DYNAMICALLY EQUIVALENT PERTURBATIONS OF LINEAR PARABOLIC EQUATIONS

RADOSŁAW CZAJA

Abstract. A family of abstract parabolic equations with sectorial operator is studied in this paper. The conditions are provided to show that the global attractors for each equation exist and coincide. Although the common dynamics is simple, the examples presented in the final part of the paper indicate that the considered family may contain a linear equation together with a large number of its nonlinear perturbations. The mentioned examples include both scalar second order equations and the celebrated Cahn-Hilliard system.

1. Introduction

We consider a family of autonomous abstract parabolic equations

$$u_t + Au = F_\lambda(u),\ t > 0, \tag{1.1}$$

in a Banach space X, where $-A$ generates a strongly continuous analytic semigroup. Our aim is to formulate a general abstract setting for the coincidence of global attractors for a wide class of nonlinear perturbations with the global attractor for the linear problem. This goal is achieved by the use of the semigroup theory for semilinear abstract parabolic equations developed in [HE], [C-D 2]. The global attractor is obtained in a metric subspace V of a certain fractional power space X^α defined by the operator A appearing in the main part of the equation (1.1) (cf. [AM], [HE]). It is then interesting to consider the situation when (1.1) is *synchronized* in the sense that all attractors $\mathcal{A}_\lambda$ coincide (cf. [HA]). We describe it in some special case, in which the dynamics is determined by the ω-limit sets of points, or even by the stationary solutions.

The above-mentioned abstract results are presented in Section 1. They are supported by some examples provided in Section 2. The first one deals with the scalar second order equation, while the second one is devoted to the Cahn-Hilliard system describing the evolution of a molten multi-component alloy. The case of binary alloys, which reduces the problem to a single equation, has been widely investigated in the literature (cf. e.g. [C-D 2], [R-H], [GR] and the references therein). In comparison with the equation, the Cahn-Hilliard system has not been studied so intensively (cf. e.g. [C-D 1], [L-Z]). Here we follow [C-D 2] to obtain the global solutions of the Cahn-Hilliard system under the assumption of the semiconvexity of the bulk free energy λ of the alloy. However, in order to obtain such a simple dynamics we

Date: 15 April 2003.

2000 *Mathematics Subject Classification.* Primary 35B41; Secondary 35K90, 35K55, 35K15.

Key words and phrases. Global attractor, dissipative semigroup, parabolic equation, Cahn-Hilliard system.

further assume that λ is convex. What surprises is the wide range of admissible perturbations as well as the lack of their impact on the global attractor. The third example indicates that the abstract results can also be applied to pseudodifferential equations.

Notation used in the paper is standard. Here we point out that $|\Omega|$ denotes the n–dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^n$, while $\mathrm{dom}(A)$ and $\mathrm{im}(A)$ stand for the domain and the range of an operator A, respectively. Moreover, $\mathcal{D}(A)$ means $\mathrm{dom}(A)$ endowed with the graph norm. We also write $\mathrm{tr}(B)$ for the trace of a matrix B, i.e. the sum of the elements in the leading diagonal. Other notation will be explained further in the text.

2. Abstract results

Consider a family of autonomous abstract parabolic equations (1.1) in a complex Banach space X, where $\lambda \in \Lambda$ (Λ is a certain set of indices). Assume that

(A.1) $A\colon X \supset \mathrm{dom}(A) \to X$ is a sectorial operator (cf. [HE, Definition 1.3.1], [CZ, Definition 2.2.2]) with compact resolvent, i.e. $(zI - A)^{-1} \in \mathcal{L}(X, X)$ is a compact operator for all $z \in \rho(A)$,
(A.2) $F_\lambda\colon X^\alpha \to X$ (where $\alpha \in [0, 1)$ is fixed from now on) is Lipschitz continuous on bounded subsets of X^α for each $\lambda \in \Lambda$, where X^α denotes the fractional power space corresponding to the operator A (cf. [HE], [AM], [C-D 2]),
(A.3) unique local X^α solutions $u_\lambda(\cdot, u_0)$ of (1.1) with $u_\lambda(0, u_0) = u_0 \in X^\alpha$ exist globally in time.

Note that (A.1) is equivalent to the requirement that $-A\colon X \supset \mathrm{dom}(A) \to X$ is an infinitesimal generator of a strongly continuous compact analytic semigroup (cf. [PA, Theorem 2.3.3]). Observe also that the existence and uniqueness of local X^α solutions for $u_0 \in X^\alpha$, mentioned in (A.3), is due to the theory given e.g. in [C-D 2], [HE] or [CZ]. Recalling [C-D 2, Theorem 3.3.1], we can state

Proposition 2.1. *Under the assumptions (A.1)-(A.3) the problem* (1.1) *defines a family of compact C^0 semigroups $\{T_\lambda(t)\colon t \geq 0\}$ on X^α such that $T_\lambda(t)u_0$ coincides with a global solution $u_\lambda(t, u_0)$ to* (1.1) *satisfying the initial condition $u_\lambda(0, u_0) = u_0$.*

Here and subsequently, $S_\lambda \subset X^\alpha$ stands for the set consisting of all stationary points of $\{T_\lambda(t)\colon\ t \geq 0\}$ on X^α.

Let us denote by V any complete metric subspace of X^α which is positively $\{T_\lambda(t)\}$-invariant for each $\lambda \in \Lambda$, i.e. $T_\lambda(t)V \subset V$, $\lambda \in \Lambda$. Throughout the remainder of this section we limit ourselves to $\{T_\lambda(t)\}$ restricted to V. Now it is a family of compact C^0 semigroups on V. Let us introduce further assumptions:

(A.4) positive orbits of points $\gamma^+_\lambda(u_0) = \{T_\lambda(t)u_0\colon\ t \geq 0\}$, $u_0 \in V$, are bounded subsets of V,
(A.5) there exist continuous *Lyapunov functions* $\mathcal{L}_\lambda\colon V \to \mathbb{R}$, $\lambda \in \Lambda$, such that for any $u_0 \in V$ the function $t \mapsto \mathcal{L}_\lambda(T_\lambda(t)u_0)$ is nonincreasing for $t > 0$ and

$$\text{if } \mathcal{L}_\lambda(T_\lambda(t)u_0) = \mathcal{L}(u_0) \text{ for all } t \geq 0, \text{ then } u_0 \in S_\lambda \cap V,$$

(A.6) there exist a metric space M and continuous functions $l_\lambda\colon V \to M$, $\lambda \in \Lambda$, which are one-to-one on $S_\lambda \cap V$ and

$$l_\lambda(T_\lambda(t)u_0) = l_\lambda(u_0),\ t \geq 0,\ u_0 \in V.$$

We underline that the Lyapunov functions in (A.5) need not have to be bounded below. Moreover, (A.5) ensures that the functions $t \mapsto \mathcal{L}_\lambda(T_\lambda(t)u_0)$ are nonincreasing for all $t \geq 0$.

Remark 2.2. If one of the assumptions (A.4)-(A.6) is satisfied with $V = X^\alpha$, then it holds for any complete metric subspace of X^α which is positively $\{T_\lambda(t)\}$-invariant for each $\lambda \in \Lambda$.

Following [LA], we recall that a *complete trajectory of a point $v \in V$ for a semigroup* $\{S(t)\colon\ t \geq 0\}$ is the curve $\phi\colon \mathbb{R} \to V$ satisfying the following conditions:

(i) $\phi(0) = v$,
(ii) $S(t)\phi(s) = \phi(s+t)$, $s \in \mathbb{R}$, $t \geq 0$.

We shall denote by Φ_λ^v (respectively Γ_λ^v) the set of all (bounded) complete trajectories of a point $v \in V$ for the semigroup $\{T_\lambda(t)\colon\ t \geq 0\}$, whereas C_λ (respectively C_λ^b) shall stand for the set of all points $v \in V$ for which there exists at least one (bounded) complete trajectory of the point v for the semigroup $\{T_\lambda(t)\colon\ t \geq 0\}$. Observe that since $\{T_\lambda(t)\colon\ t \geq 0\}$ is a C^0 semigroup, then a complete trajectory ϕ of a point v is a continuous function. Moreover, we have $T_\lambda(t)v = \phi(t)$, $t \geq 0$.

We define the *ω-limit set of a set $B \subset V$ for the semigroup* $\{T_\lambda(t)\colon\ t \geq 0\}$ by

$$\omega_\lambda(B) = \bigcap_{t \geq 0} \operatorname{cl}_V \bigcup_{s \geq t} T_\lambda(s)B.$$

If $v \in C_\lambda$ and $\phi \in \Phi_\lambda^v$, then we define the *α-limit set of the point v along the trajectory ϕ* by

$$\alpha_{\lambda,\phi}(v) = \bigcap_{t \leq 0} \operatorname{cl}_V \bigcup_{s \leq t} \{\phi(s)\}.$$

Following [S-Y, Lemma 22.3] and [C-D 2, Proposition 1.1.2], we recall

Proposition 2.3. *Assume that the assumptions (A.1)-(A.3) and (A.5) hold and $\lambda \in \Lambda$. If $v \in V$ and $\gamma_\lambda^+(v) = \{T_\lambda(t)v\colon\ t \geq 0\}$ is bounded in V, then $\omega_\lambda(v)$ is a nonempty subset of $S_\lambda \cap V$ and attracts v. Moreover, if $v \in C_\lambda$, $\phi \in \Phi_\lambda^v$ and $\gamma_{\lambda,\phi}^-(v) = \{\phi(t)\colon\ t \leq 0\}$ is bounded in V, then $\alpha_{\lambda,\phi}(v)$ is a nonempty subset of $S_\lambda \cap V$.*

We also recall that by the *global attractor for the semigroup* $\{T_\lambda(t)\colon\ t \geq 0\}$ we mean a nonempty, compact and $\{T_\lambda(t)\}$-invariant subset of V which attracts each bounded subset of V.

Proposition 2.4. *If the assumptions (A.1)-(A.5) hold and $S_\lambda \cap V$ is independent of $\lambda \in \Lambda$, then $\{T_\lambda(t)\colon\ t \geq 0\}$ on V possesses a global attractor for certain $\lambda = \lambda_0$ if and only if it possesses a global attractor for all $\lambda \in \Lambda$.*

Proof. If $\{T_{\lambda_0}(t)\colon\ t \geq 0\}$ has a global attractor, then $\bigcup_{u_0 \in V} \omega_{\lambda_0}(u_0)$ is a nonempty bounded set, since it is contained in the attractor (see [C-D 2, Corollary 1.1.1]).

Moreover, we have $\bigcup_{u_0 \in V} \omega_{\lambda_0}(u_0) = S_{\lambda_0} \cap V$ as a consequence of (A.1)-(A.5). Fix arbitrarily $\lambda \in \Lambda$. Again the assumptions (A.1)-(A.5) and the coincidence of the sets of stationary points guarantee that

$$\bigcup_{u_0 \in V} \omega_\lambda(u_0) = S_\lambda \cap V = S_{\lambda_0} \cap V$$

is a nonempty bounded subset of V. Since it attracts every point of V, we see that $\{T_\lambda(t)\colon\ t \geq 0\}$ is point dissipative. Hence the semigroup $\{T_\lambda(t)\colon\ t \geq 0\}$, being compact and point dissipative, possesses a global attractor (cf. [C-D 2, Corollary 1.1.6]). □

From the proof of the above proposition we obtain

Corollary 2.5. *Let the assumptions (A.1)-(A.5) hold and $S_\lambda \cap V$, $\lambda \in \Lambda$, be bounded subsets of V. Then the semigroup of global solutions corresponding to* (1.1) *on V possesses a global attractor $\mathcal{A}_\lambda$ for each $\lambda \in \Lambda$.*

We next focus on the situation, when (1.1) is synchronized in the sense that all attractors $\mathcal{A}_\lambda$ coincide (cf. [HA]). Theorem 2.6 below describes it in a special case where the dynamics is determined by the ω-limit sets of points, or equivalently by the stationary points (this is due to (A.6)).

Theorem 2.6. *Let the assumptions (A.1)-(A.6) hold and $S_\lambda \cap V$ be independent of $\lambda \in \Lambda$ and bounded in V. Then we have*

$$\mathcal{A}_\lambda = \bigcup_{u_0 \in V} \omega_\lambda(u_0) = S_\lambda \cap V =: \mathcal{A},\ \lambda \in \Lambda.$$

Proof. Fix $\lambda \in \Lambda$. As follows from Corollary 2.5, each semigroup $\{T_\lambda(t)\colon\ t \geq 0\}$ on V possesses a global attractor $\mathcal{A}_\lambda$. Then we know that

$$\mathcal{A}_\lambda = \bigcup_{u_0 \in C^b_\lambda} \bigcup_{\phi \in \Gamma^{u_0}_\lambda} \phi(\mathbb{R})$$

(see [LA, Proposition 2.2] or [C-D 2, Corollary 1.1.1]). It is sufficient to show that $\mathcal{A}_\lambda \subset S_\lambda \cap V$. To this end, fix $u_0 \in C^b_\lambda$ and $\phi \in \Gamma^{u_0}_\lambda$. Take any $v \in \alpha_{\lambda,\phi}(u_0) \subset S_\lambda \cap V$ and any $w \in \omega_\lambda(u_0) \subset S_\lambda \cap V$, their existence being ensured by the fact that $\phi \in \Gamma^{u_0}_\lambda$ and Proposition 2.3. From (A.6) we know that the continuous function l_λ is constant along the complete trajectory ϕ so we have $l_\lambda(v) = l_\lambda(u_0) = l_\lambda(w)$ and thus $v = w$. We shall prove that $\phi(\mathbb{R}) = \{u_0\}$. Note that (A.5) ensures that $\mathcal{L}_\lambda$ is nonincreasing along the complete trajectory ϕ. We claim that, in fact, $\mathcal{L}_\lambda$ is constant along ϕ. Suppose now, contrary to our claim, that there exist $t_1 < t_2$, $t_1, t_2 \in \mathbb{R}$, such that $\mathcal{L}_\lambda(\phi(t_1)) > \mathcal{L}(\phi(t_2))$. For t, s positive and large enough we have

$$\mathcal{L}_\lambda(\phi(-s)) \geq \mathcal{L}_\lambda(\phi(t_1)) > \mathcal{L}_\lambda(\phi(t_2)) \geq \mathcal{L}_\lambda(\phi(t)),$$

which leads to the absurd relation $\mathcal{L}_\lambda(v) > \mathcal{L}_\lambda(w) = \mathcal{L}_\lambda(v)$. Consequently,

$$\mathcal{L}_\lambda(\phi(t)) = \mathcal{L}(u_0) \text{ for all } t \in \mathbb{R}.$$

Fix $t_0 \in \mathbb{R}$. Then we know that

$$\mathcal{L}_\lambda(T_\lambda(t)\phi(t_0)) = \mathcal{L}_\lambda(\phi(t + t_0)) = \mathcal{L}_\lambda(u_0) = \mathcal{L}_\lambda(\phi(t_0)) \text{ for all } t \geq 0.$$

Thus (A.5) ensures that $\phi(t_0) \in S_\lambda \cap V$. Hence we obtain $\phi(\mathbb{R}) \subset S_\lambda \cap V$. Moreover, $\phi(\mathbb{R})$ cannot contain two distinct stationary points, since ϕ is a complete trajectory. Thus $\phi(\mathbb{R}) = \{u_0\}$. This clearly forces

$$\mathcal{A}_\lambda = S_\lambda \cap V = \bigcup_{u_0 \in V} \omega_\lambda(u_0) = \mathcal{A},\ \lambda \in \Lambda.$$

□

Note that under the assumptions (A.1)-(A.6) each bounded complete trajectory for the semigroup $\{T_\lambda(t)\colon t \geq 0\}$ is a singleton, i.e. a single stationary point. Moreover, each $\omega_\lambda(u_0)$, $u_0 \in V$, is a singleton. Nevertheless, the global attractor may still contain infinitely many elements and (A.1)-(A.6) are satisfied in a number of interesting examples as shown in Section 3.

Remark 2.7. Consider a family of parabolic abstract equations

$$\begin{cases} v_t + \mu A v = F_\lambda(v),\ t > 0, \\ v(0) = u_0, \end{cases} \tag{2.1}$$

where $\mu \in (0, \infty)$ and $\lambda \in \Lambda$ (a certain set of indices). We remark that u satisfies

$$\begin{cases} u_t + A u = \frac{1}{\mu} F_\lambda(u),\ t > 0, \\ u(0) = u_0 \end{cases} \tag{2.2}$$

if and only if $v(t) = u(\mu t)$, $t \geq 0$, satisfies (2.1). Hence the global attractor constructed for the semigroup of X^α solutions of (2.2) coincides with the global attractor for the problem (2.1).

Finally, let us sketch how to apply Theorem 2.6 in case our problem is naturally set in a real Banach space.

Remark 2.8. Owing to the requirement of the sectoriality of the operator A in (1.1) we are forced to consider the equation in a *complex* Banach space. Nevertheless, in applications, unless the operator is complex, we obtain the abstract equations (1.1) in a *real* Banach space X. However, we can complexify the operator A (cf. [AM, p. 4]) and consider $\widetilde{A}(u) = Av + iAw$, $u = v + iw \in \widetilde{X} = X + iX$. If $\widetilde{A}$ is a sectorial operator, then the linear semigroup $\{e^{-\widetilde{A}t}\colon t \geq 0\}$ preserves the space X (see [LU, Lemma 2.1.3]). Thus we are able to define the space X^α as $\mathrm{im}(\widetilde{A}^{-\alpha}|_X)$ and use the real counterpart of the theory of existence and uniqueness of X^α solutions. Therefore, if $\widetilde{A}$ satisfies (A.1) and for (1.1) the conditions (A.2) and (A.3) are satisfied, then we can still define the semigroup of X^α solutions $T_\lambda(t)u_0 = u_\lambda(t, u_0)$, $t \geq 0$, $u_0 \in X^\alpha$. Furthermore, if the assumptions (A.4)-(A.6) hold with an appropriate subspace $V \subset X^\alpha$ and $S_\lambda \cap V$ is independent of $\lambda \in \Lambda$ and bounded in V, then an analogous version of Theorem 2.6 ensures the existence of the global attractor $\mathcal{A}_\lambda = S_\lambda \cap V = \mathcal{A}$, $\lambda \in \Lambda$, for the semigroup $\{T_\lambda(t)\colon t \geq 0\}$ in V.

3. Examples

Example 3.1. Our first example will be an initial-boundary value problem for a scalar second order equation

$$\begin{cases} u_t(t,x) - \triangle u(t,x) = \mathbf{b}(x)\cdot\nabla(\lambda(u(t,x))),\ t>0,\ x\in\Omega, \\ \frac{\partial u}{\partial N}(t,x) = 0,\ t>0,\ x\in\partial\Omega, \\ u(0,x) = u_0(x),\ x\in\Omega, \end{cases} \tag{3.1}$$

where $\mathbf{b}\in C^1(\overline{\Omega})$ is a vector field such that

$$\operatorname{div}\mathbf{b}(x) = 0,\ x\in\Omega, \tag{3.2}$$

and

$$\mathbf{b}(x)\cdot N(x) = 0,\ x\in\partial\Omega. \tag{3.3}$$

Here we consider $\Omega\subset\mathbb{R}^n$, $n\geq 2$, with the C^2 boundary and $\lambda\in\Lambda = C^{1+Lip}(\mathbb{R})$.

Rewriting (3.1) in an abstract setting in $X = L^p(\Omega)$ with $p>\frac{n}{2}$, we obtain

$$\begin{cases} u_t + Au = F_\lambda(u),\ t>0, \\ u(0) = u_0, \end{cases} \tag{3.4}$$

where $F_\lambda(u) = \mathbf{b}\cdot\nabla(\lambda(u))$ and $A = -\triangle_N$ in X with the domain

$$\operatorname{dom}(A) = W^{2,p}_{\{\frac{\partial}{\partial N}\}}(\Omega) = \operatorname{cl}_{W^{2,p}(\Omega)}\left\{\phi\in C^2(\overline{\Omega})\colon \frac{\partial\phi}{\partial N} = 0 \text{ on } \partial\Omega\right\}.$$

It is well-known that $\{-\triangle_N, \{\frac{\partial}{\partial N}\}, \Omega\}$ forms a regular elliptic boundary value problem. Therefore, $-\triangle_N$ is a sectorial operator with compact resolvent (see [C-D 2, Proposition 1.2.3, Example 1.3.8]).

Fix any $\max\left(\frac{1}{2},\frac{n}{2p}\right) < \alpha < 1$ and take into consideration the fractional power space X^α. The Sobolev embeddings (cf. [HE, Theorem 1.6.1] or [CZ, Theorem 3.0.21]) now yield

$$X^\alpha\subset W^{1,p}(\Omega) \text{ and } X^\alpha\subset C(\overline{\Omega}).$$

Our considerations in X are justified, since for $u\in X^\alpha$ we have $F_\lambda(u)\in X$. Indeed, we have $b_i\in C(\overline{\Omega})$, $\lambda'(u)\in C(\overline{\Omega})$ and $\frac{\partial u}{\partial x_i}\in X$ and, in consequence,

$$\mathbf{b}\cdot\nabla(\lambda(u)) = \sum_{i=1}^n b_i\frac{\partial(\lambda(u))}{\partial x_i} = \sum_{i=1}^n b_i\frac{\partial u}{\partial x_i}\lambda'(u) = \lambda'(u)\mathbf{b}\cdot\nabla u\in X.$$

Set $\|\mathbf{b}\| = \sum_{i=1}^n \sup_\Omega |b_i|$, fix a bounded subset B of X^α and let $\phi,\psi\in B$. We have

$$\begin{gathered}\|F_\lambda(\phi)-F_\lambda(\psi)\|_X \leq \|(\lambda'(\phi)-\lambda'(\psi))\mathbf{b}\cdot\nabla\phi\|_X + \\ + \|\lambda'(\psi)\mathbf{b}\cdot\nabla(\phi-\psi)\|_X \leq c_1\|\mathbf{b}\|\left(\|\lambda'(\phi)-\lambda'(\psi)\|_{C(\overline{\Omega})}\|\phi\|_{W^{1,p}(\Omega)} + \right. \\ \left. + \|\lambda'(\psi)\|_{C(\overline{\Omega})}\|\phi-\psi\|_{W^{1,p}(\Omega)}\right) \leq L_{\lambda,B,\mathbf{b}}\|\phi-\psi\|_{X^\alpha},\end{gathered}$$

since λ' is globally Lipschitz continuous on compact subsets of $\mathbb{R}$. This shows that $F_\lambda\colon X^\alpha\to X$ is Lipschitz continuous on bounded subsets of X^α. As a consequence, there corresponds to any $u_0 \in X^\alpha$, $\alpha \in (\max(\frac{1}{2}, \frac{n}{2p}), 1)$, a unique local X^α solution $u_\lambda(\cdot, u_0)$ of (3.4) defined on the maximal interval of existence $[0, \tau_{u_0})$.

To prove the local X^α solutions exist globally in time we need an additional a priori estimate of the solutions in an auxiliary Banach space Y. We shall choose $Y = L^\infty(\Omega)$. Let us denote $v(t) = u_\lambda(t, u_0)$, $t \in [0, \tau_{u_0})$. We know that $v_t(t) \in X$, $v(t) \in W^{2,p}(\Omega)$, $\frac{\partial v}{\partial N}(t) = 0$ on $\partial\Omega$ for $0 < t < \tau_{u_0}$ and in X we have

$$v_t(t) - \triangle v(t) = \mathbf{b} \cdot \nabla(\lambda(v(t))),\ 0 < t < \tau_{u_0}. \tag{3.5}$$

Fix any $k \in \mathbb{N}$ and $0 < t < \tau_{u_0}$. Multiplying (3.5) by v^{2k-1} and integrating over Ω we obtain

$$\int_\Omega v_t v^{2k-1} dx = \int_\Omega \triangle v v^{2k-1} dx + \int_\Omega \mathbf{b} \cdot \nabla(\lambda(v)) v^{2k-1} dx.$$

Letting $g_k(s) = \int_0^s \lambda'(z) z^{2k-1} dz$, $s \in \mathbb{R}$, we have $g_k \in C^1(\mathbb{R})$ and by (3.2)–(3.3)

$$\begin{aligned} \int_\Omega \mathbf{b} \cdot \nabla(\lambda(v)) v^{2k-1} dx &= \int_\Omega \sum_{i=1}^n b_i \frac{\partial v}{\partial x_i} \lambda'(v) v^{2k-1} dx = \int_\Omega \sum_{i=1}^n b_i \frac{\partial g_k(v)}{\partial x_i} dx = \\ &= \int_{\partial\Omega} \sum_{i=1}^n b_i N_i g_k(v) dS - \int_\Omega \sum_{i=1}^n \frac{\partial b_i}{\partial x_i} g_k(v) dx = 0. \end{aligned} \tag{3.6}$$

Hence, because of the boundary condition,

$$\frac{1}{2k} \frac{d}{dt} \int_\Omega v^{2k} dx = \int_\Omega \triangle v v^{2k-1} dx = -(2k-1) \int_\Omega \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \left(v^{k-1}\right)^2 dx \leq 0. \tag{3.7}$$

Consequently, $[0, \tau_{u_0}) \ni t \mapsto \|v(t)\|^{2k}_{L^{2k}(\Omega)}$ is nonincreasing and

$$\|u_\lambda(t, u_0)\|_{L^{2k}(\Omega)} \leq \|u_0\|_{L^{2k}(\Omega)},\ u_0 \in X^\alpha,\ t \in [0, \tau_{u_0}).$$

Letting $k \to \infty$ (see [AD, Theorem 2.8]) we obtain

$$\forall_{u_0 \in X^\alpha} \forall_{0 \leq t < \tau_{u_0}} \|u_\lambda(t, u_0)\|_Y \leq \|u_0\|_Y \leq c_2 \|u_0\|_{X^\alpha}. \tag{3.8}$$

We now estimate for a fixed $0 \leq t < \tau_{u_0}$

$$\|F_\lambda(u_\lambda(t, u_0))\|_X \leq c_1 \|\mathbf{b}\| \sup_{|s| \leq \|u_\lambda(t,u_0)\|_{L^\infty(\Omega)}} |\lambda'(s)| \, \|u_\lambda(t, u_0)\|_{W^{1,p}(\Omega)}. \tag{3.9}$$

Fix $\max(\frac{1}{2}, \frac{n}{2p}) < \beta < \alpha$. The moments inequality (cf. [KR, Theorem I.5.2]) yields

$$\begin{aligned} \|u_\lambda(t, u_0)\|_{W^{1,p}(\Omega)} &\leq c_4 \|u_\lambda(t, u_0)\|_{X^\beta} \leq c_3 c_4 \|u_\lambda(t, u_0)\|_{X^\alpha}^{\frac{\beta}{\alpha}} \|u_\lambda(t, u_0)\|_X^{1-\frac{\beta}{\alpha}} \leq \\ &\leq c_5 \|u_\lambda(t, u_0)\|_Y^{1-\frac{\beta}{\alpha}} \left(1 + \|u_\lambda(t, u_0)\|_{X^\alpha}^{\frac{\beta}{\alpha}}\right), \end{aligned}$$

since $Y = L^\infty(\Omega) \subset X$. Combining this with (3.9) we conclude that

$$\begin{aligned} &\|F_\lambda(u_\lambda(t, u_0))\|_X \leq \\ &\leq c_6 \|\mathbf{b}\| \left(1 + \|u_\lambda(t, u_0)\|_{X^\alpha}^{\frac{\beta}{\alpha}}\right) \sup_{|s| \leq \|u_\lambda(t,u_0)\|_Y} |\lambda'(s)| \, \|u_\lambda(t, u_0)\|_Y^{1-\frac{\beta}{\alpha}}. \end{aligned} \tag{3.10}$$

The estimates (3.8) and (3.10) ensure by [C-D 2, Theorem 3.1.1] that

$$T_\lambda(t)u_0 = u_\lambda(t,u_0),\ t\geq 0,\ u_0\in X^\alpha,$$

forms a C^0 semigroup of global X^α solutions having positive orbits of bounded sets bounded. In particular, $\gamma_\lambda^+(u_0)=\{T_\lambda(t)u_0\colon t\geq 0\}$ is bounded for any $u_0\in X^\alpha$. So far we have checked that the conditions (A.1)-(A.4) are satisfied with $V=X^\alpha$.

Let us now define

$$\mathcal{L}(\phi)=\mathcal{L}_\lambda(\phi)=\|\phi\|^2_{L^2(\Omega)},\ \phi\in X^\alpha.$$

$\mathcal{L}$ is continuous, since if $\phi_m \underset{m\to\infty}{\longrightarrow} \phi$ in X^α, then $\phi_m \underset{m\to\infty}{\longrightarrow} \phi$ in $L^2(\Omega)$ and thus $\mathcal{L}(\phi_m) \underset{m\to\infty}{\longrightarrow} \mathcal{L}(\phi)$. Observe that $\mathcal{L}$ is nonincreasing along any trajectory (compare with (3.7) in case $k=1$). Assume that $\mathcal{L}(u_\lambda(t,u_0))=\mathcal{L}(u_0)$, $t\geq 0$. Then we have

$$0=\frac{d}{dt}\mathcal{L}(u_\lambda(t,u_0))=\frac{d}{dt}\|u_\lambda(t,u_0)\|^2_{L^2(\Omega)}=-2\sum_{i=1}^n\left\|\frac{\partial u_\lambda(t,u_0)}{\partial x_i}\right\|^2_{L^2(\Omega)},\ t>0,$$

(compare with (3.7) in case $k=1$). Thus

$$\forall_{t>0}\forall_{i\in\{1,\dots,n\}}\frac{\partial u_\lambda(t,u_0)}{\partial x_i}=0\ \text{ a.e. in }\Omega. \tag{3.11}$$

Since $u_\lambda(\cdot,u_0)\in C([0,\infty),W^{1,p}(\Omega))$, we see that (3.11) holds also for $t=0$, i.e.

$$\forall_{i\in\{1,\dots,n\}}\frac{\partial u_0}{\partial x_i}=0\ \text{ a.e. in }\Omega.$$

Hence $u_0(x)=const.$ for a.a. $x\in\Omega$. Obviously such u_0 is a stationary solution. Therefore, by uniqueness $u_\lambda(t,u_0)=u_0$, $t\geq 0$, which shows that (A.5) is valid. Nevertheless, we have also proved that if $u_\lambda(t,u_0)$, $t\geq 0$, is a stationary solution, then $t\mapsto\mathcal{L}(u_\lambda(t,u_0))$ is constant, so u_0 must be a constant function a.e. in Ω. Hence

$$S_\lambda=\{u_0\in X^\alpha\colon \exists_{const.\in\mathbb{R}}\ u_0(x)=const.\ \text{ for a.a. } x\in\Omega\}. \tag{3.12}$$

Note that S_λ is independent of λ.

Let us now define

$$l(\phi)=l_\lambda(\phi)=\frac{1}{|\Omega|}\int_\Omega\phi\,dx,\ \phi\in X^\alpha,$$

which is evidently one-to-one on S_λ. It is also a continuous functional on X^α, since

$$|l(u_n)-l(u)|\leq\frac{1}{|\Omega|}\|u_n-u\|_{L^1(\Omega)}\leq\frac{1}{|\Omega|}\|u_n-u\|_{X^\alpha}.$$

We are going to show that

$$l(u_\lambda(t,u_0))=l(u_0),\ t\geq 0,\ u_0\in X^\alpha. \tag{3.13}$$

Fix $u_0\in X^\alpha$ and let $v(t)=u_\lambda(t,u_0)$, $t\geq 0$. Then we have

$$\frac{d}{dt}\int_\Omega v(t)dx=\int_\Omega v_t(t)dx=\int_\Omega\triangle v(t)dx=\int_{\partial\Omega}\sum_{i=1}^n\frac{\partial v(t)}{\partial x_i}N_i dS=0,\ t>0,$$

where we used (3.2), (3.3) and the boundary condition for v. This shows that $l(u_\lambda(t,u_0))=c\in\mathbb{R}$ for $t>0$. Letting $t\to 0$ we obtain (3.13).

What is left to show is the existence of an appropriate closed and positively $\{T_\lambda(t)\}$–invariant subset V of X^α such that $S_\lambda \cap V$ is bounded in V. Fix $r > 0$ and set $V_r = \{u \in X^\alpha\colon |l(u)| \leq r\}$. Since $|l|$ is continuous on X^α, V_r is a closed subset of X^α. Moreover, V_r is positively $\{T_\lambda(t)\}$–invariant, since we have shown, in particular, that $l(T_\lambda(t)u_0) = l(u_0)$, $t \geq 0$, $u_0 \in V_r$. Therefore, if $u_0 \in V_r$, then $|l(T_\lambda(t)u_0)| = |l(u_0)| \leq r$, $t \geq 0$, which shows that for each $\lambda \in \Lambda$ we have $T_\lambda(t)V_r \subset V_r$ for all $t \geq 0$. We should yet prove that there exists $R > 0$ such that $S_\lambda \cap V_r \subset B_{V_r}(0, R)$, but this is obvious, since $S_\lambda \cap V_r$ consists of constant functions equibounded by r.

Therefore all assumptions of Theorem 2.6 are satisfied. According to Remark 2.7, we conclude that for any $\mu \in (0, \infty)$ and any $\lambda \in C^{1+Lip}(\mathbb{R})$ there exists exactly the same attractor for the problem

$$\begin{cases} u_t(t,x) - \mu \triangle u(t,x) = \mathbf{b}(x) \cdot \nabla(\lambda(u(t,x))),\ t > 0,\ x \in \Omega, \\ \frac{\partial u}{\partial N}(t,x) = 0,\ t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x),\ x \in \Omega, \end{cases} \tag{3.14}$$

in V_r, which appears to consist only of all constant functions such that the absolute value of the constant does not exceed $r > 0$.

Note that we can choose $\lambda = 0$ and $\mu = 1$ in particular. Therefore the dynamics of the problem (3.14) in V_r with any $\mu \in (0, \infty)$ and any $\lambda \in C^{1+Lip}(\mathbb{R})$ is exactly the same as the dynamics of the Neumann problem for the heat equation.

Remark 3.2. In the above considerations we have chosen as the phase space $L^p(\Omega)$ with $p > \frac{n}{2}$. Nevertheless, we need not to be so restrictive. Assuming that $\frac{n}{3} < p \leq \frac{n}{2}$ and $p > 1$, we can still prove the coincidence of attractors for the problem (3.14) in the subspace of X^α with $\frac{n+p}{4p} < \alpha < 1$ for any $\mu \in (0, \infty)$ and for all functions $\lambda \in C^1(\mathbb{R})$ satisfying

$$\exists_{c>0} \forall_{s,\overline{s} \in \mathbb{R}}\ |\lambda'(s) - \lambda'(\overline{s})| \leq c\,|s - \overline{s}| \left(|s|^{r-1} + |\overline{s}|^{r-1} + 1\right), \tag{3.15}$$

where $1 \leq r < \frac{p(2\alpha-1)}{n-2\alpha p}$. This is achieved by the similar argument as in the example, while the lack of embedding $X^\alpha \subset C(\overline{\Omega})$ is substituted by the embedding of X^α into an appropriate Lebesgue space. Moreover, the a priori estimate (3.8) as well as the subordination condition (3.10) are also obtained in $Y = L^q(\Omega)$ with properly chosen q.

Example 3.3. As a second example we will consider the *Cahn-Hilliard system* known as a phase separation model in the decomposition of a multicomponent alloy (cf. e.g. [C-D 1], [L-Z] and the references therein):

$$\begin{cases} \mathbf{u}_t(t,x) = -\triangle\left[\Gamma \triangle \mathbf{u}(t,x) - \nabla_{\mathbf{u}} \lambda(\mathbf{u}(t,x))\right],\ t > 0,\ x \in \Omega, \\ \nabla \mathbf{u}(t,x) N(x) = \nabla(\Delta \mathbf{u}(t,x)) N(x) = \mathbf{0},\ t > 0,\ x \in \partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{u_0}(x),\ x \in \Omega, \end{cases} \tag{3.16}$$

where $\mathbf{u}\colon [0,\infty) \times \overline{\Omega} \to \mathbb{R}^m$, $\mathbf{u}^T = (u_1, \ldots, u_m)$, $\Gamma = [\Gamma_{ij}] \in \mathbb{R}^{m \times m}$ is a symmetric and positive definite matrix, i.e.

$$\exists_{c_0>0} \forall_{\mathbf{a} \in \mathbb{R}^m}\ \mathbf{a}^T \Gamma \mathbf{a} \geq c_0 \left|\mathbf{a}\right|^2. \tag{3.17}$$

Furthermore, Ω is a bounded domain in $\mathbb{R}^n$, where $n \leq 3$, having $C^{4+\varepsilon}$ regular boundary $\partial\Omega$. Here $\nabla \mathbf{u} = \left[\frac{\partial u^i}{\partial x_j}\right]$ is a gradient $m \times n$ matrix, while N denotes an outward normal vector to $\partial\Omega$. We assume that $\lambda \in \Lambda$, where Λ denotes the set of all functions satisfying the following conditions:

(B.1) $\lambda \in C^{3+Lip}(\mathbb{R}^m)$,

(B.2) λ is bounded below, i.e. there exists $M_\lambda > 0$ such that

$$\forall_{\mathbf{u}\in\mathbb{R}^m}\ \lambda(\mathbf{u}) \geq -M_\lambda,$$

(B.3) λ is semiconvex, i.e. there exists $N_\lambda > 0$ such that

$$\forall_{\mathbf{u}\in\mathbb{R}^m}\forall_{\mathbf{a}\in\mathbb{R}^m}\ \mathbf{a}^T\lambda''(\mathbf{u})\mathbf{a} \geq -N_\lambda\left|\mathbf{a}\right|^2,$$

(B.3') λ is convex, i.e.

$$\forall_{\mathbf{u}\in\mathbb{R}^m}\forall_{\mathbf{a}\in\mathbb{R}^m}\ \mathbf{a}^T\lambda''(\mathbf{u})\mathbf{a} \geq 0.$$

Obviously, the condition (B.3') implies (B.3), but we distinguish them, since we prove most of the required properties under the weaker (and more natural physically) condition.

Let us now introduce the following notation:

$$\mathbf{C}^k = [C^k(\overline{\Omega})]^m,\ \mathbf{L}^p = [L^p(\Omega)]^m,\ \mathbf{H}^k = [H^k(\Omega)]^m,\ \mathbf{W}^{k,p} = [W^{k,p}(\Omega)]^m,$$

where $k \in \mathbb{N} \cup \{0\}$ and $1 \leq p \leq \infty$.

We consider (3.16) as an abstract Cauchy problem

$$\begin{cases} \mathbf{u}_t(t) + A\mathbf{u}(t) = \mathbf{F}_\lambda(\mathbf{u}(t)),\ t > 0, \\ \mathbf{u}(0) = \mathbf{u_0}, \end{cases} \tag{3.18}$$

in $X = \mathbf{L}^2$, where $A = \Gamma\triangle^2$ with the domain

$$\mathrm{dom}(A) = [H^4_{\{\frac{\partial}{\partial N},\frac{\partial\triangle}{\partial N}\}}(\Omega)]^m = \mathrm{cl}_{\mathbf{H}^4}\{\mathbf{u} \in \mathbf{C}^4\colon \nabla\mathbf{u}N = \nabla(\triangle\mathbf{u})N = \mathbf{0} \text{ on } \partial\Omega\}.$$

Moreover, $\mathbf{F}_\lambda\colon \mathbf{H}^2 \to X$ is given by

$$\mathbf{F}_\lambda(\mathbf{u}) = \triangle\nabla_{\mathbf{u}}\lambda(\mathbf{u}),\ \mathbf{u} \in \mathbf{H}^2.$$

Key Sobolev embeddings and Lipschitz continuity of right hand side. From the Sobolev embeddings for $n \leq 3$ we infer that

$$\mathbf{H}^2 \subset \mathbf{C}^0 \text{ and } \mathbf{H}^2 \subset \mathbf{W}^{1,4}. \tag{3.19}$$

They guarantee that $\mathbf{F}_\lambda$ is well-defined on $\mathbf{H}^2$. Since the Sobolev embeddings (3.19) hold and λ satisfies (B.1), it follows that $\mathbf{F}_\lambda$ is Lipschitz continuous on bounded subsets of $\mathbf{H}^2$ (cf. [C-D 1, p. 280]).

Sectoriality of linear operator. It is well-known that there exists $d_0 > 0$ such that $\triangle^2 + d_0 I$ is a symmetric isomorphism of $[H^4_{\{\frac{\partial}{\partial N},\frac{\partial\triangle}{\partial N}\}}(\Omega)]^m$ onto X (cf. [TR, Theorem 5.5.1]). Since Γ is a symmetric isomorphism of X onto itself and commutes with $\triangle^2 + d_0 I$, we have

$$\langle\Gamma(\triangle^2 + d_0 I)\mathbf{u}, \mathbf{v}\rangle_X = \langle(\triangle^2 + d_0 I)\mathbf{u}, \Gamma\mathbf{v}\rangle_X = \langle\mathbf{u}, (\triangle^2 + d_0 I)\Gamma\mathbf{v}\rangle_X =$$

$$= \langle\mathbf{u}, \Gamma(\triangle^2 + d_0 I)\mathbf{v}\rangle_X,\ \mathbf{u}, \mathbf{v} \in \mathrm{dom}(A).$$

Therefore, $\Gamma\triangle^2 + d_0\Gamma$ is a symmetric operator with its range being the whole X. Hence, $\Gamma\triangle^2 + d_0\Gamma$ is a self-adjoint operator in X. Since $d_0\Gamma$ is a bounded self-adjoint operator on X, we infer that $\Gamma\triangle^2$ is self-adjoint (cf. [ML, p. 119]). Fixing any $d_1 > 0$, we see by the same argument that $\Gamma\triangle^2 + d_1 I$ is a self-adjoint operator in X. Note that

$$\langle(\Gamma\triangle^2 + d_1 I)\mathbf{u}, \mathbf{u}\rangle_X = \langle\Gamma\triangle^2\mathbf{u}, \mathbf{u}\rangle_X + d_1\|\mathbf{u}\|_X^2 = \int_\Omega (\triangle\mathbf{u})^T\Gamma(\triangle\mathbf{u})dx + d_1\|\mathbf{u}\|_X^2 \geq$$
$$\geq c_0\|\triangle\mathbf{u}\|_X^2 + d_1\|\mathbf{u}\|_X^2 \geq d_1\|\mathbf{u}\|_X^2, \ \mathbf{u} \in \mathrm{dom}(A).$$

This shows that $\Gamma\triangle^2 + d_1 I$ is a positive definite self-adjoint operator, so by [C-D 2, Proposition 1.3.3] it is a positive sectorial operator in X.

Compactness of resolvent. Observe that for some positive constants c_1, c_2 and c_3 we have

$$c_1\|\mathbf{u}\|_{\mathbf{H}^4} \leq c_2\left\|(\triangle^2 + d_0 I)\mathbf{u}\right\|_X \leq$$
$$\leq \left\|(\Gamma\triangle^2 + d_1 I + (d_0\Gamma - d_1 I))\mathbf{u}\right\|_X \leq \left\|(\Gamma\triangle^2 + d_1 I)\mathbf{u}\right\|_X + c_3\|\mathbf{u}\|_X, \ \mathbf{u} \in \mathrm{dom}(A).$$

This estimate and the compactness of embedding $\mathbf{H}^4 \subset X$ ensure that the resolvent of $\Gamma\triangle^2$ is compact.

Description of key X^α spaces. Since $\Gamma\triangle^2 + d_1 I$ is a positive definite self-adjoint operator in X, we infer from [TR, Section 1.18.10], [C-D 2, p. 50] and [C-C, Proposition 2] that

$$X^\alpha = \mathcal{D}((\Gamma\triangle^2 + d_1 I)^\alpha) = [X, \mathcal{D}(\Gamma\triangle^2 + d_1 I)]_\alpha = [[L^2(\Omega)]^m, [H^4_{\{\frac{\partial}{\partial N}, \frac{\partial\triangle}{\partial N}\}}(\Omega)]^m]_\alpha =$$
$$= [[L^2(\Omega), H^4_{\{\frac{\partial}{\partial N}, \frac{\partial\triangle}{\partial N}\}}(\Omega)]_\alpha]^m \begin{cases} = [H^2_{\{\frac{\partial}{\partial N}\}}(\Omega)]^m, & \alpha = \frac{1}{2}, \\ = [H^3_{\{\frac{\partial}{\partial N}\}}(\Omega)]^m, & \alpha = \frac{3}{4}. \end{cases}$$

Local solutions. We can from now on consider the nonlinearity $\mathbf{F}_\lambda \colon X^\alpha \to X$ for $\alpha \in [\frac{1}{2}, 1)$. Then $\mathbf{F}_\lambda$ is Lipschitz continuous on bounded subsets of X^α for $\alpha \in [\frac{1}{2}, 1)$. As a consequence to any $\mathbf{u_0} \in X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$, there corresponds a unique local X^α solution of (3.18) $\mathbf{u}_\lambda(t, \mathbf{u_0})$, $0 \leq t < \tau_{\mathbf{u_0}}$, where $\tau_{\mathbf{u_0}}$ denotes the lifetime of the solution. Therefore we have already shown that conditions (A.1), (A.2) of Theorem 2.6 are satisfied.

Global solutions and boundedness of orbits of points. We are now in a position to prove that the local solutions are in fact global ones. This problem was first solved in case λ satisfied some growth conditions (see [C-D 1]) and later those limitations were overcome in [L-Z]. In case of the Cahn-Hilliard equation (i.e. $m = 1$), the global existence of X^α solutions was shown in [C-D 2]. Here we follow this monograph to show that the same method applies to the system.

Step 1. Let us consider $\mathbf{L}^2$ as a subspace of $\mathbf{H}^* = [\big(H^1(\Omega)\big)^*]^m$. Thus for $\mathbf{u} \in \mathbf{L}^2$ we have

$$\|\mathbf{u}\|_*^2 = \sum_{k=1}^m \left(\sup\{\langle w_k, u_k\rangle_{L^2(\Omega)} \colon \|w_k\|_{H^1(\Omega)} = 1\}\right)^2,$$

where $\mathbf{u}^T = (u_1, \ldots, u_m)$. Let

$$m(\mathbf{u}) = [m(u_k)] = \left[\frac{1}{|\Omega|}\int_\Omega u_k dx\right].$$

Observe that for a fixed $\mathbf{u}\in\mathbf{L}^2$ the Neumann problem

$$\begin{cases} -\triangle\mathbf{z}(x)=\mathbf{u}(x)-m(\mathbf{u}),\ x\in\Omega,\\ \nabla\mathbf{z}(x)N(x)=\mathbf{0},\ x\in\partial\Omega,\\ m(\mathbf{z})=\mathbf{0}\end{cases}\tag{3.20}$$

possesses a unique solution $\mathbf{z}\in\mathbf{H}^2$, which we shall denote by $\mathbf{z}=\mathcal{N}(\mathbf{u})$. Let us define

$$\langle\mathbf{u_1},\mathbf{u_2}\rangle_{\mathcal{N}}=\langle\mathcal{N}(\mathbf{u_1}),\mathbf{u_2}\rangle_{\mathbf{L}^2}+|\Omega|\,m(\mathbf{u_1})\cdot m(\mathbf{u_2}),\ \mathbf{u_1},\mathbf{u_2}\in\mathbf{L}^2.$$

Note that this defines a scalar product on $\mathbf{L}^2$ and thus a norm

$$\|\mathbf{u}\|^2_{\mathcal{N}}=\langle\mathbf{u},\mathbf{u}\rangle_{\mathcal{N}},\ \mathbf{u}\in\mathbf{L}^2.$$

One can easily show that the norms $\|\cdot\|_{\mathcal{N}}$ and $\|\cdot\|_*$ are equivalent on $\mathbf{L}^2$ (cf. [GR, p. 12]). Using the Poincaré inequality we obtain

$$\|\mathbf{u}\|^2_{\mathbf{L}^2}\le c_4\sqrt{\sum_{k=1}^m\|\nabla u_k\|^2_{L^2(\Omega)}}\,\|\mathbf{u}\|_{\mathcal{N}}=c_4\,\|\nabla\mathbf{u}\|_{\mathbf{L}^2}\,\|\mathbf{u}\|_{\mathcal{N}}\tag{3.21}$$

for $\mathbf{u}\in\mathbf{H}^1$ such that $m(\mathbf{u})=\mathbf{0}$.

Step 2. Fix $\alpha\in[\frac12,1)$ and $\mathbf{u_0}\in X^\alpha$. We denote

$$\mathbf{v}(t)=\mathbf{u}_\lambda(t,\mathbf{u_0}),\ t\in[0,\tau_{\mathbf{u_0}}).\tag{3.22}$$

We know that $\mathbf{v}(t)\in\mathrm{dom}(A)$, $t\in(0,\tau_{\mathbf{u_0}})$. For $0<t<\tau_{\mathbf{u_0}}$ we estimate

$$\begin{gathered} c_0\,\|\triangle\mathbf{v}(t)\|^2_{\mathbf{L}^2}\le\int_\Omega[\triangle\mathbf{v}(t)]^T\Gamma\triangle\mathbf{v}(t)dx=\\ =-\int_\Omega[\triangle\mathbf{v}(t)]^T\left(-\Gamma\triangle\mathbf{v}(t)+\nabla_{\mathbf{v}}\lambda(\mathbf{v}(t))\right)dx+\int_\Omega[\triangle\mathbf{v}(t)]^T\nabla_{\mathbf{v}}\lambda(\mathbf{v}(t))dx.\end{gathered}\tag{3.23}$$

Integrating by parts we obtain

$$\int_\Omega[\triangle\mathbf{v}(t)]^T\nabla_{\mathbf{v}}\lambda(\mathbf{v}(t))dx=-\int_\Omega\mathrm{tr}\left((\nabla\mathbf{v}(t))^T\lambda''(\mathbf{v}(t))\nabla\mathbf{v}(t)\right)dx.\tag{3.24}$$

Moreover, setting

$$K(\mathbf{v}(t))=-\Gamma\triangle\mathbf{v}(t)+\nabla_{\mathbf{v}}\lambda(\mathbf{v}(t)),\tag{3.25}$$

we get owing to integration by parts and the regularity of $\mathbf{v}(t)$

$$\int_\Omega[\triangle\mathbf{v}(t)]^TK(\mathbf{v}(t))dx=-\int_\Omega\mathrm{tr}\left((\nabla\mathbf{v}(t))^T\nabla K(\mathbf{v}(t))\right)dx.\tag{3.26}$$

Combining (3.24) and (3.26) with (3.23), we conclude that

$$c_0\,\|\triangle\mathbf{v}(t)\|^2_{\mathbf{L}^2}\le\int_\Omega\mathrm{tr}\left((\nabla\mathbf{v}(t))^T\nabla K(\mathbf{v}(t))\right)dx-\int_\Omega\mathrm{tr}\left((\nabla\mathbf{v}(t))^T\lambda''(\mathbf{v}(t))\nabla\mathbf{v}(t)\right)dx.$$

Using assumption (B.3) and the Cauchy inequality we obtain

$$\begin{aligned} c_0 \left\|\triangle \mathbf{v}(t)\right\|_{\mathbf{L}^2}^2 \leq \frac{1}{2}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(t)))^T \nabla K(\mathbf{v}(t))\right) dx + \\ + \left(N_\lambda + \frac{1}{2}\right)\int_\Omega \operatorname{tr}\left((\nabla \mathbf{v}(t))^T \nabla \mathbf{v}(t)\right) dx. \end{aligned} \tag{3.27}$$

Step 3. Let $\mathbf{v}$ and K be given by (3.22) and (3.25). Observe that integration by parts yields for $0 < t < \tau_{\mathbf{u_0}}$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(t)))^T \nabla K(\mathbf{v}(t))\right) dx = -\int_\Omega [\triangle K(\mathbf{v}(t))]^T (K(\mathbf{v}(t)))_t dx =$$

$$= -\int_\Omega [\mathbf{v}_t(t)]^T (K(\mathbf{v}(t)))_t dx = \int_\Omega [\mathbf{v}_t(t)]^T (\Gamma \triangle \mathbf{v}(t))_t dx - \int_\Omega [\mathbf{v}_t(t)]^T (\nabla_{\mathbf{v}} \lambda(\mathbf{v}(t)))_t dx.$$

Using integration by parts again we get

$$\int_\Omega [\mathbf{v}_t(t)]^T (\Gamma \triangle \mathbf{v}(t))_t dx = -\int_\Omega \operatorname{tr}\left((\nabla \mathbf{v}_t(t))^T \Gamma \nabla \mathbf{v}_t(t)\right) dx. \tag{3.28}$$

Since we also have

$$\int_\Omega [\mathbf{v}_t(t)]^T (\nabla_{\mathbf{v}} \lambda(\mathbf{v}(t)))_t dx = \int_\Omega [\mathbf{v}_t(t)]^T \lambda''(\mathbf{v}(t)) \mathbf{v}_t(t) dx, \tag{3.29}$$

this together with (3.28) yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(t)))^T \nabla K(\mathbf{v}(t))\right) dx = \\ = -\int_\Omega \operatorname{tr}\left((\nabla \mathbf{v}_t(t))^T \Gamma \nabla \mathbf{v}_t(t)\right) dx - \int_\Omega [\mathbf{v}_t(t)]^T \lambda''(\mathbf{v}(t)) \mathbf{v}_t(t) dx. \end{aligned} \tag{3.30}$$

Using assumption (B.3) and (3.17) we conclude that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(t)))^T \nabla K(\mathbf{v}(t))\right) dx \leq -c_0 \left\|\nabla \mathbf{v}_t(t)\right\|_{\mathbf{L}^2}^2 + N_\lambda \left\|\mathbf{v}_t(t)\right\|_{\mathbf{L}^2}^2. \tag{3.31}$$

Step 4. Let $\mathbf{v}$ and K be given as in (3.22) and (3.25). Note that for $0 < t < \tau_{\mathbf{u_0}}$ the equation is satisfied and $\mathbf{v}(t) \in \operatorname{dom}(A)$, so integration by parts yields

$$m(\mathbf{v}_t(t)) = \mathbf{0}. \tag{3.32}$$

Additionally, (3.32) and the continuity of the spatial average m on $\mathbf{L}^1$ give

$$m(\mathbf{v}(t)) = m(\mathbf{v}(0)),\ 0 \leq t < \tau_{\mathbf{u_0}}. \tag{3.33}$$

Since $\mathbf{v}_t(t) \in \mathbf{H}^1$ and $m(\mathbf{v}_t(t)) = \mathbf{0}$ for $0 < t < \tau_{\mathbf{u_0}}$, it follows from (3.21) that

$$\left\|\mathbf{v}_t(t)\right\|_{\mathbf{L}^2}^2 \leq c_4 \left\|\nabla \mathbf{v}_t(t)\right\|_{\mathbf{L}^2} \left\|\mathbf{v}_t(t)\right\|_{\mathcal{N}}. \tag{3.34}$$

Thus (3.31) gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(t)))^T \nabla K(\mathbf{v}(t))\right) dx \leq$$

$$\leq -c_0 \left\|\nabla \mathbf{v}_t(t)\right\|_{\mathbf{L}^2}^2 + N_\lambda c_4 \left\|\nabla \mathbf{v}_t(t)\right\|_{\mathbf{L}^2} \left\|\mathbf{v}_t(t)\right\|_{\mathcal{N}}.$$

Using Cauchy inequality and integrating over $[\varepsilon, t]$ for some fixed sufficiently small $\varepsilon > 0$ we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(t)))^T \nabla K(\mathbf{v}(t))\right) dx \leq \\ &\leq \frac{1}{2}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(\varepsilon)))^T \nabla K(\mathbf{v}(\varepsilon))\right) dx + \frac{N_\lambda^2 c_4^2}{4c_0}\int_\varepsilon^t \|\mathbf{v}_t(s)\|_{\mathcal{N}}^2\, ds. \end{aligned} \tag{3.35}$$

Combining (3.35) with (3.27) we conclude that for $0 < \varepsilon < t < \tau_{\mathbf{u_0}}$

$$\begin{aligned} c_0 \|\triangle \mathbf{v}(t)\|_{\mathbf{L}^2}^2 &\leq \frac{1}{2}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(\varepsilon)))^T \nabla K(\mathbf{v}(\varepsilon))\right) dx + \\ &+ \frac{N_\lambda^2 c_4^2}{4c_0}\int_\varepsilon^t \|\mathbf{v}_t(s)\|_{\mathcal{N}}^2\, ds + \left(N_\lambda + \frac{1}{2}\right)\|\nabla \mathbf{v}(t)\|_{\mathbf{L}^2}^2 . \end{aligned} \tag{3.36}$$

Step 5. Let $\mathbf{v}$ and K be given by (3.22) and (3.25). Using (3.32) we compute for $0 < t < \tau_{\mathbf{u_0}}$

$$\begin{aligned} &-\|\mathbf{v}_t(t)\|_{\mathcal{N}}^2 = -\langle \mathcal{N}(\mathbf{v}_t(t)), \mathbf{v}_t(t)\rangle_{\mathbf{L}^2} = \langle K(\mathbf{v}(t)) - m(K(\mathbf{v}(t))), \mathbf{v}_t(t)\rangle_{\mathbf{L}^2} = \\ &= \langle K(\mathbf{v}(t)), \mathbf{v}_t(t)\rangle_{\mathbf{L}^2} = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega \operatorname{tr}\left((\nabla \mathbf{v}(t))^T \Gamma \nabla \mathbf{v}(t)\right) dx + \int_\Omega \lambda(\mathbf{v}(t)) dx\right), \end{aligned} \tag{3.37}$$

where we used the symmetry of Γ and integration by parts. Defining

$$\mathcal{L}_\lambda(\mathbf{u}) = \frac{1}{2}\int_\Omega \operatorname{tr}\left((\nabla(\mathbf{u}))^T \Gamma \nabla(\mathbf{u})\right) dx + \int_\Omega \lambda(\mathbf{u}) dx \tag{3.38}$$

for $\mathbf{u} \in X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$, we see that $\mathcal{L}_\lambda \colon X^\alpha \to \mathbb{R}$ is continuous and $t \mapsto \mathcal{L}_\lambda(\mathbf{u}_\lambda(t, \mathbf{u_0}))$ is nonincreasing for $0 < t < \tau_{\mathbf{u_0}}$.

Step 6. We now assume that $\alpha \in [\frac{3}{4}, 1)$, $\mathbf{u_0} \in X^\alpha$ and $\mathbf{v}$, K are given as before. We know that $\mathbf{v}(t) \in [H^3_{\{\frac{\partial}{\partial N}\}}(\Omega)]^m$, $0 \leq t < \tau_{\mathbf{u_0}}$. From (3.36) and (3.37) we get

$$\begin{aligned} c_0 \|\triangle \mathbf{v}(t)\|_{\mathbf{L}^2}^2 &\leq \frac{1}{2}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(0)))^T \nabla K(\mathbf{v}(0))\right) dx + \\ &+ \frac{N_\lambda^2 c_4^2}{4c_0}\big(\mathcal{L}_\lambda(\mathbf{v}(0)) - \mathcal{L}_\lambda(\mathbf{v}(t))\big) + \left(N_\lambda + \frac{1}{2}\right)\|\nabla \mathbf{v}(t)\|_{\mathbf{L}^2}^2,\ t \in (0, \tau_{\mathbf{u_0}}). \end{aligned} \tag{3.39}$$

We shall now show that $\mathcal{L}_\lambda(\mathbf{v}(t))$ and $\|\nabla \mathbf{v}(t)\|_{\mathbf{L}^2}^2$ are bounded. Indeed, from (3.17) and the assumption (B.2) it follows that

$$\begin{aligned} \mathcal{L}_\lambda(\mathbf{v}(0)) \geq \mathcal{L}_\lambda(\mathbf{v}(t)) &\geq \frac{1}{2} c_0 \|\nabla \mathbf{v}(t)\|_{\mathbf{L}^2}^2 + \int_\Omega \lambda(\mathbf{v}(t)) dx \geq \\ &\geq \frac{1}{2} c_0 \|\nabla \mathbf{v}(t)\|_{\mathbf{L}^2}^2 - M_\lambda |\Omega| \geq -M_\lambda |\Omega| . \end{aligned} \tag{3.40}$$

Hence from (3.39), (3.40) and (3.33) we obtain

$$\begin{aligned} \|\triangle \mathbf{v}(t)\|_{\mathbf{L}^2}^2 + (m(\mathbf{v}(t)))^2 &\leq \frac{1}{2c_0}\int_\Omega \operatorname{tr}\left((\nabla K(\mathbf{v}(0)))^T \nabla K(\mathbf{v}(0))\right) dx + \\ &+ \left[\frac{N_\lambda^2 c_4^2 + 8N_\lambda + 4}{4c_0^2}\right]\big(\mathcal{L}(\mathbf{v}(0)) + M_\lambda |\Omega|\big) + (m(\mathbf{v}(0)))^2,\ t \in (0, \tau_{\mathbf{u_0}}). \end{aligned} \tag{3.41}$$

After simple computations one can see that

$$\|\triangle \mathbf{v}(t)\|^2_{\mathbf{L}^2} + (m(\mathbf{v}(t)))^2 \le \varphi_\lambda(\|\mathbf{v}(0)\|_{\mathbf{H}^3}),\ t \in (0, \tau_{\mathbf{u_0}}),$$

where $\varphi_\lambda\colon [0,\infty) \to [0,\infty)$ is a certain nondecreasing function. However, note that $\|\mathbf{u}\|_{\mathbf{H}^2}$ is equivalent to $\left(\|\triangle \mathbf{u}\|^2_{\mathbf{L}^2} + (m(\mathbf{u}))^2 \right)^{\frac{1}{2}}$ for $\mathbf{u} \in X^{\frac{1}{2}}$ (cf. [TE, Lemma III.4.2]). In consequence, for $\alpha \in [\frac{3}{4}, 1)$ we get

$$\|\mathbf{u}_\lambda(t,\mathbf{u_0})\|_{\mathbf{H}^2} \le c_5 \left(\|\triangle \mathbf{u}_\lambda(t,\mathbf{u_0})\|^2_{\mathbf{L}^2} + (m(\mathbf{u}_\lambda(t,\mathbf{u_0})))^2 \right)^{\frac{1}{2}} \le \psi_\lambda(\|\mathbf{u_0}\|_{X^\alpha}) \tag{3.42}$$

for all $t \in (0, \tau_{\mathbf{u_0}})$, where $\psi_\lambda\colon [0,\infty) \to [0,\infty)$ is a certain nondecreasing function.

Step 7. Since $\Gamma \triangle^2 + d_1 I$ is a positive sectorial operator, we estimate

$$\begin{aligned}&\|\mathbf{F}_\lambda(\mathbf{u}_\lambda(t,\mathbf{u_0})) + d_1 \mathbf{u}_\lambda(t,\mathbf{u_0})\|_{\mathbf{L}^2} \le \\ &\le \|\triangle \nabla_{\mathbf{u}} \lambda(\mathbf{u}_\lambda(t,\mathbf{u_0}))\|_{\mathbf{L}^2} + d_1 \|\mathbf{u}_\lambda(t,\mathbf{u_0})\|_{\mathbf{H}^2},\ t \in (0,\tau_{\mathbf{u_0}}).\end{aligned}$$

Let $\mathbf{v}(t)$ denote $\mathbf{u}_\lambda(t,\mathbf{u_0})$. Then we have for $t \in (0, \tau_{\mathbf{u_0}})$

$$\begin{aligned}\|\triangle \nabla_{\mathbf{v}} \lambda(\mathbf{v}(t))\|_{\mathbf{L}^2} &\le c_6 \sum_{r=1}^m \sum_{j=1}^m \sum_{k=1}^m \left\| \sum_{i=1}^n \frac{\partial v_j(t)}{\partial x_i} \frac{\partial^3 \lambda}{\partial v_r \partial v_j \partial v_k}(\mathbf{v}(t)) \frac{\partial v_k(t)}{\partial x_i} \right\|_{L^2(\Omega)} + \\ &+ \sum_{r=1}^m \sum_{j=1}^m \left\| \frac{\partial^2 \lambda}{\partial v_r \partial v_j}(\mathbf{v}(t)) \triangle v_j(t) \right\|_{L^2(\Omega)} \le c_8 \Bigg(\max_{1 \le j,k,r \le m} \sup_{|s| \le c_7 \|\mathbf{v}(t)\|_{\mathbf{H}^2}} \left| \frac{\partial^3 \lambda}{\partial v_r \partial v_j \partial v_k}(s) \right| + \\ &+ \max_{1 \le j,r \le m} \sup_{|s| \le c_7 \|\mathbf{v}(t)\|_{\mathbf{H}^2}} \left| \frac{\partial^2 \lambda}{\partial v_r \partial v_j}(s) \right| \Bigg) \left(1 + \|\mathbf{v}(t)\|^2_{\mathbf{H}^2}\right),\end{aligned}$$

where we used the Hölder inequality and the Sobolev embeddings (3.19). Hence

$$\|\mathbf{F}_\lambda(\mathbf{u}_\lambda(t,\mathbf{u_0})) + d_1 \mathbf{u}_\lambda(t,\mathbf{u_0})\|_{\mathbf{L}^2} \le \Psi_\lambda(\|\mathbf{u}_\lambda(t,\mathbf{u_0})\|_{\mathbf{H}^2}),\ t \in (0,\tau_{\mathbf{u_0}}), \tag{3.43}$$

where $\Psi_\lambda\colon [0,\infty) \to [0,\infty)$ is a certain nondecreasing function.

It follows from (3.42), (3.43) and [C-D 2, Theorem 3.1.1] that for any $\mathbf{u_0} \in X^\alpha$ with $\alpha \in [\frac{3}{4}, 1)$ we have $\tau_{\mathbf{u_0}} = \infty$. Moreover, the global X^α solutions for $\alpha \in [\frac{3}{4}, 1)$ constitute a C^0 semigroup on X^α having orbits of bounded sets bounded. Fix now $\alpha \in [\frac{1}{2}, \frac{3}{4})$. We know that the local X^α solutions exist. Hence if $\mathbf{u_0} \in X^\alpha$, then $\mathbf{u}_\lambda(t,\mathbf{u_0})$, $0 < \varepsilon \le t < \tau_{\mathbf{u_0}}$, is an $X^{\frac{3}{4}}$ solution $\mathbf{u}_\lambda(t-\varepsilon, \mathbf{u}_\lambda(\varepsilon,\mathbf{u_0}))$. Thus $\tau_{\mathbf{u_0}} = \infty$ and the relation

$$T_\lambda(t) = \mathbf{u}_\lambda(t,\mathbf{u_0}),\ t \ge 0, \tag{3.44}$$

defines a C^0 semigroup of global X^α solutions having orbits of points bounded for $\alpha \in [\frac{1}{2}, 1)$. The above considerations establish (A.3)-(A.4) with $V = X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$.

Lyapunov functions. Observe that we have already defined in (3.38) quantities that turn out to be Lyapunov functions. We have noticed that $\mathcal{L}_\lambda\colon X^\alpha \to \mathbb{R}$ is continuous and the function $t \mapsto \mathcal{L}_\lambda(\mathbf{u}_\lambda(t,\mathbf{u_0}))$ is nonincreasing for $t > 0$ with $\mathbf{u_0} \in X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$. Now we merely mention that

$$\mathcal{L}_\lambda(\mathbf{u}_\lambda(t,\mathbf{u_0})) \equiv \mathcal{L}_\lambda(\mathbf{u_0}) \ \text{ implies } \ \mathbf{u}_\lambda(t,\mathbf{u_0}) \equiv \mathbf{u_0},$$

(see [C-D 1, Lemma 1] for more details). This and previous observations ensure that (A.5) holds with $V = X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$.

Stationary solutions under assumption (B.3'). Up to now we have made our calculations under the assumptions (B.1)-(B.3). Hereafter we are going to use the stronger assumption (B.3'). We now concentrate on finding S_λ, i.e. all stationary solutions of (3.18) in X^α, $\alpha \in [\frac{1}{2}, 1)$.

Assume that $\mathbf{w} \in \mathrm{dom}(A)$ is a stationary solution of (3.18). From (3.37) we get

$$0 = \frac{d}{dt}\big(\mathcal{L}_\lambda(\mathbf{w})\big) = - \left\|\nabla K(\mathbf{w})\right\|^2_{\mathbf{L}^2}.$$

Therefore, we obtain $\nabla K(\mathbf{w}) = \mathbf{0}$ a.e. in Ω. The well-known property of the distributional derivative implies that the function under the gradient is independent of the spatial variable, so we have

$$-\Gamma \triangle \mathbf{w} + \nabla_{\mathbf{w}} \lambda(\mathbf{w}) = \mathbf{a} \text{ a.e. in } \Omega \tag{3.45}$$

with some $\mathbf{a} \in \mathbb{R}^m$. Since integration by parts gives $m(\Gamma \triangle \mathbf{w}) = \mathbf{0}$, it follows that

$$\mathbf{a} = m(\mathbf{a}) = m(\nabla_{\mathbf{w}} \lambda(\mathbf{w})). \tag{3.46}$$

Computing the scalar product of (3.45) and $\mathbf{w}$ in $\mathbf{L}^2$ we obtain

$$\langle -\Gamma \triangle \mathbf{w}, \mathbf{w}\rangle_{\mathbf{L}^2} + \langle \nabla_{\mathbf{w}} \lambda(\mathbf{w}), \mathbf{w}\rangle_{\mathbf{L}^2} = \langle m(\nabla_{\mathbf{w}} \lambda(\mathbf{w})), \mathbf{w}\rangle_{\mathbf{L}^2}. \tag{3.47}$$

Note that integration by parts gives

$$\langle -\Gamma \triangle \mathbf{w}, \mathbf{w}\rangle_{\mathbf{L}^2} = \int_\Omega \mathrm{tr}\left((\nabla \mathbf{w})^T \Gamma \nabla \mathbf{w}\right) dx.$$

Moreover, we have $\langle m(\nabla_{\mathbf{w}} \lambda(\mathbf{w})), \mathbf{w}\rangle_{\mathbf{L}^2} = \langle \nabla_{\mathbf{w}} \lambda(\mathbf{w}), m(\mathbf{w})\rangle_{\mathbf{L}^2}$. Thus (3.47) yields

$$\int_\Omega \mathrm{tr}\left((\nabla \mathbf{w})^T \Gamma \nabla \mathbf{w}\right) dx = \langle \nabla_{\mathbf{w}} \lambda(\mathbf{w}), m(\mathbf{w}) - \mathbf{w}\rangle_{\mathbf{L}^2}. \tag{3.48}$$

Computing the scalar product of (3.45) and $\triangle \mathbf{w}$ in $\mathbf{L}^2$ we get

$$-\langle \Gamma \triangle \mathbf{w}, \triangle \mathbf{w}\rangle_{\mathbf{L}^2} + \langle \nabla_{\mathbf{w}} \lambda(\mathbf{w}), \triangle \mathbf{w}\rangle_{\mathbf{L}^2} = \langle \mathbf{a}, \triangle \mathbf{w}\rangle_{\mathbf{L}^2}. \tag{3.49}$$

Rewriting the first term in (3.49) we see that

$$\langle \Gamma \triangle \mathbf{w}, \triangle \mathbf{w}\rangle_{\mathbf{L}^2} = \int_\Omega (\triangle \mathbf{w})^T \Gamma \triangle \mathbf{w} dx.$$

Lastly, integration by parts ensures that $\langle \mathbf{a}, \triangle \mathbf{w}\rangle_{\mathbf{L}^2} = 0$ and

$$\langle \nabla_{\mathbf{w}} \lambda(\mathbf{w}), \triangle \mathbf{w}\rangle_{\mathbf{L}^2} = -\int_\Omega \mathrm{tr}\left((\nabla \mathbf{w})^T \lambda''(\mathbf{w}) \nabla \mathbf{w}\right) dx.$$

These computations lead to

$$\int_\Omega (\triangle \mathbf{w})^T \Gamma \triangle \mathbf{w} dx = -\int_\Omega \mathrm{tr}\left((\nabla \mathbf{w})^T \lambda''(\mathbf{w}) \nabla \mathbf{w}\right) dx. \tag{3.50}$$

Since Γ is positive definite and we assume (B.3') here, i.e. λ is convex, we obtain

$$c_0 \left\|\triangle \mathbf{w}\right\|^2_{\mathbf{L}^2} \leq \int_\Omega (\triangle \mathbf{w})^T \Gamma \triangle \mathbf{w} dx \leq 0.$$

Hence $\triangle \mathbf{w} = \mathbf{0}$ a.e. in Ω. Applying this to (3.45) we see that $\nabla_{\mathbf{w}} \lambda(\mathbf{w}) = \mathbf{a}$ a.e. in Ω. Now it follows from (3.48) and (3.17) that $\nabla \mathbf{w} = \mathbf{0}$ a.e. in Ω. Well-known properties of distributional derivatives guarantee that $\mathbf{w}(x) = \mathbf{c} \in \mathbb{R}^m$ for a.a. $x \in \Omega$.

Therefore, we sum up our considerations describing the set of stationary solutions as

$$S_\lambda = \{\mathbf{w} \in X^\alpha \colon \exists_{\mathbf{c}\in\mathbb{R}^m}\ \mathbf{w}(x) = \mathbf{c} \text{ for a.a. } x \in \Omega\}, \tag{3.51}$$

since every function constant almost everywhere is a stationary solution of (3.18). It is worth noticing that S_λ does not depend on λ.

Functions l_λ. Let us now define $l_\lambda \colon X^\alpha \to \mathbb{R}^m$ by

$$l_\lambda(\mathbf{u}) = m(\mathbf{u}) = \left[\frac{1}{|\Omega|}\int_\Omega u_k dx\right], \text{ where } \mathbf{u}^T = (u_1, \ldots, u_m).$$

The continuity of the spatial average m on $\mathbf{L}^1$ ensures that l_λ are continuous functions. Moreover, (3.33) yields

$$l_\lambda(\mathbf{u}_\lambda(t, \mathbf{u_0})) = l_\lambda(\mathbf{u_0}),\ t \geq 0,\ \mathbf{u_0} \in X^\alpha,$$

and, by the characterization of S_λ in (3.51), the condition (A.6) holds with $V = X^\alpha$.

Appropriate subspace $V \subset X^\alpha$. We have already shown that (A.1)-(A.6) are satisfied with $V = X^\alpha$. Nevertheless, the set of stationary solutions S_λ is unbounded in X^α, so we cannot look for a global attractor in X^α. Therefore we need to find an appropriate closed and positively $\{T_\lambda(t)\}$–invariant subset V of X^α such that $S_\lambda \cap V$ is bounded in V. In the light of our previous discussion it is clear that this will be satisfied by $V_r = \{\mathbf{u} \in X^\alpha \colon\ |m(\mathbf{u})| \leq r\}$ with $r > 0$.

Therefore all assumptions of Theorem 2.6 are satisfied. According to Remark 2.7, we conclude that for any $\mu \in (0, \infty)$ and any $\lambda \in \Lambda$ there exists exactly the same attractor for the problem

$$\begin{cases} \mathbf{u}_t(t,x) + \mu\Gamma\triangle^2\mathbf{u}(t,x) = \triangle\nabla_{\mathbf{u}}\lambda(\mathbf{u}(t,x)),\ t>0,\ x\in\Omega, \\ \nabla\mathbf{u}(t,x)N(x) = \nabla(\Delta\mathbf{u}(t,x))N(x) = \mathbf{0},\ t>0,\ x\in\partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{u_0}(x),\ x\in\Omega, \end{cases} \tag{3.52}$$

in $V_r \subset X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$, which consists only of all functions constant almost everywhere in Ω such that the absolute value of the constant does not exceed $r > 0$.

Observe that Λ contains the zero function so the dynamics of the problem (3.18) with any $\lambda \in \Lambda$ is exactly the same as the dynamics of the linear parabolic problem.

Example 3.4. We also mention that from the considerations of [C-D-T] it follows that all assumptions of Theorem 2.6 are satisfied if we consider the pseudodifferential parabolic problem

$$\begin{cases} u_t(t,x) + (-\triangle_D)^\beta u(t,x) = \mathbf{b}(x)\cdot\nabla(\lambda(u(t,x))),\ t>0,\ x\in\Omega\subset\mathbb{R}^n, \\ u(t,x) = 0,\ t>0,\ x\in\partial\Omega, \\ u(0,x) = u_0(x),\ x\in\Omega, \end{cases} \tag{3.53}$$

where $\mathbf{b}\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$ is a bounded differentiable vector field such that

$$\operatorname{div}\mathbf{b}(x) = 0,\ x\in\Omega. \tag{3.54}$$

Here we consider $\partial\Omega \in C^{2+\varepsilon}$ with $\varepsilon > 0$, $\beta \in (\frac{1}{2}, 1)$ and $\lambda \in \Lambda = C^{1+Lip}(\mathbb{R})$. Hence for any $\lambda \in \Lambda$ there exists exactly the same attractor for the problem (3.53) in the whole X^α, $\alpha \in (\frac{1}{2}, \beta)$, which consists only of the zero function. Here X^α corresponds to the operator $A = (-\triangle_D)^\beta$ considered in $X = L^p(\Omega)$ with $p > n$.

Note that Λ contains the zero function so the dynamics of the problem (3.53) with any $\lambda \in \Lambda$ is exactly the same as the dynamics of the linear parabolic problem.

References

[AD] R. A. Adams, *Sobolev Spaces*, Academic Press, New York 1975.

[AM] H. Amann, *Linear and Quasilinear Parabolic Problems, Volume I*, Birkhäuser, Basel 1995.

[C-C] A. N. Carvalho, J. W. Cholewa, Local well posedness for strongly damped wave equations with critical nonlinearities, *Bull. Austral. Math. Soc.* **66** (2002), 443–463.

[C-D 1] J. W. Cholewa, T. Dłotko, Global attractor for the Cahn-Hilliard system, *Bull. Austral. Math. Soc.* **49** (1994), 277–293.

[C-D 2] J. W. Cholewa, T. Dłotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge 2000.

[C-D-T] J. W. Cholewa, T. Dłotko, A. W. Turski, Asymptotics of pseudodifferential parabolic equations, *Demonstratio Mathematica* **35** (2002), 75–91.

[CZ] R. Czaja, *Differential Equations with Sectorial Operator*, Student's Manuals of Silesian University No. 585, Katowice 2002.

[GR] C. P. Grant, *The Dynamics of Pattern Selection for the Cahn-Hilliard Equation*, PhD Thesis, University of Utah 1991.

[HA] J. K. Hale, Attracting manifolds for evolutionary equations, GaTech, Preprint CDSNS96-257.

[HE] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin 1981.

[KR] S. G. Krein, *Linear Differential Equations in Banach Space*, AMS, Providence, R.I. 1971.

[LA] O. A. Ladyženskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge 1991.

[L-Z] De-Sheng Li, Chen-Kui Zhong, Global attractor for the Cahn-Hilliard system with fast growing nonlinearity, *J. Diff. Equations* **149** (1998), 191–210.

[LU] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Berlin 1995.

[ML] W. Mlak, *Hilbert Spaces and Operator Theory*, Kluwer (PWN), Dordrecht 1991.

[PA] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York 1983.

[R-H] P. Rybka, K.-H. Hoffmann, Convergence of solutions to Cahn-Hilliard equation, *Commun. Partial Differential Equations* **24** (1999), 1055–1077.

[S-Y] G. R. Sell, Y. You, *Dynamics of Evolutionary Equations*, Springer-Verlag, New York 2002.

[TE] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York 1998.

[TR] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Deutscher Verlag der Wissenschaften, Berlin 1978.

Institute of Mathematics, Silesian University, Bankowa 14, 40-007 Katowice, Poland

E-mail address: `rczaja@ux2.math.us.edu.pl`