

# REMARKS ON THE FRACTAL DIMENSION OF BI-SPACE GLOBAL AND EXPONENTIAL ATTRACTORS

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ABSTRACT. Bi-space global and exponential attractors for the time continuous dynamical systems are considered and the bounds on their fractal dimension are discussed in the context of the smoothing properties of the system between appropriately chosen function spaces. The case when the system exhibits merely some partial smoothing properties is also considered and applications to the sample problems are given.

SUNTO. In questo lavoro sono considerate le nozioni di attrattori globali ed esponenziali “bi-space” per sistemi dinamici continui, e discusse limitazioni relative alla loro dimensione frattale in spazi di funzioni opportuni. Di particolare interesse é il caso in cui il sistema presenta un parziale effetto regolarizzante, ed alcuni esempi con questa proprietà sono mostrati.

## 1. INTRODUCTORY NOTES

Attractors for the dynamical systems governed by partial differential equations in infinite dimensional Banach spaces have been considered by several authors within past few decades; see [3, 8, 11, 16, 18, 29, 1, 10, 12, 14, 21] and references therein. A general approach leading to the description of their topological dimension have essentially been developed, allowing to obtain much of the relevant information about the asymptotic behavior of the systems corresponding to a number of physical equations in both Hilbert and Banach function space setting.

In this article we reconstruct the results of [1, 3, 11, 12] concerning the bi-space global and exponential attractors and provide simultaneously certain bounds on their fractal dimension, which are a straightforward consequence of the smoothing properties of the dynamical system acting between appropriately chosen function spaces. These mentioned properties are typical for problems that fall into a class of abstract parabolic equations in Banach spaces. Nonetheless, a situation that some partial smoothing occurs in a system can be observed in a much wider class of evolutionary problems. The specific ‘dissipative and smoothing mechanism’ can be then described, that leads to the existence of the global and exponential attractors and provides the bounds on their fractal dimension.

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2000 *Mathematics Subject Classification.* Primary 35B40; Secondary 35K90, 35K55.

*Key words and phrases.* Global attractor, exponential attractor, fractal dimension.

\* The author was supported by the fellowship of project POCI 2010/FEDER by CAMGSD, Instituto Superior Técnico, Lisbon.

\*\* The author was supported by the Postdoctoral Fellowship of the Japan Society for the Promotion of Sciences (No. PE06067).

The article is organized as follows. In Section 2 we obtain some estimate concerning fractal dimension of a precompact negatively invariant set. We reconstruct the existence of a bi-space global attractor, derive a bound for its fractal dimension and investigate the existence of a (finite dimensional) exponential attractor. Next, in Section 3, we consider global attractors with bounded fractal dimension for a semi-group  $\{S(t)\}$  governed by an abstract semilinear parabolic equation  $u_t + Au = F(u)$  in a Banach space  $X$ . We consider then some specific applications. These concern a strongly damped wave equation, including the case when the resolvent operators corresponding to a linear operator  $A$  are non-compact and nonlinear term satisfies a critical growth condition, reaction diffusion equations with subquadratically growing gradient term, and the higher order parabolic problems involving  $2m$ -th order elliptic operators in the main part and fast growing nonlinearities. Another specific application involving a (non-parabolic) evolution problem is also discussed in the context of a conserved phase-field system with thermal memory.

**Acknowledgement.** This work was carried out while the first author visited Centro de Análise Matemática, Geometria e Sistemas Dinâmicos at Instituto Superior Técnico in Lisbon, Portugal. He wishes to acknowledge the hospitality of the people from this Institution.

## 2. FRACTAL DIMENSION OF NEGATIVELY INVARIANT SETS AND BI-SPACE ATTRACTORS

In this section we estimate fractal dimension of a precompact negatively invariant set proving a certain generalization of [21, Lemma 1.3]. We discuss next the existence of a suitable notion of attractor and obtain some bounds of its fractal dimension.

Throughout this section  $V$  denotes a metric space. Recall that  $\emptyset \neq C \subset V$  is precompact in  $V$  iff each sequence of elements from  $C$  possesses a Cauchy subsequence or, equivalently, for any  $\varepsilon > 0$  there exist certain  $n \in \mathbb{N}$  and  $u_1, \dots, u_n \in C$  such that  $C \subset \bigcup_{i=1}^n B^V(u_i, \varepsilon)$ , where  $B^V(u_i, \varepsilon)$  are open balls centered at  $u_i$  of radius  $\varepsilon$ . Recall also that if  $C \neq \emptyset$  is precompact in  $V$  then its *fractal dimension* is

$$d_f^V(C) = \limsup_{\varepsilon \rightarrow 0} \log_{\frac{1}{\varepsilon}} N_\varepsilon^V(C),$$

where  $N_\varepsilon^V(C)$  denotes the smallest number of  $\varepsilon$ -balls in  $V$  needed to cover  $C$ .

Note that  $d_f^V(C) = d_f^V(\text{cl}_V C)$  and let

$$\tilde{d}_f^V(C) = \limsup_{\varepsilon \rightarrow 0} \log_{\frac{1}{\varepsilon}} \tilde{N}_\varepsilon^V(C),$$

where  $\tilde{N}_\varepsilon^V(C)$  denotes the smallest number of  $\varepsilon$ -balls in  $V$  with centers in  $C$  needed to cover  $C$ . Since  $N_\varepsilon^V(C) \leq \tilde{N}_\varepsilon^V(C) \leq N_{\frac{\varepsilon}{2}}^V(C)$ , then for a nonvoid set  $C$  precompact in  $V$  we have

$$d_f^V(C) = \tilde{d}_f^V(C).$$

Furthermore, if a subset  $V_0 \subset V$  is considered with the inherited metric and  $C \subset V_0 \subset V$ , then  $C$  is precompact in  $V_0$  and

$$d_f^{V_0}(C) = d_f^V(C).$$

The following result is a generalization of [21, Lemma 1.3].

**Lemma 2.1.** *Let  $V, W$  be normed spaces such that  $V$  is compactly embedded in  $W$  and let  $C \neq \emptyset$  be a precompact subset of  $W$ , negatively invariant under a map  $S$ , i.e.  $C \subset S(C)$ . Assume that for each  $u_0 \in C$  the map  $S$  has the following decomposition*

$$(2.1) \quad S = P(u_0) + M(u_0), \text{ where } P(u_0): C \rightarrow W, M(u_0): C \rightarrow V,$$

and

$$(2.2) \quad \exists_{0 \leq \delta < \frac{1}{2}} \exists_{\varepsilon_0 > 0} \forall_{u_0, u_1 \in C, \|u_1 - u_0\|_W \leq \varepsilon_0} \|P(u_0)u_1 - P(u_0)u_0\|_W \leq \delta \|u_1 - u_0\|_W,$$

$$(2.3) \quad \exists_{\kappa > 0} \forall_{u_0, u_1 \in C} \|M(u_0)u_1 - M(u_0)u_0\|_V \leq \kappa \|u_1 - u_0\|_W.$$

Under the above assumptions we have for any  $\nu \in (0, \frac{1}{2} - \delta)$

$$(2.4) \quad d_f^W(C) \leq \log_{\frac{1}{2(\delta+\nu)}} N_{\frac{\nu}{\kappa}}^W(B^V(0, 1)).$$

*Proof.* For  $\nu \in (0, \frac{1}{2} - \delta)$  let  $N = N_{\frac{\nu}{\kappa}}^W(B^V(0, 1))$  and consider  $0 < \varepsilon \leq \varepsilon_0$ . Note that

$$C \subset \bigcup_{1 \leq i \leq \tilde{N}_\varepsilon^W(C)} B^W(u_i, \varepsilon) \quad \text{with certain } u_1, \dots, u_{\tilde{N}_\varepsilon^W(C)} \in C.$$

If  $u \in C$ , then  $u \in B^W(u_i, \varepsilon)$  and from (2.3) we obtain

$$\|M(u_i)u - M(u_i)u_i\|_V \leq \kappa \|u - u_i\|_W < \kappa \varepsilon,$$

ensuring that

$$\frac{1}{\kappa \varepsilon} (M(u_i)u - M(u_i)u_i) \in B^V(0, 1) \subset \bigcup_{1 \leq j \leq N} B^W(w_j, \frac{\nu}{\kappa}).$$

Thus we have  $\|M(u_i)u - M(u_i)u_i - \kappa \varepsilon w_j\|_W < \nu \varepsilon$  and we use (2.2) to get

$$\|P(u_i)u - P(u_i)u_i\|_W + \|M(u_i)u - M(u_i)u_i - \kappa \varepsilon w_j\|_W < (\delta + \nu) \varepsilon.$$

Since (2.1) holds and  $C \subset S(C)$ , what was said above ensures that

$$C \subset \bigcup_{1 \leq i \leq \tilde{N}_\varepsilon^W(C)} \bigcup_{1 \leq j \leq N} B^W(S(u_i) + \kappa \varepsilon w_j, (\delta + \nu) \varepsilon).$$

If we now increase the radius of the balls twice we will obtain

$$C \subset \bigcup_{1 \leq i \leq \tilde{N}_\varepsilon^W(C)} \bigcup_{1 \leq j \leq N} B^W(\tilde{w}_{ij}, 2(\delta + \nu) \varepsilon) \quad \text{with centers } \tilde{w}_{ij} \in C,$$

which shows that for every  $0 < \varepsilon \leq \varepsilon_0$  we have

$$(2.5) \quad \tilde{N}_{2(\delta+\nu)\varepsilon}^W(C) \leq \tilde{N}_\varepsilon^W(C) \cdot N.$$

Induction argument ensures that for any  $k \in \mathbb{N}$

$$\tilde{N}_{[2(\delta+\nu)]^k \varepsilon_0}^W(C) \leq \tilde{N}_{\varepsilon_0}^W(C) \cdot N^k.$$

Since for small  $\varepsilon \in (0, \varepsilon_0]$  there is  $k \in \mathbb{N}$  such that  $[2(\delta + \nu)]^{k+1} \varepsilon_0 < \varepsilon \leq [2(\delta + \nu)]^k \varepsilon_0$ , the estimate (2.4) now follows from the relation

$$\log_{\frac{1}{\varepsilon}} \tilde{N}_\varepsilon^W(C) \leq \log_{\frac{1}{[2(\delta+\nu)]^k \varepsilon_0}} \tilde{N}_{\varepsilon_0}^W(C) \cdot N^{k+1}. \quad \square$$

In what follows  $\{S(t)\}$  denotes a family of maps such that

$$(2.6) \quad S(t): V \rightarrow V, t \geq 0, S(0) = Id \text{ and } S(t)S(s) = S(t+s) \text{ for } s, t \geq 0.$$

It is reasonable to consider the situation when orbits of bounded sets eventually enter another metric space  $W$  and discuss some minimal smoothness conditions on  $\{S(t)\}$  that lead to the existence of an attracting compact invariant set.

We denote by  $\rho_W$  (resp.  $d_W$ ) the metric (resp. the Hausdorff semidistance) in  $W$ . If there is  $B_0$  bounded in  $V$  which *absorbs* bounded subsets of  $V$ , that is

$$S(t)B \subset B_0 \text{ for each } B \text{ bounded in } V \text{ and all } t \geq t_B > 0,$$

then following [1] we will consider the two *asymptotic smoothness conditions*:

$$(2.7) \quad \begin{aligned} &\text{any sequence } \{S(t_n)v_n\} \text{ with } t_n \rightarrow \infty \text{ and } \{v_n\} \subset B_0 \\ &\text{has a subsequence convergent in the metric of } W, \end{aligned}$$

$$(2.8) \quad \begin{aligned} &\text{for each sequence } \{S(t_n)v_n\} \text{ with } t_n \rightarrow \infty, \{v_n\} \subset B_0 \text{ there} \\ &\text{is a subsequence } \{S(t_{n_k})v_{n_k}\} \text{ and a certain point } w \in V \cap W \\ &\text{such that } \lim_{k \rightarrow \infty} \rho_W(S(t)S(t_{n_k})v_{n_k}, S(t)w) = 0 \text{ for all } t \geq 0. \end{aligned}$$

**Proposition 2.2.** ([1]) *Suppose  $B_0$  is bounded in  $V$ , absorbs bounded subsets of  $V$  and  $W$  is a metric space containing  $B_0$ .*

- (i) *If (2.7) holds, then there is a nonvoid set  $\mathbf{A} \subset \text{cl}_W B_0$ , compact in  $W$  and attracting sets bounded in  $V$  with respect to the Hausdorff semidistance  $d_W$ .*
- (ii) *If (2.8) holds, then  $\mathbf{A}$  is additionally a subset of  $V$  invariant under  $\{S(t)\}$ . Moreover,  $\mathbf{A}$  is closed in  $V$  provided that*

$$\{w_m\} \subset V \cap W, \rho_V(w_m, w) \rightarrow 0 \text{ and } \rho_W(w_m, \tilde{w}) \rightarrow 0 \text{ imply } w = \tilde{w}.$$

*Proof.* Following [1], the result is a consequence of the definition of

$$(2.9) \quad \mathbf{A} = \{w \in W : S(t_n)v_n \rightarrow w \text{ in } W \text{ for some } \{v_n\} \subset B_0 \text{ and } t_n \rightarrow \infty\}. \quad \square$$

Coming back to [3] (see also [1] and references therein) we recall that a nonvoid set  $\mathbf{A} \subset V \cap W$ , which is invariant under  $\{S(t)\}$ , closed in  $V$ , compact in  $W$  and attracts bounded subsets of  $V$  with respect to the Hausdorff semidistance  $d_W$  is called a *global  $(V - W)$  attractor* for  $\{S(t)\}$ . Proposition 2.2 then implies

**Corollary 2.3.** *Let  $W$  be a metric space such that  $V \subset W$  and  $\rho_V(w_m, w) \rightarrow 0$  implies  $\rho_W(w_m, w) \rightarrow 0$ . Suppose that  $B_0$  is bounded in  $V$ ,  $B_0$  absorbs bounded subsets of  $V$  and (2.8) holds.*

*Under these assumptions there exists a global  $(V - W)$  attractor  $\mathbf{A}$  for  $\{S(t)\}$  and if, in addition,  $V$  is a reflexive Banach space and  $W$  is a normed space, then  $\mathbf{A}$  is bounded in  $V$ .*

*Proof.* By Proposition 2.2 the set  $\mathbf{A}$  in (2.9) is a  $(V - W)$  global attractor and  $\mathbf{A}$  is bounded in  $V$  as a consequence of weak convergence properties.  $\square$

If  $V = W$ , then a global  $(V - V)$  attractor is a notion of a *global attractor* in [16] and of a *compact invariant  $B$ -attractor* in [18]. In this case Proposition 2.2 implies

**Corollary 2.4.** *Suppose  $B_0$  is bounded in  $V$  and absorbs bounded subsets of  $V$ .*

- (i) *If (2.8) holds with  $W = V$ , then there exists a global attractor (a compact invariant  $B$ -attractor) for  $\{S(t)\}$ .*
- (ii) *If (2.7) holds with  $W = V$  and the map  $V \ni v \rightarrow S(t)v \in V$  is continuous<sup>1</sup> for each  $t \in [0, \infty)$ , then there is a global attractor (a compact invariant  $B$ -attractor)  $\mathbf{A}$  for  $\{S(t)\}$ .*

With further assumptions on  $\{S(t)\}$  fractal dimension of  $\mathbf{A}$  can also be estimated and boundedness of  $\mathbf{A}$  in  $V$  will follow even if  $V$  is not reflexive. In the light of Lemma 2.1 and Corollary 2.3 the following results hold.

**Theorem 2.5.** *If  $V, W$  are normed spaces and the assumptions of Corollary 2.3 hold, then there exists a global  $(V - W)$  attractor  $\mathbf{A}$ .*

*If, in addition,  $V$  is compactly embedded in  $W$  and there exists  $t_0 > 0$  such that*

$$S(t_0) = P(t_0)(u_0) + M(t_0)(u_0), \quad u_0 \in \mathbf{A},$$

*where  $P(t_0)(u_0): \mathbf{A} \rightarrow W$ ,  $M(t_0)(u_0): \mathbf{A} \rightarrow V$  satisfy*

$$(2.10) \quad \forall_{u_0, u_1 \in \mathbf{A}, \|u_1 - u_0\|_W \leq \varepsilon_0} \|P(t_0)(u_0)u_1 - P(t_0)(u_0)u_0\|_W \leq \delta \|u_1 - u_0\|_W,$$

$$(2.11) \quad \forall_{u_0, u_1 \in \mathbf{A}} \|M(t_0)(u_0)u_1 - M(t_0)(u_0)u_0\|_V \leq \kappa \|u_1 - u_0\|_W,$$

*with some  $0 \leq \delta < \frac{1}{2}$ ,  $\varepsilon_0 > 0$  and  $\kappa > 0$ , then for every  $\nu \in (0, \frac{1}{2} - \delta)$  we have*

$$(2.12) \quad d_f^W(\mathbf{A}) \leq \log_{\frac{1}{2(\delta+\nu)}} N_{\frac{\nu}{\kappa}}^W(B^V(0, 1)).$$

**Corollary 2.6.** *If  $V, W$  are normed spaces and the assumptions of Corollary 2.3 hold, then there exists a global  $(V - W)$  attractor  $\mathbf{A}$ .*

*If, in addition,  $V$  is compactly embedded in  $W$  and*

$$(2.13) \quad \exists_{t_0 > 0} \exists_{\kappa > 0} \forall_{u_1, u_2 \in \mathbf{A}} \|S(t_0)u_1 - S(t_0)u_2\|_V \leq \kappa \|u_1 - u_2\|_W,$$

*then  $\mathbf{A}$  is compact in  $V$  and satisfies*

$$(2.14) \quad d_f^V(\mathbf{A}) \leq d_f^W(\mathbf{A}) \leq \log_{\frac{1}{2\nu}} N_{\frac{\nu}{\kappa}}^W(B^V(0, 1)), \quad \nu \in (0, \frac{1}{2}).$$

In the similar vein, following the ideas of [9, 12, 24], we prove

**Proposition 2.7.** *Let  $V$  be a Banach space compactly embedded in a Banach space  $W$ . Assume that  $V_0$  is a subset of  $V$  and let  $B_0$  be a bounded set of  $V_0$  absorbing bounded subsets of  $V_0$ . Suppose that there exists  $t_0 \geq t_{B_0}$  such that*

$$(2.15) \quad S(t_0) = P(t_0) + M(t_0),$$

*where  $P(t_0): B_0 \rightarrow W$ ,  $M(t_0): B_0 \rightarrow V$  satisfy with some  $0 \leq \delta < \frac{1}{2}$ ,  $\kappa > 0$ ,  $0 < \theta < 1$  and  $\mu > 0$  the following estimates*

$$(2.16) \quad \forall_{u_1, u_2 \in B_0} \|P(t_0)u_1 - P(t_0)u_2\|_W \leq \delta \|u_1 - u_2\|_W,$$

$$(2.17) \quad \forall_{u_1, u_2 \in B_0} \|M(t_0)u_1 - M(t_0)u_2\|_V \leq \kappa \|u_1 - u_2\|_W,$$

$$(2.18) \quad \forall_{t_1, t_2 \in [t_0, 2t_0]} \forall_{u_1, u_2 \in B_0} \|S(t_1)u_1 - S(t_2)u_2\|_W \leq \mu(|t_1 - t_2|^\theta + \|u_1 - u_2\|_W).$$

<sup>1</sup>See [26], where continuity assumption is replaced by the requirement that  $S(t)$  is a closed map.

Then for any  $\nu \in (0, \frac{1}{2} - \delta)$  there exists a nonvoid set  $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_\nu \subset B_0$ , positively invariant under  $\{S(t)\}$ , precompact in  $W$  and satisfying conditions

- (i)  $\exists_{\omega > 0} \forall_{B \text{ bounded in } V_0} \lim_{t \rightarrow \infty} e^{\omega t} d_W(S(t)B, \widehat{\mathcal{M}}) = 0$ ,
- (ii)  $d_f^W(\widehat{\mathcal{M}}) \leq \frac{1}{\theta} (1 + \log_{\frac{1}{2(\delta+\nu)}} N_{\frac{\nu}{\kappa}}^W(B^V(0, 1)))$ .

*Proof.* Fix  $\nu \in (0, \frac{1}{2} - \delta)$  and note that  $S(t)B_0 \subset B_0$  for  $t \geq t_0$ . Since  $B_0$  is bounded in  $W$ , from [12, Proposition 1] we deduce that there is a nonvoid set  $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_\nu \subset B_0$ , precompact in  $W$  satisfying  $S(t_0)\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{M}}$ ,  $d_f^W(\widetilde{\mathcal{M}}) \leq \log_{\frac{1}{2(\delta+\nu)}} N_{\frac{\nu}{\kappa}}^W(B^V(0, 1))$  and such that, for certain  $C, \xi > 0$ , we have  $d_W(S(nt_0)B_0, \widetilde{\mathcal{M}}) \leq Ce^{-\xi n}$  for each  $n \in \mathbb{N}$ . Setting  $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_\nu := \bigcup_{s \in [t_0, 2t_0]} S(s)\widetilde{\mathcal{M}} \subset B_0$  we then have that  $S(t)\widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}$  for  $t \geq 0$ . As a result of (2.18) the set  $\widehat{\mathcal{M}}$  is also precompact in  $W$  and

$$d_f^W(\widehat{\mathcal{M}}) \leq \frac{1}{\theta} d_f^{\mathbb{R} \times W}([t_0, 2t_0] \times \widetilde{\mathcal{M}}) \leq \frac{1}{\theta} (1 + d_f^W(\widetilde{\mathcal{M}})).$$

If  $B$  is a bounded subset of  $V_0$ , then  $S(t_B)B \subset B_0$ . If furthermore  $t \geq t_B + 3t_0$ , then  $t - t_B = n_t t_0 + 2t_0 + r_t$  with certain  $r_t \in [0, t_0]$ ,  $n_t \in \mathbb{N}$ , and via (2.18) we get

$$\begin{aligned} d_W(S(t)B, \widehat{\mathcal{M}}) &= d_W(S(t - t_B)S(t_B)B, \widehat{\mathcal{M}}) \leq d_W(S(t - t_B)B_0, \widehat{\mathcal{M}}) \\ &\leq d_W(S(t_0)S(n_t t_0)B_0, S(t_0)\widetilde{\mathcal{M}}) \leq \mu d_W(S(n_t t_0)B_0, \widetilde{\mathcal{M}}) \leq C_B e^{-\frac{\xi}{t_0} t}, \end{aligned}$$

which completes the proof.  $\square$

Generalizing the notion of an exponential attractor in [11] we will say that a nonvoid set  $\mathcal{M} \subset V \cap W$  is an *exponential  $(V - W)$  attractor* for  $\{S(t)\}$  if  $\mathcal{M}$  is positively invariant under  $\{S(t)\}$ , closed in  $V$ , compact in  $W$ ,  $d_f^W(\mathcal{M}) < \infty$  and

$$(2.19) \quad \exists_{\omega > 0} \forall_{B \text{ bounded in } V} \lim_{t \rightarrow \infty} e^{\omega t} d_W(S(t)B, \mathcal{M}) = 0.$$

**Corollary 2.8.** *If  $V$  is reflexive, the assumptions of Proposition 2.7 hold with  $V_0 = V$  and  $S(t) : W \supset \text{cl}_W B_0 \rightarrow W$  is continuous for each  $t > 0$ , then, for  $\nu \in (0, \frac{1}{2} - \delta)$ ,*

- (i)  $\mathcal{M}_\nu := \text{cl}_W \widehat{\mathcal{M}}_\nu$  is an exponential  $(V - W)$  attractor bounded in  $V$ ,
- (ii) there exists a finite dimensional global  $(V - W)$  attractor  $\mathbf{A} \subset \mathcal{M}_\nu$ .

**Corollary 2.9.** *Let  $V$  be a Banach space compactly embedded in a Banach space  $W$  and let  $B_0$  be a bounded and closed set in  $V$  absorbing bounded subsets of  $V$ . Suppose that (2.18) holds and*

$$(2.20) \quad \forall_{t > 0} \exists_{\kappa(t) > 0} \forall_{u_1, u_2 \in B_0} \|S(t)u_1 - S(t)u_2\|_V \leq \kappa(t) \|u_1 - u_2\|_W.$$

Then for any  $\nu \in (0, \frac{1}{2})$

- (i) there exists an exponential  $(V - V)$  attractor  $\mathcal{M}_\nu \subset B_0$  satisfying

$$(2.21) \quad d_f^V(\mathcal{M}_\nu) \leq \frac{1}{\theta} \left( 1 + \log_{\frac{1}{2\nu}} N_{\frac{\nu}{\kappa_0}}^W(B^V(0, 1)) \right),$$

where  $\kappa_0 = \kappa(t_0)$ ,

- (ii) there exists a finite dimensional global  $(V - V)$  attractor  $\mathbf{A}$  contained in  $\mathcal{M}_\nu$ .

*Proof.* (i) The argument is the same as in the proof of Proposition 2.7, but we also set  $\mathcal{M}_\nu := \text{cl}_V S(1)\widehat{\mathcal{M}}_\nu$  and observe by (2.20) that  $\mathcal{M}_\nu \subset \text{cl}_V B_0 = B_0$  is compact in  $V$  and positively invariant under  $\{S(t)\}$ . Moreover,  $d_f^V(\mathcal{M}_\nu) \leq d_f^W(\widehat{\mathcal{M}}_\nu)$  and for  $t \geq t_B + 3t_0 + 1$  we also have

$$d_V(S(t)B, \mathcal{M}_\nu) = d_V(S(1)S(t-1)B, S(1)\widehat{\mathcal{M}}_\nu) \leq \kappa(1)d_W(S(t-1)B, \widehat{\mathcal{M}}_\nu),$$

which ensures that (2.19) holds with  $V = W$ . This completes the proof of (i).

Part (ii) now follows easily via Corollary 2.4.  $\square$

### 3. APPLICATIONS

In this section we present the results concerning bi-space attractors with bounded fractal dimension for the semigroup governed by the Cauchy problem for an abstract parabolic equation. We also consider some specific examples, like a strongly damped wave equation involving a critically growing nonlinearity and a linear main part with non-compact resolvent, reaction-diffusion equations with subquadratically growing gradient term and  $2m$ -th order parabolic problems with fast growing nonlinearities. We finally discuss a (non-parabolic) evolution problem, which will be a conserved phase-field system with thermal memory.

**3.1. Abstract semilinear parabolic problems.** Throughout this subsection  $X$  denotes a Banach space,  $A: X \supset D(A) \rightarrow X$  is a positive sectorial operator in  $X$  and  $X^\sigma$ ,  $\sigma \geq 0$ , are the associated fractional power spaces.

It is known that  $-A$  generates in  $X = X^0$  a  $C^0$  analytic semigroup  $\{e^{-At}\}$  and

$$(3.1) \quad \|e^{-At}\|_{L(X, X^\sigma)} \leq c_\sigma \frac{e^{-at}}{t^\sigma}, \quad \sigma \geq 0, t > 0,$$

where  $a > 0$  is such that  $\text{Re}\sigma(A) > a$  and  $c_\sigma$  are certain positive constants.

In this subsection we fix  $\alpha \in [0, 1)$  and assume that  $F: X^\alpha \rightarrow X^0$  satisfies

$$(3.2) \quad \forall B \text{ bounded in } X^\alpha \exists L_{\alpha, B} > 0 \forall v, w \in B \|F(v) - F(w)\|_{X^0} \leq L_{\alpha, B} \|v - w\|_{X^\alpha}.$$

We remark that the inequality in (3.2) is now true with  $\alpha$  replaced by any  $\beta \in [\alpha, 1)$ . We consider the Cauchy problem

$$(3.3) \quad \begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0 \in X^\alpha. \end{cases}$$

It is known from [17] that there is a certain  $\tau_{u_0} > 0$  such that in  $C([0, \tau_{u_0}), X^\alpha)$  there exists a unique mild solution  $u(\cdot, u_0)$  of (3.3) and either  $\tau_{u_0} = \infty$ , that is  $u(\cdot, u_0)$  exists globally in time, or  $\tau_{u_0} < \infty$  and  $\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t, u_0)\|_{X^\alpha} = \infty$ . Such  $u(\cdot, u_0)$  satisfies variation of constants formula

$$(3.4) \quad u(t, u_0) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s, u_0))ds, \quad t \in [0, \tau_{u_0}),$$

and has additional regularity properties; namely

$$u(\cdot, u_0) \in C([0, \tau_{u_0}), X^\alpha) \cap C^1((0, \tau_{u_0}), X^\alpha) \cap C((0, \tau_{u_0}), X^1).$$

In what follows such  $u(\cdot, u_0)$  will be called an  $X^\alpha$  solution of (3.3).

We will assume throughout this subsection that all  $X^\alpha$  solutions of (3.3) exist globally in time, in which case (3.3) defines a  $C^0$  semigroup  $\{S(t)\}$  on  $X^\alpha$  such that

$$S(t)u_0 := u(t, u_0), \quad t \geq 0, \quad u_0 \in X^\alpha.$$

From (3.1), (3.2) and (3.4) we obtain the following auxiliary relation.

**Lemma 3.1.** *If  $t > 0$ ,  $\beta \in [\alpha, 1)$ ,  $\mathcal{B}$  is bounded in  $X^\alpha$  and  $S(s)u_1, S(s)u_2 \in \mathcal{B}$  for each  $s \in [0, t]$ , then*

$$\|S(t)u_1 - S(t)u_2\|_{X^\beta} \leq \frac{c_{\beta-\alpha}}{t^{\beta-\alpha}} \|u_1 - u_2\|_{X^\alpha} + \int_0^t \frac{c_\beta L_{\beta, \mathcal{B}}}{(t-s)^\beta} \|S(s)u_1 - S(s)u_2\|_{X^\beta} ds.$$

**Theorem 3.2.** *Suppose that the resolvent of  $A$  is compact,  $B_0$  is bounded in  $X^\alpha$  and absorbs bounded subsets of  $X^\alpha$ .*

*Then, for each  $\beta \in (\alpha, 1)$ ,*

(i) *there exists a global  $(X^\alpha - X^\beta)$  attractor  $\mathbf{A}$  for  $\{S(t)\}$  satisfying*

$$(3.5) \quad d_f^{X^\beta}(\mathbf{A}) \leq d_f^{X^\alpha}(\mathbf{A}) \leq \log_2 N_{\frac{1}{4\kappa_\beta}}^{X^\alpha}(B^{X^\beta}(0, 1)),$$

*with  $\kappa_\beta$  specified in (3.7),*

(ii) *there exists an exponential  $(X^\alpha - X^\beta)$  attractor  $\mathcal{M}$  containing  $\mathbf{A}$ .*

*Proof.* Fix  $\beta \in (\alpha, 1)$  and note that  $\{S(t)\}$  is a  $C^0$  semigroup on  $X^\beta$ . By assumption,  $X^\sigma$  is compactly embedded into  $X^\gamma$  for each  $\sigma > \gamma$ . From [8, Lemma 3.2.1] it follows that  $\tilde{B}_0 := \text{cl}_{X^\beta} S(t_{B_0})B_0$  is compact in  $X^\beta$  and absorbs bounded subsets of  $X^\beta$ .

(i) Corollary 2.4(ii) applies with  $V = W = X^\beta$  and there is a global  $(X^\beta - X^\beta)$  attractor  $\mathbf{A}$ , which as a set does not depend on  $\beta$ . We observe that  $\mathbf{A}$  is a global  $(X^\alpha - X^\beta)$  attractor because  $S(t)B \subset \tilde{B}_0$  for any  $B$  bounded in  $X^\alpha$  and all  $t$  sufficiently large.

Applying Lemma 3.1 with  $\mathcal{B} = \mathbf{A}$ ,  $\beta \in (\alpha, 1)$ ,  $u_1, u_2 \in \mathbf{A}$ , and  $t = t^*$  satisfying

$$(3.6) \quad \frac{c_\beta L_{\beta, \mathbf{A}} t^{*1-\beta}}{2^{1-2\beta+\alpha}} \left( \frac{1}{1-\beta+\alpha} + \frac{1}{1-\beta} \right) = \frac{1}{2}$$

we obtain via Volterra inequality (see [8, formulas (1.2.21) and (1.2.30)]) that

$$(3.7) \quad \|S(t^*)u_1 - S(t^*)u_2\|_{X^\beta} \leq 2c_{\beta-\alpha} t^{*\alpha-\beta} \|u_1 - u_2\|_{X^\alpha} =: \kappa_\beta \|u_1 - u_2\|_{X^\alpha}.$$

In particular, Corollary 2.6 applies with  $V = X^\beta$ ,  $W = X^\alpha$  and with the constant  $\kappa = \kappa_\beta = 2c_{\beta-\alpha} t^{*\alpha-\beta}$ . Part (i) is thus proved.

(ii) By [8, Corollary 3.3.2] and the properties of  $\tilde{B}_0$ , the positive orbit  $\gamma^+(\tilde{B}_0)$  of  $\tilde{B}_0$  is bounded in  $X^\beta$ . Applying Lemma 3.1 with  $\mathcal{B} = \gamma^+(\tilde{B}_0)$ ,  $u_1, u_2 \in \tilde{B}_0$ ,  $0 < t < T$  and using the Volterra inequality we get

$$(3.8) \quad \|S(t)u_1 - S(t)u_2\|_{X^\beta} \leq t^{\alpha-\beta} \text{const}(T, \alpha, \beta, \tilde{B}_0) \|u_1 - u_2\|_{X^\alpha}, \quad 0 < t < T.$$

In particular there is a constant  $c > 0$  independent of  $u_1, u_2 \in \tilde{B}_0$  and  $t \in [\tau_1, \tau_2] \subset (0, \infty)$  such that

$$(3.9) \quad \|S(t)u_1 - S(t)u_2\|_{X^\alpha} \leq c \|u_1 - u_2\|_{X^\alpha}, \quad u_1, u_2 \in \tilde{B}_0, \quad t \in [\tau_1, \tau_2].$$



From [8, formula (2.2.3)] it follows for  $u \in \tilde{B}_0$  and  $t_1, t_2 \in [\tau_1, \tau_2]$  that

$$(3.10) \quad \|S(t_1)u - S(t_2)u\|_{X^\alpha} \leq \tilde{c} |t_1 - t_2|^\theta,$$

where  $\theta \in (0, 1)$  and  $\tilde{c} > 0$  do not depend on  $t_1, t_2 \in [\tau_1, \tau_2]$  and  $u \in \tilde{B}_0$ . Therefore, Corollary 2.9 applies with  $V = X^\beta$ ,  $W = X^\alpha$  and there exists an exponential  $(X^\beta - X^\alpha)$  attractor  $\mathcal{M}$ . Since  $S(t)B \subset \tilde{B}_0$  for any  $B$  bounded in  $X^\alpha$  and all  $t$  sufficiently large,  $\mathcal{M}$  is an exponential  $(X^\alpha - X^\beta)$  attractor and the proof is complete.  $\square$

Repeating the argument leading to (3.7) in the proof of Theorem 3.2 and using Lemma 2.1 with  $S = S(t^*)$ ,  $P(u_0) \equiv 0$ ,  $M(u_0) \equiv S(t^*)$ , we obtain

**Corollary 3.3.** *If the resolvent of  $A$  is compact and  $\mathbf{A}$  is a nonvoid bounded and invariant subset of  $X^\alpha$ , then  $\mathbf{A}$  is precompact in  $X^\sigma$ ,  $\sigma \in [\alpha, 1)$ , and*

$$(3.11) \quad d_f^{X^\beta}(\mathbf{A}) \leq d_f^{X^\alpha}(\mathbf{A}) \leq \log_2 N_{\frac{1}{4\kappa_\beta}}^{X^\alpha}(B^{X^\beta}(0, 1)), \quad \beta \in (\alpha, 1),$$

with  $\kappa_\beta$  as in (3.7).

**Remark 3.4.** We remark that the estimate (3.11) requires the knowledge of the  $\varepsilon$ -entropy  $\log_2 N_\varepsilon^{X^\alpha}(B^{X^\beta}(0, 1))$ , for which in the case when abstract fractional power spaces are involved the explicit estimates may not be easily available. However, in applications, fractional power scale  $X^\alpha$ ,  $\alpha \geq 0$ , is often characterized with the aid of some known function spaces, for which such estimates appear within the references (see e.g. [30, §4.10.3]).

**3.2. Second order parabolic equations in bounded domains.** Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with smooth boundary and consider the second order problem

$$(3.12) \quad u_t = \Delta u + f(x, u, \nabla u), \quad t > 0, \quad x \in \Omega, \quad u(0, x) = u_0(x), \quad x \in \Omega,$$

with Dirichlet homogeneous boundary condition.

Following [28] (see also [19]) assume for  $f \in C(\bar{\Omega} \times \mathbb{R}^{N+1}, \mathbb{R})$  that

$$(3.13) \quad f \text{ is locally Lipschitz with respect to each variable separately,}$$

$$(3.14) \quad sf(x, s, \mathbf{0}) \leq 0 \text{ for all } x \in \bar{\Omega}, \quad s \in \mathbb{R}, \quad |s| \geq K \text{ with a certain } K > 0,$$

and that for some continuous map  $h: [0, \infty) \rightarrow [0, \infty)$  and certain  $\gamma \in (0, 2)$

$$(3.15) \quad |f(x, s, \mathbf{p})| \leq h(|s|)(1 + |\mathbf{p}|^\gamma) \text{ whenever } x \in \bar{\Omega}, \quad s \in \mathbb{R}, \quad \mathbf{p} \in \mathbb{R}^N.$$

We consider in this example  $A = -\Delta$  in  $X^0 = L^p(\Omega)$ ,  $p > N$ , with the domain  $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  and choose  $\alpha \in (\frac{1}{2} + \frac{N}{2p}, 1)$ . The resolvent of  $A$  is compact and, since  $X^\alpha \hookrightarrow C^1(\bar{\Omega})$ , the function

$$F: X^\alpha \rightarrow X^0, \quad F(u)(x) = f(x, u(x), \nabla u(x)) \text{ for } u \in X^\alpha, \quad x \in \Omega,$$

fulfils (3.2) with  $X^\alpha = [L^p(\Omega), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]_\alpha =: W_0^{2\alpha,p}(\Omega)$  and  $X^0 = L^p(\Omega)$ .

Due to (3.14) and comparison argument we have (see [28] and references therein)

$$(3.16) \quad \|u(t, u_0)\|_{L^\infty(\Omega)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, K\}, \quad u_0 \in X^\alpha, \quad t \in (0, \tau_{u_0}),$$

$$(3.17) \quad \limsup_{t \rightarrow \infty} \sup_{u_0 \in B^{X^\alpha}(0, r)} \|u(t, u_0)\|_{L^\infty(\Omega)} \leq K, \quad r > 0.$$

Due to (3.15) one obtains as in [2] that, for certain  $0 < \eta < 1$  and  $\beta > \alpha$ ,

$$(3.18) \quad \|F(u(t, u_0))\|_{L^p(\Omega)} \leq c(\|u_0\|_{L^\infty(\Omega)})(1 + \|u(t, u_0)\|_{X^\beta}^\eta), \quad t \in (0, \tau_{u_0}).$$

From this we infer that all  $X^\alpha$  solutions exist globally in time and for the associated  $C^0$  semigroup  $\{S(t)\}$  on  $X^\alpha$  there is  $B_0$  bounded in  $X^\alpha$  and absorbing bounded subsets of  $X^\alpha$  (see [8, Corollary 4.1.3]); in particular Theorem 3.2 applies.

**Corollary 3.5.** *If (3.13), (3.14), (3.15) hold and  $\alpha \in (\frac{1}{2} + \frac{N}{2p}, 1)$ ,  $p > N$ , then for each  $\beta \in (\alpha, 1)$  there is a global  $(W_0^{2\alpha, p}(\Omega) - W_0^{2\beta, p}(\Omega))$  attractor  $\mathbf{A}$  satisfying*

$$(3.19) \quad d_f^{W_0^{2\beta, p}(\Omega)}(\mathbf{A}) \leq d_f^{W_0^{2\alpha, p}(\Omega)}(\mathbf{A}) \leq \log_2 N \frac{W_0^{2\alpha, p}(\Omega)}{4\kappa_\beta} (B^{W_0^{2\beta, p}(\Omega)}(0, 1)),$$

which is contained in an exponential  $(W_0^{2\alpha, p}(\Omega) - W_0^{2\beta, p}(\Omega))$  attractor.

When  $f = f(x, u)$  does not depend on the gradient we choose any  $p > N$ ,  $\alpha \in (\frac{N}{2p}, \frac{1}{2})$  and note that  $\phi(x) \in [-K, K]$  for  $\phi \in \mathbf{A}$  and  $x \in \Omega$ . If  $L$  is a (uniform for  $x \in \bar{\Omega}$ ) Lipschitz constant for  $f$  with respect to  $s \in [-K, K]$  and  $c_\Omega$  is a constant from the Poincaré inequality, then for  $\beta = \frac{1}{2}$  we have  $X^\beta = X^{\frac{1}{2}} = W_0^{1, p}(\Omega)$  and

$$\|f(\cdot, u_1(\cdot)) - f(\cdot, u_2(\cdot))\|_{L^p(\Omega)} \leq c_\Omega L \|u_1 - u_2\|_{W_0^{1, p}(\Omega)} =: L_{\frac{1}{2}, \mathbf{A}} \|u_1 - u_2\|_{W_0^{1, p}(\Omega)}$$

whenever  $u_1, u_2 \in \mathbf{A}$ . In this case (3.6)-(3.7) apply with  $L_{\beta, \mathbf{A}} = c_\Omega L$  and we get (3.19) with  $\kappa_\beta = 2c_{\frac{1}{2}-\alpha}(2^{3-\alpha}c_{\frac{1}{2}}c_\Omega L(1+\alpha)(1+2\alpha)^{-1})^{1-2\alpha}$ .

**3.3. Wave equation with damping operator  $(-\Delta_D)^{\frac{1}{2}}$ .** Following [4, 5], we consider<sup>2</sup>

$$(3.20) \quad \begin{cases} u_{tt} + \eta(-\Delta_D)^{\frac{1}{2}}u_t + (-\Delta_D)u = f(u), & t > 0, x \in \Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = v_0(x), & x \in \Omega, \\ u(t, x) = 0, & t \geq 0, x \in \partial\Omega, \end{cases}$$

where  $\eta > 0$ ,  $\Omega$  is a bounded  $C^2$  smooth domain in  $\mathbb{R}^N$ ,  $N \geq 3$ , and  $\Delta_D$  is the Dirichlet Laplacian in  $L^2(\Omega)$  with the domain  $H^2(\Omega) \cap H_0^1(\Omega)$ . We assume that

$$(3.21) \quad |f(s) - f(\bar{s})| \leq c|s - \bar{s}|(1 + |s|^{\rho-1} + |\bar{s}|^{\rho-1}), \quad s, \bar{s} \in \mathbb{R}, \quad \text{with } \rho \in (1, \frac{N+2}{N-2})$$

and

$$(3.22) \quad \limsup_{|s| \rightarrow \infty} \frac{f(s)}{s} \leq 0.$$

We can rewrite (3.20) in the form (3.3) in  $X^0 = H_0^1(\Omega) \times L^2(\Omega)$  with

$$A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ -\Delta_D \varphi + \eta(-\Delta_D)^{\frac{1}{2}}\psi \end{bmatrix} \quad \text{for } \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(A), \quad F(\begin{bmatrix} \varphi \\ \psi \end{bmatrix}) = \begin{bmatrix} 0 \\ f(\varphi) \end{bmatrix},$$

where

$$D(A) = X^1 = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega).$$

<sup>2</sup>Instead of  $(-\Delta_D)^{\frac{1}{2}}$ , chosen here for better clarity of argument, one can consider the damping operator  $(-\Delta_D)^\theta$  with  $\theta \in [\frac{1}{2}, 1)$ , but the analysis would have to be adapted.

From [7, 4] we know that  $A$  is a sectorial positive operator with compact resolvent and bounded imaginary powers. Therefore, we have (see [4, Proposition 3])

$$X^\alpha = [X^0, X^1]_\alpha = D((-\Delta_D)^{\frac{1+\alpha}{2}}) \times D((-\Delta_D)^{\frac{\alpha}{2}}), \quad \alpha \in (0, 1).$$

Set  $\alpha = \frac{N-2}{N+2}$ . By (3.21) it follows from [5, Lemma 2] that  $F$  satisfies (3.2). Recall from [5, Theorem 5] that (3.22) ensures the existence of a  $C^0$  semigroup  $\{S(t)\}$  of global  $X^\alpha$  solutions to (3.20), which possesses a global  $(X^\alpha - X^\alpha)$  attractor. Thus Theorem 3.2 applies. Furthermore, it follows from [5] that  $\{S(t)\}$  can be considered as a semigroup on  $X^0$  and the attractor attracts bounded subsets of  $X^0$ .

Summarizing, we obtain

**Corollary 3.6.** *If  $\rho \in (1, \frac{N+2}{N-2})$ ,  $\alpha = \frac{N-2}{N+2}$  and (3.21),(3.22) hold, then for each  $\beta \in (\alpha, 1)$  there exists a global  $(X^0 - X^\beta)$  attractor  $\mathbf{A}$  contained in an exponential  $(X^0 - X^\beta)$  attractor and satisfying*

$$(3.23) \quad d_f^{X^\beta}(\mathbf{A}) \leq d_f^{X^\alpha}(\mathbf{A}) \leq \log_2 N_{\frac{1}{4\kappa\beta}}^{X^\alpha}(B^{X^\beta}(0, 1)).$$

**3.4. Wave equation with damping operator  $-\Delta_D$ .** Following [5, 24, 25] we will consider in this subsection the problem

$$(3.24) \quad \begin{cases} u_{tt} - \Delta_D u_t - \Delta_D u = f(u), & t > 0, \quad x \in \Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = v_0(x), & x \in \Omega, \\ u(t, x) = 0, & t \geq 0, \quad x \in \partial\Omega, \end{cases}$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^3$  and  $f \in C^2(\mathbb{R}, \mathbb{R})$  satisfies the (critical) growth condition

$$(3.25) \quad \exists_{c>0} |f''(s)| \leq c(1 + |s|^3), \quad s \in \mathbb{R},$$

and the dissipativeness condition

$$(3.26) \quad \limsup_{|s| \rightarrow \infty} \frac{f(s)}{s} < \lambda_1,$$

with  $\lambda_1$  being the first eigenvalue of the negative Dirichlet Laplacian  $-\Delta_D$  in  $L^2(\Omega)$ .

As shown in [4] the problem (3.24) can be viewed as the Cauchy problem for an abstract parabolic equation of the form (3.3) with initial data in a product space  $X^0 = H_0^1(\Omega) \times L^2(\Omega)$ . Here, defining in  $X^0$

$$A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ -\Delta_D(\varphi + \psi) \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(A),$$

and

$$D(A) = X^1 = \left\{ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in H_0^1(\Omega) \times H_0^1(\Omega) : \varphi + \psi \in H^2(\Omega) \cap H_0^1(\Omega) \right\},$$

we remark that  $A$  with the domain  $X^1$  is a positive sectorial operator in  $X^0$  possessing bounded imaginary powers but the resolvent of  $A$  is non-compact.

The approach of [5], which uses a suitable decomposition of  $f$  and a nonlinear variation of constants formula (so called Alekseev's formula), ensures that a  $C^0$  semigroup  $\{S(t)\}$  is associated to (3.24) in  $H_0^1(\Omega) \times L^2(\Omega)$  and  $\{S(t)\}$  possesses a global attractor  $\mathbf{A}$ .

In [25] a significant progress was made, since for  $f$  satisfying (3.25) and (3.26) a higher regularity of the attractor was proved. It was merely mentioned in [25, Remark 3.2] that  $\mathbf{A}$  has finite fractal dimension, which appeared in connection with the previous work [24], where dimension  $d_f^{H_0^1(\Omega) \times L^2(\Omega)}(\mathcal{M})$  of an exponential attractor  $\mathcal{M}$  containing  $\mathbf{A}$  was estimated from above in a subcritical case by  $1 + \log_2 N_{\frac{1}{8\kappa}}^{H_0^1(\Omega) \times L^2(\Omega)}(B^{H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)}(0, 1))$ . In what follows the estimate of this type will be derived for  $d_f^{H_0^1(\Omega) \times L^2(\Omega)}(\mathbf{A})$  in a critical case within the analytic semigroup approach.

We first establish the auxiliary result involving the damped wave operator  $A$ .

**Lemma 3.7.** *—  $A$  with the domain  $Z^1 := H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega)$  generates a  $C^0$  analytic semigroup in  $Z^0 := H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$ , which is exponentially decaying. Furthermore, the associated fractional power spaces  $Z^\sigma$ ,  $\sigma \geq 0$ , satisfy the embedding inequality*

$$\| [\varphi] \psi \|_{H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)} \leq b_\sigma \| [\varphi] \psi \|_{Z^\sigma}, \quad [\varphi] \psi \in Z^\sigma, \quad \text{whenever } \sigma > \frac{1}{2}.$$

*Proof.* The first assertion comes from [32, Proposition 2.2]. As for the embedding property we observe that, for each  $[\varphi] \psi \in Z^1$ ,

$$\| \psi \|_{H_0^1(\Omega)} \leq c \| \Delta_D \psi \|_{L^2(\Omega)}^{\frac{1}{2}} \| \psi \|_{L^2(\Omega)}^{\frac{1}{2}} \leq \hat{c} \| A [\varphi] \psi \|_{Z^0}^{\frac{1}{2}} \| [\varphi] \psi \|_{Z^0}^{\frac{1}{2}}$$

and

$$\begin{aligned} \| \Delta_D \varphi \psi \|_{L^2(\Omega)} &= \left( \| \Delta_D \psi \|_{L^2(\Omega)} + \| \Delta_D \varphi \|_{L^2(\Omega)} - \| \Delta_D \psi \|_{L^2(\Omega)} \right)^{\frac{1}{2}} \| \Delta_D \varphi \psi \|_{L^2(\Omega)}^{\frac{1}{2}} \\ &\leq \left( \| \Delta_D \psi \|_{L^2(\Omega)} + \| \Delta_D(\varphi + \psi) \|_{L^2(\Omega)} \right)^{\frac{1}{2}} \| \Delta_D \varphi \psi \|_{L^2(\Omega)}^{\frac{1}{2}} \leq \tilde{c} \| A [\varphi] \psi \|_{Z^0}^{\frac{1}{2}} \| [\varphi] \psi \|_{Z^0}^{\frac{1}{2}}. \end{aligned}$$

Hence, using [17, Exercise 1.4.11], we obtain for  $\sigma > \frac{1}{2}$

$$(3.27) \quad \left\| \begin{bmatrix} -\Delta_D & 0 \\ 0 & (-\Delta_D)^{\frac{1}{2}} \end{bmatrix} [\varphi] \psi \right\|_{L^2(\Omega) \times L^2(\Omega)} \leq C \| [\varphi] \psi \|_{Z^\sigma}, \quad [\varphi] \psi \in Z^\sigma,$$

which completes the proof.  $\square$

**Theorem 3.8.** *If (3.25)-(3.26) hold, then the global  $(H_0^1(\Omega) \times L^2(\Omega) - H_0^1(\Omega) \times L^2(\Omega))$  attractor  $\mathbf{A}$  for the semigroup associated to (3.24) in  $H_0^1(\Omega) \times L^2(\Omega)$  is a bounded subset of  $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$  and*

$$d_f^{H_0^1(\Omega) \times L^2(\Omega)}(\mathbf{A}) \leq \log_2 N_{\frac{1}{8\kappa}}^{H_0^1(\Omega) \times L^2(\Omega)}(B^{H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)}(0, 1)),$$

where  $\kappa$  is given in (3.33).

*Proof.* Boundedness of the attractor in  $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$  comes from [25].

If  $\{e^{-At}\}$  is an analytic semigroup generated by  $-A$  in  $X^0$ , then  $\{e^{-At}\}$  restricted to  $Z^0$  (which will be denoted the same) coincides with the analytic semigroup generated by  $-A$  in  $Z^0$ . In both settings  $\{e^{-At}\}$  is exponentially decaying, so that

$$(3.28) \quad \| e^{-At} \|_{L(X^0)} \leq c_0 e^{-at}, \quad \| e^{-At} \|_{L(Z^0)} \leq c_0 e^{-at}, \quad t \geq 0,$$

for certain  $c_0 \geq 1$  and  $a > 0$ . In particular, in both settings  $\text{Re}\sigma(A) > 0$ .

Letting  $v = u_t$  note from [4, 5] that the variation of constants formula

$$(3.29) \quad \begin{bmatrix} u \\ v \end{bmatrix} (t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) = e^{-At} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-A(t-s)} \begin{bmatrix} 0 \\ f(u(s, u_0, v_0)) \end{bmatrix} ds, \quad t \geq 0,$$

holds in  $X^0$  but also in  $Z^0$  provided that in the latter case  $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Z^0$ . Therefore, if  $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \in \mathbf{A}$  ( $\mathbf{A}$  being invariant and bounded in  $Z^0$ ) and

$$M(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} := \int_0^t e^{-A(t-s)} \begin{bmatrix} 0 \\ f(u(s, u_0, v_0)) \end{bmatrix} ds, \quad t > 0,$$

then

$$(3.30) \quad \begin{aligned} & \|M(t) \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - M(t) \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{Z^\sigma} \\ & \leq c_\sigma \int_0^t \frac{1}{(t-s)^\sigma} (\|f(u(s, u_1, v_1)) - f(u(s, u_2, v_2))\|_{L^2(\Omega)}) ds \\ & \leq c_\sigma L(1-\sigma)^{-1} t^{1-\sigma} \sup_{s \in [0, t]} (\|u(s, u_1, v_1) - u(s, u_2, v_2)\|_{L^2(\Omega)}) \\ & \leq c_\sigma L \lambda_1^{-\frac{1}{2}} (1-\sigma)^{-1} t^{1-\sigma} \sup_{s \in [0, t]} \|\begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}) - \begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix})\|_{X^0}, \quad \sigma \in (\frac{1}{2}, 1), \end{aligned}$$

where the embedding  $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ , local Lipschitz continuity of  $f: \mathbb{R} \rightarrow \mathbb{R}$  and Poincaré inequality  $\lambda_1 \|\phi\|_{L^2(\Omega)}^2 \leq \|(-\Delta_D)^{\frac{1}{2}} \phi\|_{L^2(\Omega)}^2$  were used.<sup>3</sup>

Denoting next

$$P(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} := e^{-At} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad t > 0,$$

and using (3.28) we obtain

$$(3.31) \quad \|P(t) \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - P(t) \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{X^0} \leq c_0 e^{-at} \|\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{X^0}, \quad t > 0.$$

We also estimate  $\sup_{s \in [0, t]} \|\begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}) - \begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix})\|_{X^0}$  using (3.28)-(3.29) to get

$$(3.32) \quad \begin{aligned} & \|\begin{bmatrix} u \\ v \end{bmatrix}(t, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}) - \begin{bmatrix} u \\ v \end{bmatrix}(t, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix})\|_{X^0} \leq c_0 \|\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{X^0} \\ & \quad + \int_0^t c_0 \|f(u(s, u_1, v_1)) - f(u(s, u_2, v_2))\|_{L^2(\Omega)} ds \\ & \leq c_0 \|\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{X^0} + \int_0^t c_0 L \lambda_1^{-\frac{1}{2}} \|\begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}) - \begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix})\|_{X^0} ds, \quad t \geq 0. \end{aligned}$$

Since the Gronwall's inequality ensures

$$\sup_{s \in [0, t]} \|\begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}) - \begin{bmatrix} u \\ v \end{bmatrix}(s, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix})\|_{X^0} \leq c_0 \|\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{X^0} e^{c_0 L \lambda_1^{-\frac{1}{2}} t}, \quad t > 0,$$

from this and (3.30) we obtain for  $\sigma \in (\frac{1}{2}, 1)$

$$\|M(t) \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - M(t) \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{Z^\sigma} \leq c_0 c_\sigma L \lambda_1^{-\frac{1}{2}} (1-\sigma)^{-1} t^{1-\sigma} \|\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\|_{X^0} e^{c_0 L \lambda_1^{-\frac{1}{2}} t}, \quad t > 0.$$

<sup>3</sup>Due to  $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$  regularity of  $\mathbf{A}$  the set  $\{u(s, u_0, v_0)(x), x \in \Omega, s \geq 0, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathbf{A}\}$  is contained in a certain interval  $[-r, r]$  and  $f|_{[-r, r]}: [-r, r] \rightarrow \mathbb{R}$  is Lipschitz continuous with a Lipschitz constant  $L$ .

Hence, Lemma 2.1 applies with  $V = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ ,  $W = H_0^1(\Omega) \times L^2(\Omega)$  and  $S = S(t^*)$ , for which we choose  $\delta = \nu = \frac{1}{8}$  and

$$(3.33) \quad t^* = a^{-1} \ln 8c_0, \quad \kappa = b_\sigma c_0 c_\sigma L \lambda_1^{-\frac{1}{2}} (1 - \sigma)^{-1} t^{*1-\sigma} e^{c_0 L \lambda_1^{-\frac{1}{2}} t^*}, \quad \sigma \in \left(\frac{1}{2}, 1\right). \quad \square$$

**3.5. Higher order parabolic problems with fast growing nonlinearities.** Consider the well-known Cahn Hilliard equation

$$(3.34) \quad u_t = \Delta(-\Delta u + f(u)), \quad t > 0, \quad x \in \Omega,$$

with the initial-boundary conditions

$$(3.35) \quad \begin{aligned} u(0, x) &= u_0(x), \quad x \in \Omega, \\ \mathcal{B}_0 u &= \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{and} \quad \mathcal{B}_1 u = \frac{\partial(\Delta u)}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

in a bounded smooth domain  $\Omega \subset \mathbb{R}^N$  where  $N \leq 3$ ; see [29] and references therein.

As in [20, 8] we assume that  $f \in C^{2+Lip}(\mathbb{R}, \mathbb{R})$  satisfies

$$(3.36) \quad \int_0^z f(s) ds \geq -M, \quad -f'(z) \leq \lambda \quad \text{for} \quad z \in \mathbb{R}$$

with some positive constants  $M, \lambda$  and

$$(3.37) \quad \forall_{M' \geq 0} \exists_{\beta > 0} \forall_{|m| \leq M'} \forall_{v \in \mathbb{R}} v f(v + m) \geq -\beta.$$

We remark that one can take a polynomial  $f(v) = \sum_{k=1}^{2p-1} a_k v^k$  with  $a_{2p-1}$  positive and an arbitrary  $p \in \mathbb{N}$ .

In this example we consider  $A = \Delta^2 + I$  in  $X = L^2(\Omega)$  with the domain

$$D(A) = H_{2, \{\mathcal{B}_0, \mathcal{B}_1\}}^4(\Omega) = \{\phi \in H_2^4(\Omega) : \mathcal{B}_0 \phi = \mathcal{B}_1 \phi = 0\}.$$

It is known that  $A$  is a sectorial positive operator with compact resolvent and

$$X^{\frac{1}{2}} = H_{2, \{\mathcal{B}_0\}}^2(\Omega) \hookrightarrow L^\infty(\Omega) \cap H_4^1(\Omega).$$

In this setting we have that the nonlinear map

$$F(u)(x) = \Delta(f(u(x))) + u(x), \quad u \in H_{2, \{\mathcal{B}_0\}}^2(\Omega), \quad x \in \Omega,$$

satisfies the Lipschitz condition (3.2) with  $\alpha = \frac{1}{2}$ ,  $X^{\frac{1}{2}} = H_{2, \{\mathcal{B}_0\}}^2(\Omega)$  and  $X^0 = L^2(\Omega)$ .

Recall from [8, §6.4] that to the Cahn-Hilliard problem (3.34)-(3.35) corresponds a  $C^0$  semigroup  $\{S(t)\}$  of the global  $X^{\frac{1}{2}}$  solutions. Define also a complete metric subspace  $H_\gamma$  of  $H_{2, \{\mathcal{B}_0\}}^2(\Omega)$  by

$$H_\gamma = \left\{ \phi \in H_{2, \{\mathcal{B}_0\}}^2(\Omega) : \frac{1}{|\Omega|} \left| \int_\Omega \phi(x) dx \right| \leq \gamma \right\}, \quad \gamma > 0.$$

As shown in [8, Proposition 6.4.3], for every  $\gamma > 0$  there exists a nonvoid invariant set  $\mathbf{A}_\gamma$ , compact in  $H_{2, \{\mathcal{B}_0\}}^2(\Omega)$ , and attracting bounded subsets of  $H_\gamma$  with respect to the Hausdorff semidistance in  $H_{2, \{\mathcal{B}_0\}}^2(\Omega)$ .

Thus Corollary 3.3 applies and we conclude that

**Corollary 3.9.** *If (3.36)-(3.37) hold, then for any  $\beta \in (\frac{1}{2}, 1)$  the set  $\mathbf{A}_\gamma$  is compact in  $H_{2,\{\mathcal{B}_0,\mathcal{B}_1\}}^{4\beta}(\Omega)$  and*

$$(3.38) \quad d_f^{H_{2,\{\mathcal{B}_0,\mathcal{B}_1\}}^{4\beta}(\Omega)}(\mathbf{A}_\gamma) \leq d_f^{H_{2,\{\mathcal{B}_0\}}^2(\Omega)}(\mathbf{A}_\gamma) \leq \log_2 N_{\frac{1}{4\kappa\beta,\gamma}}^{H_{2,\{\mathcal{B}_0\}}^2(\Omega)}(B_{H_{2,\{\mathcal{B}_0,\mathcal{B}_1\}}^{4\beta}(\Omega)}(0,1)).$$

Now let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $A = \sum_{|\sigma| \leq 2m} a_\sigma D^\sigma$  be a linear  $2m$ -th order differential operator and let  $B_j = \sum_{|\sigma| \leq m_j} b_\sigma^j D^\sigma$  ( $j = 0, \dots, m-1$ ) be boundary operators such that the triple  $(A, \{B_j\}, \Omega)$  forms a *regular elliptic boundary value problem* (see e.g. [8, pp. 29-30]). Suppose, in addition, that  $A$  in  $L^2(\Omega)$  with the domain  $H_{2,\{B_j\}}^{2m}(\Omega)$  is selfadjoint and positive definite; e.g.  $A = (-\Delta)^m$  and  $B_j = \frac{\partial^j}{\partial \nu^j}$  for  $j = 0, \dots, m-1$ .

Following [6, 31], consider an initial-boundary value problem of the form

$$(3.39) \quad \begin{cases} u_t + Au = f(u), & t > 0, x \in \Omega \subset \mathbb{R}^N, N > 2m > 2, \\ B_0 u = \dots = B_{m-1} u = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0 \in H_{2,\{B_j\}}^m(\Omega), \end{cases}$$

where the nonlinear term satisfies (3.22) and

$$(3.40) \quad f \in C^1(\mathbb{R}, \mathbb{R}), \quad \lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{|s|^{\frac{4m}{N-2m}}} = 0.$$

It was shown in [6, Theorem 4.1] that under the assumptions (3.22) and (3.40) the problem (3.39) is globally well-posed in  $H_{2,\{B_j\}}^m(\Omega)$  and the associated semigroup  $\{S(t)\}$  has a global  $(H_{2,\{B_j\}}^m(\Omega) - H_{2,\{B_j\}}^m(\Omega))$  attractor  $\mathbf{A}$ .

We now refer to the specific condition in [6, Theorem 2.1 (ii)], which implies that there are certain  $\tau_0, C', \varsigma > 0$  such that for each  $u_1, u_2 \in \mathbf{A}$

$$(3.41) \quad \|S(\tau_0)u_1 - S(\tau_0)u_2\|_{H_{2,\{B_j\}}^{m+2m\varsigma}(\Omega)} \leq C' \tau_0^{-\varsigma} \|u_1 - u_2\|_{H_{2,\{B_j\}}^m(\Omega)},$$

where  $\varsigma > 0$  can be chosen less but arbitrarily close to  $\frac{1}{2}$  accordingly to the results of [6, Lemma 4.1 and Theorem 4.1]. The following conclusion is now straightforward.

**Corollary 3.10.** *If (3.22) and (3.40) hold, then there is a global  $(H_{2,\{B_j\}}^m(\Omega) - H_{2,\{B_j\}}^m(\Omega))$  attractor  $\mathbf{A}$  for (3.39), compact in  $H_{2,\{B_j\}}^{m+2m\varsigma}(\Omega)$ , and satisfying the estimate*

$$d_f^{H_{2,\{B_j\}}^{m+2m\varsigma}(\Omega)}(\mathbf{A}) \leq d_f^{H_{2,\{B_j\}}^m(\Omega)}(\mathbf{A}) \leq \log_2 N_{\frac{\tau_0^\varsigma}{4C'}}^{H_{2,\{B_j\}}^m(\Omega)}(B_{H_{2,\{B_j\}}^{m+2m\varsigma}(\Omega)}(0,1)).$$

**3.6. A problem with memory.** The conserved phase-field system with thermal memory for the *temperature variation field*  $\vartheta$  and for the *order parameter*  $\chi$  reads as follows

$$(3.42) \quad (\vartheta + \chi)_t = \int_0^\infty k(s) \Delta \vartheta(t-s) ds, \quad t > 0, x \in \Omega,$$

$$(3.43) \quad \chi_t = \Delta(-\Delta\chi + \alpha\chi_t + f(\chi) - \vartheta), \quad t > 0, x \in \Omega,$$

with the initial-boundary conditions

$$(3.44) \quad \begin{aligned} \vartheta(0, x) = \vartheta_0(x), \quad \chi(0, x) = \chi_0(x) \text{ and } \vartheta(-s, x) = \vartheta_1(s, x), \quad s > 0, \quad x \in \Omega, \\ \mathcal{B}_0 \vartheta = \mathcal{B}_0 \chi = 0 \text{ and } \mathcal{B}_0(-\Delta \chi + \alpha \chi_t + f(\chi) - \vartheta) = 0 \text{ on } \partial \Omega, \end{aligned}$$

where  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^N$ ,  $N \leq 3$ , and  $\mathcal{B}_0$  is the operator introduced in (3.35). Here  $\alpha > 0$  is the *viscosity parameter* and  $k$  is the smooth, nonnegative, nontrivial, summable *memory kernel*, which accounts for the thermal memory. The presence of the memory requires the introduction of the given past history  $\vartheta_1: \Omega \times (0, \infty) \rightarrow \mathbb{R}$ . Throughout this section we assume that

$$(3.45) \quad f \in C^2(\mathbb{R}, \mathbb{R}), \quad rf(r) \geq c_0 r^4 - c_1 \quad \text{and} \quad |f''(r)| \leq c_2(1 + |r|), \quad r \in \mathbb{R},$$

for some  $c_0 > 0$  and  $c_1, c_2 \geq 0$ . We remark that any double well potential derivative, i.e.  $f(r) = r^3 - cr$ ,  $c > 0$ , fulfils the above assumptions.

In order to prove that our problem generates a dynamical system, following the approach in [22], we need to introduce an additional variable  $\eta$ , usually called the *summed past history*, and defined as

$$\eta(t)(s) = - \int_0^s \Delta (e(t - \tau) - \chi(t - \tau)) d\tau, \quad (t, s) \in [0, \infty) \times [0, \infty), \quad x \in \Omega,$$

where  $e = \vartheta + \chi$  is the *enthalpy density*. It is immediate to check that  $\eta$  formally satisfies the first order hyperbolic equation

$$\eta_t(t)(s) = -\eta_s(t)(s) - \Delta (e(t) - \chi(t)), \quad (t, s) \in (0, \infty) \times (0, \infty), \quad x \in \Omega.$$

Concerning the boundary and initial conditions to associate with the equation above, on account of (3.44), we deduce

$$(3.46) \quad \begin{aligned} \eta(0)(s) &= - \int_0^s \Delta \vartheta_1(\tau, x) d\tau, \quad s \geq 0, \quad x \in \Omega, \\ \mathcal{B}_0 \eta(t)(s) &= 0 \text{ on } \partial \Omega, \quad (t, s) \in (0, \infty) \times (0, \infty). \end{aligned}$$

Moreover, we observe that, under reasonable assumptions on the past history and the memory kernel, a formal integration by parts yields

$$- \int_0^\infty k(s) \Delta (e(t - s) - \chi(t - s)) ds = \int_0^\infty \mu(s) \eta(t)(s) ds, \quad t > 0, \quad x \in \Omega,$$

where we have set

$$\mu(s) = -k'(s), \quad s > 0.$$

The assumption on  $0 \leq \mu \in C^1((0, \infty), \mathbb{R}) \cap L^1((0, \infty), \mathbb{R})$  is

$$(3.47) \quad \exists_{\sigma > 0} \forall_{s > 0} \mu'(s) + \sigma \mu(s) \leq 0.$$

Notice in particular that (3.47) entails the exponential decay of  $\mu$  as  $s \rightarrow \infty$ . In order to deepen the dissipative properties of our system, following [9] (see also [23]), it is necessary to introduce the *rescaled kernel*

$$\tilde{\mu}(s) = \frac{1}{\varepsilon_0^2} \mu \left( \frac{1}{\varepsilon_0} s \right), \quad s > 0,$$



where  $\varepsilon_0 > 0$  is a sufficiently small number called a *relaxation time* of the system. It is important to point out that for the function  $\tilde{\mu}$  assumption (3.47) still holds, by replacing  $\sigma$  by  $\frac{\sigma}{\varepsilon_0}$ .

We next consider the operator  $B = -\Delta$  in  $H = \{u \in L^2(\Omega) : \int_{\Omega} u(x)dx = 0\}$  with the domain  $D(B) = \{u \in H_{2,\{\mathcal{B}_0\}}^2(\Omega) : \int_{\Omega} u(x)dx = 0\}$ . It is known that  $B$  is a strictly positive operator, so that we can define  $V = D(B^{1/2})$ .

We introduce the memory space defined as the weighted Lebesgue space

$$\mathcal{M} = L_{\tilde{\mu}}^2((0, \infty), V'),$$

where  $V'$  is the adjoint space of  $V$ . Moreover, we let  $T$  be the linear operator on  $\mathcal{M}$  with domain

$$D(T) = \{\eta \in \mathcal{M} : \eta_s \in \mathcal{M}, \eta(0) = 0\},$$

defined by

$$(3.48) \quad T\eta = -\eta_s,$$

where  $\eta_s$  is the distributional derivative of  $\eta$ .

Then we can reformulate the original initial-boundary value problem as the following integro-partial differential system in the variables  $(e, \chi, \eta) = (e(t), \chi(t), \eta(t))$

$$(3.49) \quad e_t + \int_0^{\infty} \tilde{\mu}(s)\eta(s)ds = 0, \quad t > 0, \quad x \in \Omega,$$

$$(3.50) \quad \chi_t = \Delta(-\Delta\chi + \alpha\chi_t + f(\chi) - e + \chi), \quad t > 0, \quad x \in \Omega,$$

$$(3.51) \quad \eta_t = T\eta - \Delta(e - \chi), \quad t > 0, \quad x \in \Omega,$$

with the boundary and initial conditions (3.44), (3.46) and

$$(3.52) \quad \eta(t)(0) = 0, \quad t \geq 0.$$

The proper phase space for our problem will be then

$$\mathcal{H} = L^2(\Omega) \times H_2^1(\Omega) \times \mathcal{M}.$$

As showed in [23, Theorem 3.4], phase-field problem (3.49)-(3.51) generates a  $C^0$  semigroup  $\{S(t)\}$  of global  $\mathcal{H}$  solutions.

In order to prove dissipativeness, we need to restrict the phase-space. Taking into account the mass conservation for  $e$  and  $\chi$ , we introduce the complete metric space

$$\mathcal{H}_{\beta,\gamma} = \left\{ (e, \chi, \eta) \in \mathcal{H} : \frac{1}{|\Omega|} \left| \int_{\Omega} e(x)dx \right| \leq \beta \text{ and } \frac{1}{|\Omega|} \left| \int_{\Omega} \chi(x)dx \right| \leq \gamma \right\}, \quad \beta, \gamma \geq 0.$$

Moreover, it is essential to construct a compactly embedded subspace of  $\mathcal{H}_{\beta,\gamma}$ . To this purpose we also define for any  $\eta \in \mathcal{M}$  the *tail* of  $\eta$  in  $\mathcal{M}$ , that is the function

$$\mathcal{T}_{\eta} : [1, \infty) \rightarrow [0, \infty), \quad \mathcal{T}_{\eta}(\tau) = \int_{(0, \frac{1}{\tau}) \cup (\tau, \infty)} \tilde{\mu}(s) \|\eta(s)\|_{V'}^2 ds, \quad \tau \geq 1.$$

Then we introduce the vector space

$$\mathcal{L} = \left\{ \eta \in L_{\tilde{\mu}}^2((0, \infty), H) : \eta \in D(T), \sup_{\tau \geq 1} \tau \mathcal{T}_{\eta}(\tau) < \infty \right\}.$$

As proved in [15],  $\mathcal{L}$  is a Banach space endowed with the norm

$$\|\eta\|_{\mathcal{L}} = \left( \|\eta\|_{L_{\bar{\mu}}^2((0,\infty),H)}^2 + \|T\eta\|_{\mathcal{M}}^2 + \sup_{\tau \geq 1} \tau \mathcal{T}_{\eta}(\tau) \right)^{\frac{1}{2}}, \quad \eta \in \mathcal{L}.$$

Thus, on account of an immediate generalization of [27, Lemma 5.5], we have

$$\mathcal{Z} = H_2^1(\Omega) \times H_{2,\{B_0\}}^2(\Omega) \times \mathcal{L} \hookrightarrow \mathcal{H} \text{ and } \mathcal{Z}_{\beta,\gamma} = \mathcal{Z} \cap \mathcal{H}_{\beta,\gamma} \hookrightarrow \mathcal{H}_{\beta,\gamma}$$

with compact embeddings. Moreover,  $\mathcal{Z}_{\beta,\gamma}$  is positively invariant under  $\{S(t)\}$ . On account of the above results, we are now in a position to proceed in the asymptotic analysis of the problem. As showed in [23, Corollary 5.2] we have

**Lemma 3.11.** *There exists a ball  $B_1 = B_1(\beta, \gamma)$  in  $\mathcal{Z}_{\beta,\gamma}$ , which absorbs bounded sets in  $\mathcal{Z}_{\beta,\gamma}$  under  $\{S(t)\}$ .*

Consequently (see [23, Theorem 9.2]), we can establish the next lemma, which provides conditions of Proposition 2.7.

**Lemma 3.12.** *For any  $\delta \in (0, \frac{1}{2})$  there exist  $t_0 \geq t_{B_1}$ ,  $\kappa > 0$  and a decomposition*

$$(3.53) \quad S(t_0) = P(t_0) + M(t_0),$$

such that

$$(3.54) \quad \|P(t_0)u_1 - P(t_0)u_2\|_{\mathcal{H}} \leq \delta \|u_1 - u_2\|_{\mathcal{H}}, \quad u_1, u_2 \in B_1,$$

$$(3.55) \quad \|M(t_0)u_1 - M(t_0)u_2\|_{\mathcal{Z}} \leq \kappa \|u_1 - u_2\|_{\mathcal{H}}, \quad u_1, u_2 \in B_1,$$

and there exist  $0 < \theta \leq 1$  and  $\lambda > 0$  such that

$$(3.56) \quad \|S(t_1)u_1 - S(t_2)u_2\|_{\mathcal{H}} \leq \lambda(|t_1 - t_2|^{\theta} + \|u_1 - u_2\|_{\mathcal{H}})$$

for all  $t_1, t_2 \in [t_0, 2t_0]$  and  $u_1, u_2 \in B_1$ .

Thus, Proposition 2.7 applies and we can state the following result.

**Proposition 3.13.** *For any  $\beta, \gamma \geq 0$ ,  $\delta \in (0, \frac{1}{2})$  and  $\nu \in (0, \frac{1}{2} - \delta)$  there exist  $\omega > 0$  and a nonvoid set  $\widehat{\mathcal{M}} \subset B_1$  such that  $\widehat{\mathcal{M}}$  is positively invariant under  $\{S(t)\}$ , precompact in  $\mathcal{H}_{\beta,\gamma}$  with*

$$d_f^{\mathcal{H}_{\beta,\gamma}}(\widehat{\mathcal{M}}) \leq \frac{1}{\theta} \left( 1 + \log_{\frac{1}{2(\delta+\nu)}} N_{\frac{\nu}{\kappa}}^{\mathcal{H}}(B^{\mathcal{Z}}(0, 1)) \right),$$

and for any bounded subset  $B$  of  $\mathcal{Z}_{\beta,\gamma}$

$$\lim_{t \rightarrow \infty} e^{\omega t} d_{\mathcal{H}_{\beta,\gamma}}(S(t)B, \widehat{\mathcal{M}}) = 0.$$

Note that by [22, Theorem 4.1] there exists a bounded and closed set  $B_0 \subset \mathcal{H}_{\beta,\gamma}$  absorbing bounded subsets of  $\mathcal{H}_{\beta,\gamma}$  such that  $B_1 \subset B_0$  and with some constants  $C > 0$  and  $\xi > 0$  we have

$$\|S(t)u_1 - S(t)u_2\|_{\mathcal{H}} \leq C e^{\xi t} \|u_1 - u_2\|_{\mathcal{H}}, \quad u_1, u_2 \in B_0, \quad t > 0.$$

Furthermore, it follows from [23, Theorem 6.1] that there exists  $\omega_1 > 0$  such that for every bounded subset  $B$  of  $\mathcal{H}_{\beta,\gamma}$  we have

$$(3.57) \quad \lim_{t \rightarrow \infty} e^{\omega_1 t} d_{\mathcal{H}_{\beta,\gamma}}(S(t)B, B_1) = 0.$$

Using (3.57) and the compactness of the embedding  $\mathcal{Z}_{\beta,\gamma}$  into  $\mathcal{H}_{\beta,\gamma}$ , we obtain the following consequence of Corollary 2.4 (i).

**Corollary 3.14.** *For any  $\beta, \gamma \geq 0$ ,  $\{S(t)\}$  has a global  $(\mathcal{H}_{\beta,\gamma} - \mathcal{H}_{\beta,\gamma})$  attractor  $\mathbf{A}$ .*

By (3.57) and the transitivity of exponential attraction (see [13, Theorem 3.1]) we also get

**Corollary 3.15.** *If  $\beta, \gamma \geq 0$ ,  $\delta \in (0, \frac{1}{2})$  and  $\nu \in (0, \frac{1}{2} - \delta)$ , then  $\mathcal{M} = \text{cl}_{\mathcal{H}} \widehat{\mathcal{M}}$  is an exponential  $(\mathcal{H}_{\beta,\gamma} - \mathcal{H}_{\beta,\gamma})$  attractor for  $\{S(t)\}$  and contains the finite dimensional global  $(\mathcal{H}_{\beta,\gamma} - \mathcal{H}_{\beta,\gamma})$  attractor  $\mathbf{A}$ .*

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