v(t_0)) = z(v(t_0 - \omega)) > z(v(t_0)),$$

which is a contradiction.

Consider now $\frac{\omega^+}{\omega^-}$ irrational. Note that $\mathbb{R} \ni t \to z(v(t)) \in \mathbb{R}$ as a monotone function has at most a countable number of points of discontinuity. Therefore we choose $t_1 \leq t_0 - \omega^+ < t_0$ such that $v(t_1)$ does not have multiple zeros and observe that

$$z(v(t_1)) > z(v(t_0)).$$

To obtain a contradiction it suffices to find $t_2 > t_0$ such that $z(v(t_2)) = z(v(t_1))$. Below we show that

$$\forall_{\varepsilon>0} \exists_{t_2>t_0} \left\| (p^+(t_2; a^+), p^-(t_2 + \tau; a^-)) - (p^+(t_1; a^+), p^-(t_1 + \tau; a^-)) \right\|_{X^\alpha \times X^\alpha} < \varepsilon,$$

which implies that $\|v(t_1) - v(t_2)\|_{C^1(S^1)} < \varepsilon$ and thus $z(v(t_1)) = z(v(t_2))$ for sufficiently small $\varepsilon > 0$. Fix $\varepsilon > 0$. Since p^+ and p^- are continuous and periodic, the uniform continuity implies that there exists $\delta > 0$ such that for any points $(t, s), (\hat{t}, \hat{s}) \in \mathbb{R}^2$ we know that $\left|(t, s) - (\hat{t}, \hat{s})\right| < \delta$ implies

$$\left\| (p^+(t; a^+), p^-(s + \tau; a^-)) - (p^+(\hat{t}; a^+), p^-(\hat{s} + \tau; a^-)) \right\|_{X^\alpha \times X^\alpha} < \varepsilon.$$

Choose $n_0 \in \mathbb{N}$ such that $\frac{\omega^-}{n_0} < \delta$ and set $s_0 = \frac{t_0 - t_1}{\omega^+} \geq 1$. Denoting by $[\cdot]$ the floor function, consider the $n_0 + 1$ real positive numbers

$$([s_0] + 1)\frac{\omega^+}{\omega^-}, 2([s_0] + 1)\frac{\omega^+}{\omega^-}, \ldots, (n_0 + 1)([s_0] + 1)\frac{\omega^+}{\omega^-}$$

and their fractional parts, which are pairwise different, since $\frac{\omega^+}{\omega^-}$ is irrational. At least two of the numbers, say $m_1\frac{\omega^+}{\omega^-} > m_2\frac{\omega^+}{\omega^-}$, have to have fractional parts closer than $\frac{1}{n_0}$. Therefore, we know that $m_1 - m_2 \geq [s_0] + 1 > s_0$ and

$$(m_1 - m_2)\frac{\omega^+}{\omega^-} = l + r,$$

where $l \in \mathbb{N} \cup \{0\}$ and $0 < |r| < \frac{1}{n_0}$.

Hence the distance between the points $(t_1 + (m_1 - m_2)\omega^+, t_1 + (m_1 - m_2)\omega^+)$ and $(t_1 + (m_1 - m_2)\omega^+, t_1 + l\omega^-)$ is less than $\frac{\omega^-}{n_0} < \delta$. In conclusion, for $t_2 = t_1 + (m_1 - m_2)\omega^+ > t_0$ we have

$$\begin{aligned}\big\|(p^+(t_2; a^+), p^-(t_2 + \tau; a^-)) - (p^+(t_1; a^+), p^-(t_1 + \tau; a^-))\big\|_{X^\alpha \times X^\alpha}\\ = \|(p^+(t_2; a^+), p^-(t_2 + \tau; a^-))\\ -(p^+(t_1 + (m_1 - m_2)\omega^+; a^+), p^-(t_1 + l\omega^- + \tau; a^-))\|_{X^\alpha \times X^\alpha} < \varepsilon.\end{aligned}$$

This ends the proof of the first assertion.

The second claim follows from the compactness of $\{p^+(t; a^+) - p^-(s; a^-) \colon s, t \in \mathbb{R}\}$ in X^α (as a continuous image of a two-dimensional torus), since to each point v of this set there corresponds a ball $B_{X^\alpha}(v, 2\varepsilon_v)$ in which the zero number is constant and we choose a finite subcover from $\bigcup_v B_{X^\alpha}(v, \varepsilon_v)$. □

The above lemmas allow us to combine the inequalities (5.27) and (6.11). As a particular result, we deduce that there is no homoclinic connection for a hyperbolic periodic orbit.

Theorem 7.3. *If* $u_0 \in (W^u(\Pi^-) \cap W^s_{loc}(\Pi^+)) \setminus (\Pi^- \cup \Pi^+)$ *and* $2N^\pm = z(p_t^\pm(0; a^\pm))$, *then*

$$N^- \geq N^+ \quad \textit{and} \quad i(\Pi^-) \geq i(\Pi^+) + 1, \tag{7.2}$$

which excludes a homoclinic connection for a hyperbolic periodic orbit.

Moreover, if $i(\Pi^+) = 2N^+$, *then we even have*

$$N^- \geq N^+ + 1. \tag{7.3}$$

Proof. Let $\varepsilon_0 > 0$ be the minimum of $\varepsilon_1, \ldots, \varepsilon_n$ from Lemmas 7.1, 7.2. Since (7.1) implies for large $t > 0$

$$\big\|u(t; u_0) - p^+(t; a^+)\big\|_{X^\alpha} < \varepsilon_0 \text{ and } \big\|u(-t; u_0) - p^-(-t; a^-)\big\|_{X^\alpha} < \varepsilon_0,$$

we have

$$\begin{aligned}z(u_0 - a^-) &\geq z(u(t; u_0) - p^+(t; a^+) + p^+(t; a^+) - p^-(t; a^-))\\ &= z(p^+(t; a^+) - p^-(t; a^-)) = z(p^-(-t; a^-) - p^+(-t; a^+))\\ &= z(u(-t; u_0) - p^-(-t; a^-) + p^-(-t; a^-) - p^+(-t; a^+)) \geq z(u_0 - a^+).\end{aligned}$$

Here we have assumed that $a^- \neq a^+$, but if $a^- = a^+$ we have $z(u_0 - a^-) = z(u_0 - a^+)$ immediately. By Theorem 5.2 we have

$$z(u_0 - a^+) \geq \begin{cases} i(\Pi^+) + 1 = 2N^+ & \text{if } i(\Pi^+) = 2N^+ - 1,\\ i(\Pi^+) + 2 = 2N^+ + 2 & \text{if } i(\Pi^+) = 2N^+,\end{cases}$$

and by Theorem 6.2 we have

$$z(u_0 - a^-) \leq \begin{cases} i(\Pi^-) - 1 = 2N^- - 2 & \text{if } i(\Pi^-) = 2N^- - 1, \\ i(\Pi^-) = 2N^- & \text{if } i(\Pi^-) = 2N^-. \end{cases}$$

This leads straightforward to (7.2) and (7.3). □

8. Transversal intersection of stable and unstable manifolds of hyperbolic periodic orbits

We are going to show the transversal intersection of stable and unstable manifolds of hyperbolic periodic orbits following the approach used for equilibria by M. Chen, X.-Y. Chen and J. K. Hale in [5]. Assume that

$$u_0 \in S(\tau)W^u_{loc}(\Pi^-) \cap W^s_{loc}(\Pi^+)$$

with some $\tau \geq 0$. Our purpose is to show that

$$T_{u_0}S(\tau)W^u_{loc}(\Pi^-) + T_{u_0}W^s_{loc}(\Pi^+) = X^\alpha.$$

Without loss of generality we assume that $u_0 \notin \Pi^- \cup \Pi^+$. We consider the linearization of (1.1) around $u(\cdot; u_0)$

$$\begin{cases} v_t = v_{xx} + \hat{b}(t,x)v + \hat{d}(t,x)v_x, \ t > s, \ x \in S^1, \\ v(s) = \psi \end{cases} \tag{8.1}$$

and the corresponding evolution operators $\hat{T}(t,s) \in \mathcal{L}(X^\alpha, X^\alpha)$, $t \geq s$. Therefore, we have $\hat{T}(t,s)\psi = v(t; s, \psi)$. Note that each operator $\hat{T}(t,s)$ is injective and its adjoint is also injective. Let $p^\pm(\cdot; a^\pm)$ be periodic solutions with $a^\pm \in \Pi^\pm$ such that $2N^\pm = z(p^\pm_t(0; a^\pm))$ and

$$u(t; u_0) - p^\pm(t; a^\pm) \to 0 \text{ as } t \to \pm\infty.$$

First we concentrate on the description of the tangent space to the local stable manifold. To this end, let us consider the linearization of (1.1) around $p^+(\cdot; a^+)$

$$\begin{cases} w^+_t = w^+_{xx} + b^+(t,x)w^+ + d^+(t,x)w^+_x, \ t > s, \ x \in S^1, \\ w^+(s) = \psi \end{cases}$$

and the corresponding evolution operators $T^+(t,s) \in \mathcal{L}(X^\alpha, X^\alpha)$, $t \geq s$. We denote by $T^+_{\omega^+} = T^+(\omega^+, 0)$ the period map and as in Section 4 we put its multipliers in a sequence $\{\lambda_j\}_{j\geq 0}$ such that they appear according to their algebraic multiplicity and are ordered by $|\lambda_j| \geq |\lambda_{j+1}|$.

Since the coefficients of the equations converge to each other, following the part of the proof of Lemma 5.1 we get $U_n = \hat{T}((n+1)\omega^+, n\omega^+) - T^+_{\omega^+} \to 0$ in $\mathcal{L}(X^\alpha, X^\alpha)$. We also have the formula

$$v((n+1)\omega^+) = \hat{T}((n+1)\omega^+, n\omega^+)v(n\omega^+) = T^+_{\omega^+}v(n\omega^+) + U_n v(n\omega^+)$$

and the assumptions (B.1)-(B.2) of [5] are satisfied.

For any $\psi \in X^\alpha$ and any $m \in \mathbb{N}$ we write

$$\rho_\infty(m,\psi) = \limsup_{n\to\infty} \left\| v(n\omega^+; m\omega^+, \psi) \right\|_{X^\alpha}^{\frac{1}{n}} = \limsup_{n\to\infty} \left\| \hat{T}(n\omega^+, m\omega^+)\psi \right\|_{X^\alpha}^{\frac{1}{n}}.$$

Let Λ be the set of all nonnegative numbers r such that

$$\sigma(T^+_{\omega^+}) \cap \{z \in \mathbb{C}\colon\ |z| = r\} \neq \emptyset.$$

We also set $r_j = |\lambda_j|$, $j \geq 0$. For any integer $j \geq 0$ and $m \in \mathbb{N}$ we define the spaces

$$F_j^+(m) = \{\psi \in X^\alpha\colon \rho_\infty(m,\psi) \leq r_j\}.$$

From [5, Corollary B.3] it follows that for every $\psi \in X^\alpha$ and $m \in \mathbb{N}$ there exists $r \in \Lambda$ such that

$$\lim_{n\to\infty} \left\| v(n\omega^+; m\omega^+, \psi) \right\|_{X^\alpha}^{\frac{1}{n}} = r.$$

Therefore, we have

$$X^\alpha = F_0^+(m) \supset F_1^+(m) \supset F_2^+(m) \supset \dots.$$

Since it is well-known that linear equations like (8.1) do not possess nontrivial solutions decaying faster than any exponential (see [15]), as in [5, Theorem 3.1] we see that for all $\psi \neq 0$ and $m \in \mathbb{N}$ we have $\rho_\infty(m,\psi) > 0$. In other words, we get

$$\bigcap_{j=0}^{\infty} F_j^+(m) = \{0\},$$

because $\rho_\infty(m,0) = 0$, $m \in \mathbb{N}$.

From the asymptotic behavior of the solution $v(n\omega^+; m\omega^+, \psi)$ we are able to characterize the zero number $z(\psi)$ of the initial condition ψ. Assume that

$$\lim_{n\to\infty} \left\| v(n\omega^+; m\omega^+, \psi) \right\|_{X^\alpha}^{\frac{1}{n}} = r_j.$$

We have the following two cases. If j is odd and $r_j > r_{j+1}$, then $\psi \in F_j^+(m) \backslash F_{j+1}^+(m)$ and by [5, Theorems B.2, B.4] there exists $\phi \in E_{\frac{j+1}{2}}(\Pi^+)$, $\|\phi\|_{X^\alpha} = 1$ and a subsequence $\{t_{n_k}\} \subset \{n\omega^+ \colon n \geq m\}$ such that

$$\frac{v(t_{n_k}; m\omega^+, \psi)}{\|v(t_{n_k}; m\omega^+, \psi)\|_{X^\alpha}} \to \phi.$$

Therefore, we have for large $k \in \mathbb{N}$

$$z(\psi) \geq z(v(t_{n_k}; m\omega^+, \psi)) = z\left(\frac{v(t_{n_k}; m\omega^+, \psi)}{\|v(t_{n_k}; m\omega^+, \psi)\|_{X^\alpha}} \right) = z(\phi) = j+1.$$

If j is even, then $\psi \in F_j^+(m) \setminus F_{j+1}^+(m)$ and by [5, Theorems B.2,B.4] there exists $\phi \in E_{\frac{j}{2}}(\Pi^+)$, $\|\phi\|_{X^\alpha} = 1$ and a subsequence $\{t_{n_k}\} \subset \{n\omega^+ \colon n \geq m\}$ such that

$$\frac{v(t_{n_k}; m\omega^+, \psi)}{\|v(t_{n_k}; m\omega^+, \psi)\|_{X^\alpha}} \to \phi.$$

Therefore, we have for large $k \in \mathbb{N}$

$$z(\psi) \geq z(v(t_{n_k}; m\omega^+, \psi)) = z\left(\frac{v(t_{n_k}; m\omega^+, \psi)}{\|v(t_{n_k}; m\omega^+, \psi)\|_{X^\alpha}} \right) = z(\phi) = j.$$

Fix $m \in \mathbb{N}$, $\psi \in F_k^+(m)$, $\psi \neq 0$ and let $j \geq k$ be such that $\psi \in F_j^+(m) \setminus F_{j+1}^+(m)$. Suppose that k is even. If j is even, then $z(\psi) \geq j \geq k$, whereas if j is odd, then $z(\psi) \geq j+1 \geq k$. Suppose now that k is odd. If j is even, then $z(\psi) \geq j \geq k+1$ due to the difference of parity of k and j. Moreover, if j is odd, then $z(\psi) \geq j+1 \geq k+1$.

Since the local stable manifold $W^s_{loc}(\Pi^+)$ of the hyperbolic periodic orbit Π^+ coincides locally with the local center-stable manifold $W^{cs}_{loc}(a^+)$ of a fixed point $a^+ \in \Pi^+$ for the map $S(\omega^+)$, it follows from [5, Theorem C.4] that

$$T_{u_0} W^s_{loc}(\Pi^+) = \{\psi \in X^\alpha \colon \limsup_{n\to\infty} \left\| v(n\omega^+; 0, \psi) \right\|^{\frac{1}{n}}_{X^\alpha} \leq 1\}.$$

If $i(\Pi^+) = 2N^+$, then $r_{2N^+} = \lambda_{2N^+} = 1$ and by [5, Theorem B.7] for sufficiently large $m_0 \in \mathbb{N}$ we see that $T_{u(m_0\omega^+;u_0)} W^s_{loc}(\Pi^+) = F^+_{2N^+}(m_0)$ is isomorphic to $\mathrm{cl}_{X^\alpha}(\mathrm{span}\{p_t^+(0;a^+)\} \oplus E_{N^++1}(\Pi^+) \oplus \ldots)$. Therefore, $z(\psi) \geq 2N^+$ for $\psi \in T_{u(m_0\omega^+;u_0)} W^s_{loc}(\Pi^+) \setminus \{0\}$ and $\mathrm{codim}\, T_{u(m_0\omega^+;u_0)} W^s_{loc}(\Pi^+) = 2N^+$.

If $i(\Pi^+) = 2N^+ - 1$, then $r_{2N^+-1} = \lambda_{2N^+-1} = 1$ and by [5, Theorem B.7] for sufficiently large $m_0 \in \mathbb{N}$ we see that $T_{u(m_0\omega^+;u_0)} W^s_{loc}(\Pi^+) = F^+_{2N^+-1}(m_0)$ is isomorphic to $\mathrm{cl}_{X^\alpha}(E_{N^+}(\Pi^+) \oplus E_{N^++1}(\Pi^+) \oplus \ldots)$. Therefore, $z(\psi) \geq 2N^+$ for $\psi \in T_{u(m_0\omega^+;u_0)} W^s_{loc}(\Pi^+)$ and $\mathrm{codim}\, T_{u(m_0\omega^+;u_0)} W^s_{loc}(\Pi^+) = 2N^+ - 1$.

Since the adjoint operator of the evolution operator $\hat{T}(m_0\omega^+, 0)$ is injective (thus $\hat{T}(m_0\omega^+, 0)$ has dense range by [13, Theorem 7.3.3]) and $T_{u_0} W^s_{loc}(\Pi^+)$ is the preimage of $T_{u(m_0\omega^+;u_0)} W^s_{loc}(\Pi^+)$ under $\hat{T}(m_0\omega^+, 0)$, we see that if $i(\Pi^+) = 2N^+$, then

$$\mathrm{codim}\, T_{u_0} W^s_{loc}(\Pi^+) = \mathrm{codim}\, F^+_{2N^+}(0) = 2N^+$$

and $z(\psi) \geq 2N^+$ for $\psi \in T_{u_0} W^s_{loc}(\Pi^+) \setminus \{0\}$, whereas if $i(\Pi^+) = 2N^+ - 1$, then

$$\mathrm{codim}\, T_{u_0} W^s_{loc}(\Pi^+) = \mathrm{codim}\, F^+_{2N^+-1}(0) = 2N^+ - 1$$

and $z(\psi) \geq 2N^+$ for $\psi \in T_{u_0} W^s_{loc}(\Pi^+) \setminus \{0\}$.

In both cases of $i(\Pi^+)$ we now consider $r_{2N^++1} < 1$ and the subspace $F^+_{2N^++1}(0)$ of $T_{u_0} W^s_{loc}(\Pi^+)$, which is isomorphic to $\mathrm{cl}_{X^\alpha}(E_{N^++1}(\Pi^+) \oplus \ldots)$. Then we have $z(\psi) \geq 2N^+ + 2$ for $\psi \in F^+_{2N^++1}(0) \setminus \{0\}$ and $\mathrm{codim}\, F^+_{2N^++1}(0) = 2N^+ + 1$.

From the above considerations the following result follows.

Lemma 8.1. *We have*

$$z(v) \geq 2N^+,\ v \in T_{u_0} W^s_{loc}(\Pi^+) \setminus \{0\}.$$

Moreover, there exists a subspace W^+ of $T_{u_0} W^s_{loc}(\Pi^+)$ such that

$$z(v) \geq 2N^+ + 2,\ v \in W^+ \setminus \{0\}$$

and $\mathrm{codim}\, W^+ = 2N^+ + 1$.

Using Lemma 8.1 we finally prove the main result of the paper.

Theorem 8.2. *The stable and unstable manifolds of two hyperbolic periodic orbits $\Pi^\pm$ for the problem* (1.1) *have a transversal intersection*

$$W^u(\Pi^-) \pitchfork W^s_{loc}(\Pi^+),$$

i.e. if $u_0 \in S(\tau) W^u_{loc}(\Pi^-) \cap W^s_{loc}(\Pi^+)$ with some $\tau \geq 0$, then

$$T_{u_0} S(\tau) W^u_{loc}(\Pi^-) + T_{u_0} W^s_{loc}(\Pi^+) = X^\alpha.$$

Proof. The proof is in the same vein as the proof of Lemma 8.1 and is based on the description of the tangent space to the global unstable manifold.

First assume $i(\Pi^+) = 2N^+ - 1$. Thus we have $\operatorname{codim} W^s_{loc}(\Pi^+) = 2N^+ - 1$. From the inequality $N^- \geq N^+$ (see Theorem 7.3) we see, in a similar way as in the local stable manifold case, that for sufficiently large $m_0 \in \mathbb{N}$ there exists a subspace $F^-_{m_0,N^+}$ of $T_{u(-m_0\omega^-;u_0)}S(\tau)W^u_{loc}(\Pi^-)$ such that

$$\dim F^-_{m_0,N^+} = \dim \big(E_0(\Pi^-) \oplus \ldots \oplus E_{N^+-1}(\Pi^-)\big) = 2N^+ - 1,$$

$$z(v) \leq 2N^+ - 2,\ v \in F^-_{m_0,N^+} \setminus \{0\}.$$

Now note that $F^-_{0,N^+} = \hat{T}(0, -m_0\omega^-)F^-_{m_0,N^+}$ is a subspace of $T_{u_0}S(\tau)W^u_{loc}(\Pi^-)$ with

$$\dim F^-_{0,N^+} = \dim F^-_{m_0,N^+} \ \text{ and } \ z(v) \leq 2N^+ - 2,\ v \in F^-_{0,N^+} \setminus \{0\},$$

since the evolution operator $\hat{T}(\cdot,\cdot)$ for the linearized equation around the solution $u(\cdot;u_0)$ is injective and does not increase the zero number. Thus we have

$$\dim F^-_{0,N^+} = 2N^+ - 1 = \operatorname{codim} W^s_{loc}(\Pi^+)$$

and $F^-_{0,N^+} \cap T_{u_0}W^s_{loc}(\Pi^+) = \{0\}$, since $z(v) \leq 2N^+ - 2$ for $v \in F^-_{0,N^+} \setminus \{0\}$ and by Lemma 8.1 we have $z(v) \geq 2N^+$ for $v \in T_{u_0}W^s_{loc}(\Pi^+) \setminus \{0\}$. This proves that

$$T_{u_0}S(\tau)W^u_{loc}(\Pi^-) + T_{u_0}W^s_{loc}(\Pi^+) = X^\alpha$$

in this case.

Let now $i(\Pi^+) = 2N^+$. Then by Theorem 7.3 we have $N^- \geq N^+ + 1$ and we see again that for sufficiently large $m_0 \in \mathbb{N}$ there exists a subspace $F^-_{m_0,N^+}$ of $T_{u(-m_0\omega^-;u_0)}S(\tau)W^u_{loc}(\Pi^-)$ such that

$$\dim F^-_{m_0,N^+} = \dim \big(E_0(\Pi^-) \oplus \ldots \oplus E_{N^+}(\Pi^-)\big) = 2N^+ + 1,$$

$$z(v) \leq 2N^+,\ v \in F^-_{m_0,N^+} \setminus \{0\}.$$

As before, note that $F^-_{0,N^+} = T(0, -m_0\omega^-)F^-_{m_0,N^+}$ is a subspace of $T_{u_0}S(\tau)W^u_{loc}(\Pi^-)$ with

$$\dim F^-_{0,N^+} = \dim F^-_{m_0,N^+} \ \text{ and } \ z(v) \leq 2N^+,\ v \in F^-_{0,N^+} \setminus \{0\}.$$

Thus we get

$$\dim F^-_{0,N^+} = 2N^+ + 1 = \operatorname{codim} W^+,$$

where W^+ is the subspace of $T_{u_0}W^s_{loc}(\Pi^+)$ given in Lemma 8.1. Moreover, we have

$$F^-_{0,N^+} \cap W^+ = \{0\},$$

since $z(v) \leq 2N^+$ for $v \in F^-_{0,N^+} \setminus \{0\}$ and $z(v) \geq 2N^+ + 2$ for $v \in W^+ \setminus \{0\}$, by Lemma 8.1. Therefore, we obtain $F^-_{0,N^+} \oplus W^+ = X^\alpha$. This shows that

$$T_{u_0}S(\tau)W^u_{loc}(\Pi^-) + T_{u_0}W^s_{loc}(\Pi^+) = X^\alpha$$

also in this case and concludes the proof. □

9. Concluding remarks

As pointed out in the introduction, the transversality between stable and unstable manifolds is one of the main ingredients for the structural stability of the semiflow generated by (1.1). Furthermore, from the point of view of applications the discussion of structural stability is essential for dynamical systems. As concluding remarks we overview here some of the available results dealing with this topic.

The discussion of structural stability involves the characterization of the semiflow on the global attractor $\mathcal{A}_f$ and its dependence on the nonlinearity $f \in C^2$ considered as a parameter. In this infinite-dimensional setting it requires the comparison between different global attractors using topological equivalence (for a reference see [12]). If $\mathcal{A}_f$ and $\mathcal{A}_g$ denote the global attractors corresponding to the semiflows generated by (1.1) with nonlinearities f and g, respectively, $\mathcal{A}_f$ and $\mathcal{A}_g$ are *orbit equivalent*, $\mathcal{A}_f \cong \mathcal{A}_g$, if there exists a homeomorphism $h\colon \mathcal{A}_f \to \mathcal{A}_g$ taking orbits of $(1.1)_f$ to orbits of $(1.1)_g$ preserving the time direction. Then, a global attractor $\mathcal{A}_f$ is *structurally stable* if there exists a neighborhood $\mathcal{N}(f)$ of f (in the space of C^2 functions with the adequate topology) such that $\mathcal{A}_g \cong \mathcal{A}_f$ for all $g \in \mathcal{N}(f)$. Therefore, a structurally stable global attractor $\mathcal{A}_f$ is invariant up to a homeomorphism under small perturbations of the nonlinearity f.

Structural stability is known to hold in the class of Morse-Smale semiflows (see [23] for details). The class of semiflows considered enjoys a number of properties related to the existence of global attractors and includes many semiflows generated by partial differential equations, like in our case, and delay or functional differential equations. For completeness we recall that such a semiflow has the Morse-Smale property if: *(i)* the corresponding global attractor $\mathcal{A}_f$ has only a finite number of equilibria and periodic orbits, which are all hyperbolic, *(ii)* all the stable and unstable manifolds of these critical elements are transversal and *(iii)* the nonwandering set $\Omega(f)$ (the set of points of $\mathcal{A}_f$ for which any neighborhood U is visited after an arbitrarily large time $T > 0$ by an orbit starting in U) contains only the equilibria and the periodic orbits. Property *(iii)* is expected to hold for (1.1) under condition *(i)* due to the zero number decay property and the result [7, Theorem 1] mentioned in the introduction. Therefore, the verification of the Morse-Smale property of $\mathcal{A}_f$ should involve only the confirmation of property *(i)*, which holds generically in the above space of C^2 functions, and property *(ii)* regarding the transversality between stable and unstable manifolds of the critical elements. Here, in view of our transversality result, it only remains to check transversality for pairs of critical elements where at least one is an equilibrium.

In restricted classes of nonlinear functions f we are able to exhibit the Morse–Smale property. In fact, this is the case when $f = f(u, u_x)$ does not depend explicitly on x. Then, S^1-equivariance of (1.1) forces the global attractor $\mathcal{A}_f$ to be invariant under the S^1-action and reduces the number of possibilities for the critical elements of the semiflow. In the generic situation, the set of critical elements is composed of a finite set of hyperbolic homogeneous equilibria, corresponding to the solutions of $f(e, 0) = 0$, and a finite set of hyperbolic rotating waves as mentioned in the introduction. From the results in [8] it follows that, in the class of nonlinearities $f = f(u, u_x)$, the generic global attractor $\mathcal{A}_f$ is structurally stable.

The situation is quite different in the class of x-dependent nonlinearities $f = f(x, u, u_x)$. As pointed out before, the time periodic orbits need not be rotating waves. Furthermore, $\mathcal{A}_f$ may contain hyperbolic nonhomogeneous equilibria and orbits homoclinic to equilibria. The existence of these homoclinic orbits follows from a result in [29] asserting that the flow of any planar vector field can be realized locally by a flow embedding in a two-dimensional invariant subspace of (1.1). Therefore, nontransversal intersection between stable and unstable manifolds of equilibria actually takes place. Also, the occurrence of nonhomogeneous equilibria is a distinguished important feature, and their heteroclinic connections need to be analyzed.

Finally, to understand the global geometry of $\mathcal{A}_f$ when structural stability fails, it is important to consider local bifurcation problems that explore the boundary of the structurally stable set of global attractors. Here, in view of the result [29, Theorem 2], the set of codimension one bifurcations for vector fields in the plane should play a role.

In conclusion, all these considerations just show that the geometry of the global attractor of (1.1), in general, is far from understood.

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