

ASYMPTOTICS OF PARABOLIC EQUATIONS WITH POSSIBLE BLOW-UP

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ABSTRACT. In this paper we describe the long-time behaviour of solutions of parabolic equations in the case when some of them may blow up in a finite or infinite time. This is done by providing a maximal compact invariant set attracting each initial data for which the corresponding solution does not blow up. The abstract result is applied to the Frank-Kamenetskii equation and the N -dimensional Navier-Stokes system where small external force is considered.

1. INTRODUCTION

In this paper we study the asymptotic behaviour of parabolic equations when some solutions may blow up in a finite or infinite time. Hereafter we consider X^α solutions as in [C-D1] and earlier in [HE] with the modification given in [MI]. In our investigation we make use of the theory of semilinear abstract parabolic equations given in e.g. [HE], [HA] or [CZ].

The situation that for some initial data the corresponding solution blows up often occurs in physical applications. For a detailed mathematical description of this phenomenon we refer the reader to [G-V]. Here we only mention a particular problem with a parameter $\lambda > 0$

$$\begin{cases} u_t = \Delta u + \lambda e^u, & t > 0, \\ u(0) = u_0, \end{cases}$$

which comes from combustion theory and is known under the name of *solid fuel ignition model*, *exponential reaction-diffusion equation* or, lastly, *Frank-Kamenetskii equation*, the latter name being used from now on in this paper. Among many other results concerning this problem (see e.g. [GE], [J-L], [B-E]) it was shown in [FU] that for specially chosen initial data u_0 the corresponding solution blows up. On the other hand, there also exist parabolic problems with so far unknown global solvability for all initial data. A typical example in this group is the famous N -dimensional Navier-Stokes system, $N \geq 3$, that has been investigated now for more than a century.

The mentioned specific circumstances do not allow us to describe the asymptotics by using the notion of a global attractor in the large space

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of initial data as in [HA], [LA], [C-D1]. Instead, we consider the maximal compact invariant set attracting each initial condition for which the corresponding solution does not blow up. We first describe all these ideas abstractly considering in Section 2 a Cauchy problem for a semilinear sectorial equation in a Banach space. Next in Section 3 we discuss two particular examples: the one-dimensional version of Frank-Kamenetskii equation and the N -dimensional Navier-Stokes system in the case of small external force.

2. ABSTRACT PARABOLIC PROBLEMS WITH POSSIBLE BLOW-UP

Consider an abstract parabolic problem

$$(2.1) \quad \begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0, \end{cases}$$

where X is a Banach space and $A: X \supset \text{dom}(A) \rightarrow X$ is a positive sectorial operator with compact resolvent. Moreover, assume that $F: X^\alpha \rightarrow X$ ($\alpha \in [0, 1)$ is fixed from now on) is Lipschitz continuous on bounded subsets of the fractional power space $X^\alpha = \text{dom}(A^\alpha)$ (cf. [HE], [AM]).

Under the above assumptions the theory of semilinear parabolic equations given e.g. in [HE] ensures for each $u_0 \in X^\alpha$ the existence of a unique local X^α solution defined on a maximal interval of existence $[0, \tau_{u_0})$. Therefore we know that

$$u \in C([0, \tau_{u_0}), X^\alpha) \cap C((0, \tau_{u_0}), X^1) \cap C^1((0, \tau_{u_0}), X)$$

and (2.1) is satisfied in X . Moreover, we have either $\tau_{u_0} = \infty$ or $\tau_{u_0} < \infty$ and

$$\limsup_{t \rightarrow \tau_{u_0}} \|u(t, u_0)\|_{X^\alpha} = \infty.$$

Since the problem (2.1) is autonomous, the uniqueness of solutions allows us to construct a local semiflow on X^α . We thus have

$$u(s, u(t, u_0)) = u(s + t, u_0), \quad u_0 \in X^\alpha, \quad s, t \geq 0, \quad s + t < \tau_{u_0},$$

and the solutions of (2.1) are continuous with respect to their initial data on compact time intervals (cf. [HE, Theorem 3.4.1] or [C-D1, Proposition 2.3.2]).

For our further investigations let us define a metric space

$$(2.2) \quad V = \{u_0 \in X^\alpha : \sup_{t \in [0, \infty)} \|u(t, u_0)\|_{X^\alpha} < \infty\}$$

and assume that V is nonvoid.

Consider a C^0 -semigroup $T(t): V \rightarrow V$ defined by

$$(2.3) \quad T(t)u_0 = u(t, u_0), \quad t \geq 0, \quad u_0 \in V.$$

Note that we do not know in advance whether V is a closed subset of X^α . Thus it is unknown if V , being the natural phase space for the problem (2.1), is a complete metric space or not. Therefore the assumption of the compact resolvent does not have to imply the compactness of the semigroup

$\{T(t): t \geq 0\}$ on V . Nevertheless, we shall show below that this semigroup is *asymptotically smooth* in the sense of [HA].

For $B \subset V$ we denote its *orbit* by $\gamma^+(B) = \bigcup_{t \geq 0} T(t)B$, while its ω -*limit set* is given by

$$\omega(B) = \bigcap_{t \geq 0} \text{cl}_{X^\alpha} T(t)\gamma^+(B).$$

We also abbreviate $\gamma^+(\{v\})$ and $\omega(\{v\})$ to $\gamma^+(v)$ and $\omega(v)$, respectively.

Remark 2.1. *If $B \subset V$ and $\gamma^+(B)$ is bounded, then $\text{cl}_{X^\alpha} \gamma^+(B) \subset V$ and $T(t)B$ is precompact in X^α for any $t > 0$.*

Proof. Let $v_0 \in \text{cl}_{X^\alpha} \gamma^+(B)$. Then there exist $v_n \in B$ and $t_n \geq 0$ such that $T(t_n)v_n \rightarrow v_0$ in X^α . Since

$$\forall_{s \geq 0} \|u(s, T(t_n)v_n)\|_{X^\alpha} = \|u(s + t_n, v_n)\|_{X^\alpha} \leq R_{\gamma^+(B)},$$

the norm $\|u(s, v_0)\|_{X^\alpha}$ cannot blow up so that $v_0 \in V$. Also, if $\gamma^+(B)$ is bounded in X^α , then $T(t)B$ with $t > 0$ is in fact bounded in $X^{\alpha+\varepsilon}$, which, via compactness of the embeddings $X^\beta \subset X^\alpha$, $\beta > \alpha$, ensures that it is precompact in X^α . \square

Proposition 2.2. *The C^0 -semigroup $\{T(t): t \geq 0\}$ on V is asymptotically smooth, i.e. each nonvoid closed (in V) bounded and positively invariant set $W \subset V$ contains a nonvoid compact subset $\omega(W)$, which attracts W .*

Proof. Fix a nonvoid closed (in V) bounded and positively invariant set $W \subset V$. Since $\gamma^+(W) \subset W$, we infer from Remark 2.1 that $\text{cl}_{X^\alpha} T(1)\gamma^+(W)$ is a compact subset of V . We thus see that

$$\omega(W) = \bigcap_{t \geq 0} \text{cl}_{X^\alpha} T(t)\gamma^+(W) \subset \text{cl}_{X^\alpha} T(1)\gamma^+(W) \subset V$$

is compact as a closed subset of a compact space and is nonvoid as an intersection of a centered family of closed sets in a compact space. Also $\omega(W) \subset W$, since by our assumptions on W and Remark 2.1 we have

$$\omega(W) \subset \text{cl}_{X^\alpha} W \cap V = \text{cl}_V W = W.$$

We now prove that $\omega(W)$ attracts W . We shall show that

$$d(T(t)W, \omega(W)) \xrightarrow[t \rightarrow \infty]{} 0,$$

where d denotes the Hausdorff semidistance. Contrary to our claim suppose that

$$(2.4) \quad \exists_{\varepsilon > 0} \exists_{t_n \rightarrow \infty} \forall_{n \in \mathbb{N}} d(T(t_n)W, \omega(W)) > \varepsilon.$$

Since almost all the elements of the sequence $\{T(t_n)x_n\}$ belong to the compact set $\text{cl}_{X^\alpha} T(1)\gamma^+(W)$ it contains a subsequence convergent to an element of $\omega(W)$, which contradicts the inequality (2.4). \square

Remark 2.3. *It is worth noting that Remark 2.1 implies immediately that if $u_0 \in V$, then $\text{cl}_{X^\alpha} \gamma^+(u_0)$ is a compact positively invariant subset of V and the ω -limit set $\omega(u_0)$ is a nonvoid compact connected and invariant subset of V , which attracts u_0 (cf. [LA, Theorem 2.1]). More generally, since by Proposition 2.2 the semigroup is asymptotically smooth, we infer that for $\emptyset \neq B \subset V$ with $\gamma^+(B)$ bounded, the ω -limit set $\omega(B)$ is a nonvoid compact and invariant subset of V attracting B (see [C-D1, Proposition 1.1.1]).*

Under an additional assumption of the *point dissipativeness* of the semigroup $\{T(t): t \geq 0\}$ we now ensure the existence of a nonvoid compact and invariant set, which attracts each point of V . Indeed, if B_0 is a nonvoid bounded subset of V attracting points of V , then any ε -neighbourhood $\mathcal{N}(B_0) = \bigcup_{u \in B_0} B(u, \varepsilon)$ of B_0 in V absorbs each point of V . Consequently,

$$\tilde{B}_0 = \{v \in \mathcal{N}(B_0): \gamma^+(v) \subset \mathcal{N}(B_0)\}$$

is a nonvoid bounded positively invariant subset of V absorbing each point of V . From this and Remark 2.3 it follows that $\omega(\tilde{B}_0)$ is a nonvoid compact and invariant subset of V attracting each point of V .

Required dissipativeness property can be easily controlled if there exists a Lyapunov function. Recall that by a *Lyapunov function on V* we mean a continuous function $\mathcal{L}: V \rightarrow \mathbb{R}$ such that for any $u_0 \in V$

- (i) the function $t \mapsto \mathcal{L}(u(t, u_0))$ is nonincreasing in $(0, \infty)$,
- (ii) if $\mathcal{L}(u(\cdot, u_0)) \equiv \mathcal{L}(u_0)$, then $u_0 \in \mathcal{E}$,

where by \mathcal{E} we denote the set of all stationary solutions of (2.1). Recall also that from the existence of a Lyapunov function on V we infer that $\omega(u_0) \subset \mathcal{E}$ for each $u_0 \in V$. Therefore, if there exists a Lyapunov function on the metric space V given in (2.2), then the set \mathcal{E} is nonvoid and if it is bounded, it is also compact and attracts each point of V . Thus we get

Corollary 2.4. *Suppose that $\{T(t): t \geq 0\}$ is defined on V given in (2.2). Assume further that $\{T(t): t \geq 0\}$ is point dissipative (for example, there exists a Lyapunov function on V and \mathcal{E} is bounded). Then for any $u_0 \in X^\alpha$ the X^α solution $u(\cdot, u_0)$ of (2.1) either blows up (in a finite or infinite time) or stays bounded and approaches a nonvoid compact and invariant set.*

We have shown above that the semigroup is asymptotically smooth and we have also stated natural conditions ensuring its point dissipativeness. However, these two properties do not guarantee the existence of a compact global attractor in V . It would be true if we knew the semigroup on V was compact (cf. [C-D1, Corollary 1.1.6]), which, as we have already discussed, may not be the case, or if orbits of bounded sets were bounded (cf. [C-D1, Theorem 1.1.2]). Unfortunately, the latter condition may be difficult to check in specific examples. As will be seen in Section 3 it is much easier to examine the boundedness of the set of all (hypothetical) bounded complete orbits of points.

Following [LA], we recall that a *complete trajectory* of a point $v \in V$ for the semigroup $\{T(t): t \geq 0\}$ is the curve $\phi: \mathbb{R} \rightarrow V$ satisfying the following conditions:

- (i) $\phi(0) = v$,
- (ii) $T(t)\phi(s) = \phi(s + t)$, $s \in \mathbb{R}$, $t \geq 0$.

If \mathcal{S} denotes the set of all points in V for which there exists at least one *bounded* complete trajectory for the semigroup $\{T(t): t \geq 0\}$, then $T(t)\mathcal{S} = \mathcal{S}$ for each $t > 0$. Also the following result holds.

Theorem 2.5. *Suppose that the semigroup $\{T(t): t \geq 0\}$ is defined by (2.3) on the nonvoid metric space V given in (2.2). Then \mathcal{S} is a nonvoid invariant subset of V , which attracts each subset of V with bounded orbit. If \mathcal{S} is bounded, then it is compact and maximal bounded invariant subset of V . Additionally, if orbits of bounded subsets of V are bounded, then \mathcal{S} is a compact global attractor in V .*

Proof. As a consequence of Remark 2.3, \mathcal{S} is nonvoid whenever V is nonvoid. Moreover, $\omega(B) \subset \mathcal{S}$ for any $\emptyset \neq B \subset V$ such that $\gamma^+(B)$ is bounded. It is next sufficient to note that if \mathcal{S} is bounded, then – since it is also invariant – we obtain that $\text{cl}_{X^\alpha} \mathcal{S}$ is a compact subset of V (see Remark 2.1). The proof is thus complete. \square

Corollary 2.6. *Suppose A is a sectorial operator in a Banach space X and A has compact resolvent. Assume that $F: X^\alpha \rightarrow X$, with $\alpha \in [0, 1)$ fixed, is Lipschitz continuous on bounded subsets of X^α . Let all (hypothetical) bounded complete orbits of points be uniformly bounded in X^α . Then for any $u_0 \in X^\alpha$ the X^α solution $u(\cdot, u_0)$ of (2.1) either blows up (in a finite or infinite time) or stays bounded and approaches a maximal compact invariant set.*

We recall that similarly as in the case of an ω -limit set we know that the α -limit set of $u_0 \in \mathcal{S}$ along a bounded complete trajectory ϕ of u_0 ,

$$\alpha_\phi(u_0) = \bigcap_{t \leq 0} \text{cl}_{X^\alpha} \bigcup_{s \leq t} \{\phi(s)\},$$

is a nonvoid compact subset of V . If, in addition, there exists a Lyapunov function on V , then $\alpha_\phi(u_0) \subset \mathcal{E}$. In the latter case abstract conditions for the boundedness of \mathcal{S} can be formulated.

Proposition 2.7. *Assume that there exists a Lyapunov function \mathcal{L} on V . Then the following conditions are equivalent:*

- (a) \mathcal{S} is a bounded subset of V ,
- (b) \mathcal{E} is a bounded subset of V and one of the equivalent conditions holds:
 - (i) if $v_n \in \mathcal{S}$ and $\|v_n\|_{X^\alpha} \rightarrow \infty$, then $|\mathcal{L}(v_n)| \rightarrow \infty$ (cf. [HA, Definition 3.8.1]),
 - (ii) if $B \subset \mathcal{S}$ and $\mathcal{L}(B)$ is a bounded subset of \mathbb{R} , then B is bounded.

Proof. We shall prove that (b) implies (a). Therefore we assume (b) and suppose contrary to our claim that there exist $v_n = \phi_n(t_n)$, where $t_n \in \mathbb{R}$, ϕ_n are bounded complete trajectories, say of some u_n , and $\|v_n\|_{X^\alpha} \rightarrow \infty$ as $n \rightarrow \infty$. We choose $\alpha_n \in \alpha_{\phi_n}(u_n)$ and $\omega_n \in \omega(u_n)$ being the elements of \mathcal{E} . Since the Lyapunov function \mathcal{L} is nonincreasing along each complete trajectory, we see that

$$\max\{\mathcal{L}(e) : e \in \mathcal{E}\} \geq \mathcal{L}(\alpha_n) \geq \mathcal{L}(v_n) \geq \mathcal{L}(\omega_n) \geq \min\{\mathcal{L}(e) : e \in \mathcal{E}\},$$

where the maximum and the minimum exist due to the compactness of \mathcal{E} . But this is impossible, because of (i). This shows that \mathcal{S} is bounded. \square

3. EXAMPLES

Example 3.1. Let us consider the Dirichlet problem for *Frank-Kamenetskii equation*

$$(3.1) \quad \begin{cases} u_t = \Delta u + \lambda e^u, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and $\Omega = B(0, 1) \subset \mathbb{R}^N$. This problem occurs in models of thermal explosions, especially in the description of thermal self-ignition of chemically active mixture contained in some vessel. We refer the reader to [FK], [GE, §15] and [B-E] for more details.

Rewriting the problem (3.1) in an abstract setting we consider

$$(3.2) \quad \begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0 \end{cases}$$

in the Hilbert space $X = L^2(\Omega)$, where $A = -\Delta_D : X \supset \text{dom}(A) \rightarrow X$ with its domain $\text{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well-known that this operator is a positive sectorial operator in X with compact resolvent. Although the Frank-Kamenetskii equation is also interesting and complex for dimensions $3 \leq N \leq 9$ (especially because of an infinite number of stationary solutions for $\lambda = 2(N-2)$, see e.g. [J-L], [B-E, Theorem 2.19], [F-P], [N-S]), we restrict our attention to $N = 1$ for the purpose of this example. Nevertheless, we still use the general notation.

Fix $\frac{3}{4} < \alpha < 1$ so that $X^\alpha \subset C^1(\overline{\Omega})$. Evidently, if $u \in X^\alpha$, then we have $u \in C^1(\overline{\Omega})$, $F(u) = \lambda e^u \in C^1(\overline{\Omega})$, and $F : X^\alpha \rightarrow X$ is Lipschitz continuous on bounded sets. Consequently, (3.1) generates a local semiflow of X^α solutions. Denoting then by V the set of all initial data for which the solution stays bounded in X^α we see that

$$(3.3) \quad \mathcal{L}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \lambda e^u dx$$

is a Lyapunov function on V .

Stationary solutions and their Morse indices. If $w \in \text{dom}(A)$ is a stationary solution, then w satisfies

$$(3.4) \quad \begin{cases} -\Delta w = \lambda e^w, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases}$$

which is also known in the literature under the names of *Emden-Fowler equation* or *Gelfand problem*. From the regularity theory of elliptic operators (cf. [TR, Theorem 5.4.1]) it follows that the stationary solutions are smooth and in particular they belong to $C^2([-1, 1])$. Also, all solutions of (3.4) are positive, and thus radially symmetric by the result of [G-N-N]. We recall (see [B-E, Theorem 2.19]) that there exists $\lambda^* > 0$, $\lambda^* \approx 0.878$, such that

- (a) for each $\lambda \in (0, \lambda^*)$ there are two solutions,
- (b) for $\lambda = \lambda^*$ there is a unique solution,
- (c) for $\lambda > \lambda^*$ there are no solutions.

Moreover, each solution w has to satisfy

$$(3.5) \quad w(x) = w(0) - 2 \ln \cosh \left(\frac{1}{2} \sqrt{2\lambda e^{w(0)}} x \right), \quad -1 \leq x \leq 1.$$

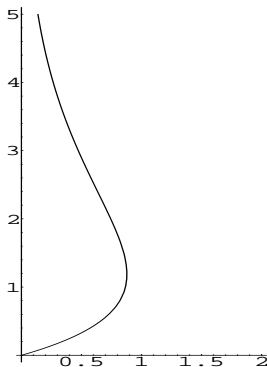


FIGURE 1. Dependence between $w(0)$ and λ

If $\lambda \in (0, \lambda^*)$, then from [FU] we infer that there exists the minimal solution. Let us denote it by w^+ and the maximal solution by w^- . We know that there exists $\gamma > 0$ such that

$$\gamma \rho(x) \leq w^-(x) - w^+(x), \quad x \in \overline{\Omega},$$

where $\rho(x)$ is the distance from x to $\partial\Omega$. Consequently, the curves shown in Figure 1 represent the solution curves

$$\{(\lambda, w) \in (0, \infty) \times C(\overline{\Omega}) : (\lambda, w) \text{ satisfies (3.4)}\}.$$

We remark that, as shown in [FU, Theorem 6] (see also the refinement in [FI, Remark 2.5]), if $\lambda \in (0, \lambda^*)$, $w^-(x) \leq u_0(x)$ for $x \in \overline{\Omega}$ and $u_0 \not\equiv w^-$, then the solution u of (3.1) blows up in a finite time.

Consider now the linearization of (3.4) at w

$$(3.6) \quad \begin{cases} \Delta v + \lambda e^w v = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases}$$

and observe that $\sigma(-\Delta - \lambda e^w)$ consists only of real eigenvalues. We say that w is a *hyperbolic stationary solution* if $0 \notin \sigma(-\Delta - \lambda e^w)$. Furthermore, the number of negative eigenvalues of $-\Delta - \lambda e^w$ is called the *Morse index* $\text{ind}(w)$ of the stationary solution w . Although $0 \in \sigma(-\Delta - \lambda^* e^w)$, it is known that if $\lambda \in (0, \lambda^*)$, then w^- and w^+ are both hyperbolic stationary solutions. Additionally, we have $\text{ind}(w^+) = 0$ and $\text{ind}(w^-) = 1$ (for details see [C-R, Proposition 2.15] and [N-S, Section 2]).

Unstable manifold of w^- and description of \mathcal{S} . Let $\lambda \in (0, \lambda^*)$. Note that w^- is a hyperbolic fixed point in the sense of [C-C-H, p. 357]. Then the *unstable set* $W^u(w^-)$ is a C^1 submanifold of X^α with

$$\dim W^u(w^-) = \text{ind}(w^-) = 1$$

(see [HE, Theorem 6.1.9], [C-C-H, Appendix C]) and by [B-F1, Theorem 2.1] we have

$$(3.7) \quad \forall_{w \in W^u(w^-)} z(w - w^-) < \dim W^u(w^-) = 1,$$

where $z(g)$ denotes the number of sign changes of a continuous function g .

The existence of a Lyapunov function excludes the existence of (non-constant) homoclinic orbits. Thus we restrict our attention to heteroclinic orbits. Let ϕ be a (hypothetical) complete trajectory of $u_0 \in X^\alpha$ such that $\phi(t) \rightarrow w^-$ as $t \rightarrow -\infty$ and $\phi(t) \rightarrow w^+$ as $t \rightarrow \infty$ in $X^\alpha \subset C^1(\overline{\Omega})$. Since the complete trajectory ϕ does not blow up, we obtain from (3.7)

$$\forall_{t \in \mathbb{R}} \forall_{x \in \overline{\Omega}} \phi(t)(x) \leq w^-(x).$$

If we show that

$$(3.8) \quad \exists_{t^* < 0} \forall_{t \leq t^*} \forall_{x \in \overline{\Omega}} w^+(x) \leq \phi(t)(x),$$

by the monotonicity we will then have

$$\forall_{t \in \mathbb{R}} \forall_{x \in \overline{\Omega}} w^+(x) \leq \phi(t)(x) \leq w^-(x).$$

Hence for the boundedness of \mathcal{S} it is sufficient to prove (3.8).

Contrary to our claim, assume that there exist $t_n \rightarrow -\infty$, $y_n \in \overline{\Omega}$ such that

$$(3.9) \quad \phi(t_n)(y_n) < w^+(y_n).$$

In fact we can assume that there exists $x_n \rightarrow x_0$ with $x_0 \in \overline{\Omega}$ and

$$(3.10) \quad \phi(t_n)(x_n) = w^+(x_n), \quad n \in \mathbb{N}.$$

Indeed, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have $\phi(t_n)(0) > w^+(0)$. Since (3.9) holds, the Darboux property ensures the existence of $x_n \in \overline{\Omega}$ such that $\phi(t_n)(x_n) = w^+(x_n)$. By the compactness of $\overline{\Omega}$ we may choose a convergent subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, which we have required.

We consider two cases. If $x_0 \in (-1, 1)$, then $\phi(t_n)(x_n)$ tends to $w^-(x_0)$ and $w^-(x_0) = w^+(x_0)$, which is impossible. Therefore we have $x_0 \in \{-1, 1\}$. From (3.5) we get

$$(w^+)'(1) > (w^-)'(1) \text{ and } (w^+)(-1) < (w^-)'(-1).$$

Let us consider $x_0 = 1$. By (3.10) and the mean value theorem we have

$$\frac{w^+(x_n) - w^+(1)}{x_n - 1} = \frac{\phi(t_n)(x_n) - \phi(t_n)(1)}{x_n - 1} = [\phi(t_n)]'(\xi_n).$$

Since the left hand side tends to $(w^+)'(1)$ and the right hand side to $(w^-)'(1)$, we get $(w^+)'(1) = (w^-)'(1)$, a contradiction. The same reasoning applies to the case $x_0 = -1$. This ends the proof of the boundedness of \mathcal{S} .

Observe that if we consider the solution semigroup for $u \in X^\alpha$ satisfying the inequalities $w^+(x) \leq u(x) \leq w^-(x)$ for all $x \in \Omega$, then this semigroup is compact. Hence for this semigroup there exists a compact *connected* global attractor. In particular there must exist a heteroclinic orbit connecting w^- to w^+ .

In fact there exists a unique heteroclinic orbit connecting these two equilibria. This can be established using the argument in [B-F3, Lemma 3.5]. Thus we obtain the description of $\mathcal{S} = \{w^-, w^+, \phi(\mathbb{R})\}$, where ϕ is the only complete trajectory connecting w^- to w^+ .

We conclude that \mathcal{S} is a maximal compact invariant set attracting any subset of V possessing bounded orbit. In particular, if $u_0 \in X^\alpha$, then the corresponding X^α solution either blows up or stays bounded and approaches a maximal compact invariant set.

Example 3.2. Consider the N -dimensional Navier-Stokes system for incompressible viscous fluid flow under the assumption of small perturbation. We shall show that in this case there exists a Lyapunov function. In our considerations we follow the paper [C-D2, Section 4]. We consider the problem

$$(3.11) \quad \begin{cases} u_t = \nu \Delta u - \nabla p - (u \cdot \nabla)u + f, & t > 0, x \in \Omega \subset \mathbb{R}^N, \\ \operatorname{div} u = 0, & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases}$$

where $N \geq 2$, $\nu > 0$ is a viscosity constant and Ω is a bounded domain in \mathbb{R}^N with boundary $\partial\Omega$ of class $C^{2+\varepsilon}$.

For any $f \in [L^p(\Omega)]^N$, $p > N$, the system can be viewed as an abstract Cauchy problem

$$(3.12) \quad \begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0 \end{cases}$$

in the space

$$X = \operatorname{cl}_{[L^p(\Omega)]^N} \{ \phi \in [C_0^\infty(\Omega)]^N : \operatorname{div} \phi = 0 \text{ in } \Omega \}$$

using the projector P given by the decomposition of $[L^p(\Omega)]^N$ into the spaces of divergence free vector fields and scalar function gradients (see [F-M] and [G-M]). Namely, we define $A = -\nu P\Delta: X \supset \text{dom}(A) \rightarrow X$ with

$$\text{dom}(A) = X \cap \{\phi \in [W^{2,p}(\Omega)]^N : \phi = 0 \text{ on } \partial\Omega\},$$

which is a sectorial operator with compact resolvent, and $F: X^\alpha \rightarrow X$ by

$$(3.13) \quad F(u) = -P(u \cdot \nabla)u + Pf, \quad u \in X^\alpha.$$

Restricting further α to the interval $[\frac{1}{2}, 1)$ we observe that F in (3.13) is well-defined and is Lipschitz continuous on bounded subsets of X^α .

Recall that for sufficiently small $f \in [L^p(\Omega)]^N$ (especially if the external force f is zero)

$$(3.14) \quad \text{there exists a stationary solution } w \in \text{dom}(A) \text{ of the Navier-Stokes system such that } \|w\|_{[W^{1,\infty}(\Omega)]^N} < \frac{\nu}{C_\Omega^2},$$

where C_Ω is a constant in the Poincaré inequality.

Lyapunov function on V and description of \mathcal{S} . Assuming further (3.14) we define V as in (2.2) and consider the functional

$$(3.15) \quad \mathcal{L}(u) = \frac{1}{2} \|u - w\|_{[L^2(\Omega)]^N}^2, \quad u \in V.$$

We shall show that \mathcal{L} is a Lyapunov function on V . Since $p > N \geq 2$, it follows that \mathcal{L} is continuous on V . Fix $u_0 \in V$. Letting $u(t) = u(t, u_0)$, $t \geq 0$, we have

$$(3.16) \quad (u - w)_t = -A(u - w) - P((u - w) \cdot \nabla)w - P(u \cdot \nabla)(u - w), \quad t > 0.$$

From [F-M] it follows that

$$Pv = P_2v, \quad v \in [L^p(\Omega)]^N,$$

where P_2 is a selfadjoint bounded projection operator on $[L^2(\Omega)]^N$. Hence for $v_1, v_2, v_3 \in \text{dom}(A)$ we have

$$\langle P(v_1 \cdot \nabla)v_2, v_3 \rangle_{[L^2(\Omega)]^N} = \langle P_2(v_1 \cdot \nabla)v_2, v_3 \rangle_{[L^2(\Omega)]^N} = \langle (v_1 \cdot \nabla)v_2, v_3 \rangle_{[L^2(\Omega)]^N}$$

and

$$\langle Av_1, v_1 \rangle_{[L^2(\Omega)]^N} = -\nu \langle \Delta v_1, v_1 \rangle_{[L^2(\Omega)]^N}.$$

Multiplying (3.16) by $u - w$ in $[L^2(\Omega)]^N$ we obtain

$$(3.17) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|u - w\|_{[L^2(\Omega)]^N}^2 &\leq -\nu \sum_{i=1}^N \|\nabla(u_i - w_i)\|_{L^2(\Omega)}^2 + \\ &+ \|w\|_{[W^{1,\infty}(\Omega)]^N} \|u - w\|_{[L^2(\Omega)]^N}^2 \leq \\ &\leq \left(-\frac{\nu}{C_\Omega^2} + \|w\|_{[W^{1,\infty}(\Omega)]^N} \right) \|u - w\|_{[L^2(\Omega)]^N}^2. \end{aligned}$$

The above inequality proves that $t \mapsto \mathcal{L}(u(t, u_0))$ is nonincreasing for $t > 0$. Moreover, we see that there exists $a > 0$ such that

$$\|u - w\|_{[L^2(\Omega)]^N} \leq \|u_0 - w\|_{[L^2(\Omega)]^N} e^{-at}, \quad t \geq 0.$$

This shows in particular that

$$(3.18) \quad \lim_{t \rightarrow \infty} u(t, u_0) = w \text{ in } [L^2(\Omega)]^N.$$

Our task will be, however, to justify the convergence in X^α . Meanwhile, assume that $\mathcal{L}(u(\cdot, u_0)) \equiv \mathcal{L}(u_0)$. Then we have as in (3.17)

$$0 = \frac{d}{dt} \mathcal{L}(u(t, u_0)) \leq \left(-\frac{\nu}{C_\Omega^2} + \|w\|_{[W^{1,\infty}(\Omega)]^N} \right) \|u(t, u_0) - w\|_{[L^2(\Omega)]^N}^2 \leq 0.$$

This implies $u_0 = w \in \mathcal{E}$ and consequently $\mathcal{E} = \{w\}$.

Suppose that ϕ is a (hypothetical) bounded complete trajectory of u_0 . Since $\alpha_\phi(u_0) = \{w\}$ and $\omega(u_0) = \{w\}$, we have

$$\mathcal{L}(w) \geq \mathcal{L}(\phi(s)) \geq \mathcal{L}(w), \quad s \in \mathbb{R},$$

so that $\phi(\mathbb{R}) = \{w\}$. Hence we obtain $\mathcal{S} = \{w\}$.

We thus infer that if (3.14) holds, then for any $u_0 \in X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$, the X^α solution $u(\cdot, u_0)$ of the Navier-Stokes system either blows up in X^α or stays bounded and approaches in X^α the maximal compact invariant set $\{w\}$.

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