Let $X$ be a Banach space. We ask whether there exists a constant $\nu(X) < +\infty$ (depending only on $X$) such that:

For any set $\Omega \neq \emptyset$, any algebra $F \subset 2^\Omega$, and any function $\nu : F \to X$ satisfying

\[ \| \nu(A \cup B) - \nu(A) - \nu(B) \| \leq 1 \]

for $A, B \in F$, $A \cap B = \emptyset$, there exists a vector measure $\mu : F \to X$ such that

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for $A \in F$. If the above condition is valid, then we say that $X$ has the SVM property.
Let $X$ be a Banach space. We ask whether there exists a constant $\nu(X) < +\infty$ (depending only on $X$) such that: for any set $\Omega \neq \emptyset$, any algebra $\mathcal{F} \subset 2^\Omega$, and any function $\nu: \mathcal{F} \to X$ satisfying
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Formulation of the problem

Let $X$ be a Banach space. We ask whether there exists a constant $\nu(X) < +\infty$ (depending only on $X$) such that: for any set $\Omega \neq \emptyset$, any algebra $\mathcal{F} \subset 2^\Omega$, and any function $\nu: \mathcal{F} \to X$ satisfying

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If the above condition is valid, then we say that $X$ has the SVM property.
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The Kalton–Roberts theorem

Theorem (Kalton & Roberts, 1983). There exists an absolute constant $K < 45$ with the following property: for any set $\Omega \neq \emptyset$, any algebra $\mathcal{A} \subset 2^\Omega$, and any function $\nu: \mathcal{A} \to \mathbb{R}$ satisfying

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for $A, B \in \mathcal{A}$, $A \cap B = \emptyset$, there exists an additive set function $\mu: \mathcal{A} \to \mathbb{R}$ such that

$$|\nu(A) - \mu(A)| \leq K$$

for $A \in \mathcal{A}$. This means, in our terminology, that the space $\mathbb{R}$ has the SVM property. As an obvious consequence, the finite-dimensional spaces $\mathbb{R}^n$, as well as the space $\ell_\infty$, also have the SVM property.
Theorem (Kalton & Roberts, 1983). There exists an absolute constant $K < 45$ with the following property: for any set $\Omega \neq \emptyset$, any algebra $\mathcal{A} \subset 2^\Omega$, and any function $\nu: \mathcal{A} \to \mathbb{R}$ satisfying

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Motivation
The Kalton–Roberts theorem

**Theorem (Kalton & Roberts, 1983).** There exists an absolute constant $K < 45$ with the following property: for any set $\Omega \neq \emptyset$, any algebra $\mathcal{A} \subset 2^\Omega$, and any function $\nu : \mathcal{A} \to \mathbb{R}$ satisfying

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This means, in our terminology, that the space $\mathbb{R}$ has the SVM property. As an obvious consequence, the finite-dimensional spaces $\mathbb{R}^n$, as well as the space $\ell_\infty$, also have the SVM property.
Let $\kappa$ be a cardinal number. We say that a Banach space $X$ has the \textbf{$\kappa$-SVM property} if and only if there exists a constant $\nu(\kappa, X) < \infty$ (depending only on $\kappa$ and $X$) such that given any algebra $\mathcal{F} \subset 2^\Omega$ of cardinality less than $\kappa$, and any map $\nu: \mathcal{F} \to X$ satisfying

$$\|\nu(A \cup B) - \nu(A) - \nu(B)\| \leq 1 \quad \text{for } A, B \in \mathcal{F}, \ A \cap B = \emptyset,$$

there exists a vector measure $\mu: \mathcal{F} \to X$ such that

$$\|\nu(A) - \mu(A)\| \leq \nu(\kappa, X) \quad \text{for } A \in \mathcal{F}.$$
SVM character

Let \( \kappa \) be a cardinal number. We say that a Banach space \( X \) has the \( \kappa \)-SVM property if and only if there exists a constant \( v(\kappa, X) < \infty \) (depending only on \( \kappa \) and \( X \)) such that given any algebra \( \mathcal{F} \subset 2^\Omega \) of cardinality less than \( \kappa \), and any map \( \nu: \mathcal{F} \to X \) satisfying

\[
\| \nu(A \cup B) - \nu(A) - \nu(B) \| \leq 1 \quad \text{for } A, B \in \mathcal{F}, \ A \cap B = \emptyset,
\]

there exists a vector measure \( \mu: \mathcal{F} \to X \) such that

\[
\| \nu(A) - \mu(A) \| \leq v(\kappa, X) \quad \text{for } A \in \mathcal{F}.
\]

If \( X \) is a Banach space which does not have the SVM property, then by the SVM character of \( X \) we mean the minimal cardinal number \( \kappa \) such that \( X \) does not have the \( \kappa \)-SVM property, and we denote it by \( \tau(X) \).
SVM character

Let $\kappa$ be a cardinal number. We say that a Banach space $X$ has the $\kappa$-SVM property if and only if there exists a constant $\nu(\kappa, X) < \infty$ (depending only on $\kappa$ and $X$) such that given any algebra $\mathcal{F} \subset 2^{\Omega}$ of cardinality less than $\kappa$, and any map $\nu: \mathcal{F} \to X$ satisfying

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there exists a vector measure $\mu: \mathcal{F} \to X$ such that

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If $X$ is a Banach space which does not have the SVM property, then by the SVM character of $X$ we mean the minimal cardinal number $\kappa$ such that $X$ does not have the $\kappa$-SVM property, and we denote it by $\tau(X)$.

Remark. Note that $\tau(X)$ is properly defined for every Banach space not enjoying the SVM property.
Let $\kappa$ be a cardinal number. We say that a Banach space $X$ has the **$\kappa$-SVM property** if and only if there exists a constant $\nu(\kappa, X) < \infty$ (depending only on $\kappa$ and $X$) such that given any algebra $\mathcal{F} \subset 2^\Omega$ of cardinality less than $\kappa$, and any map $\nu : \mathcal{F} \to X$ satisfying

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there exists a vector measure $\mu : \mathcal{F} \to X$ such that

$$\|\nu(A) - \mu(A)\| \leq \nu(\kappa, X) \quad \text{for } A \in \mathcal{F}.$$ 

If $X$ is a Banach space which does not have the SVM property, then by the **SVM character** of $X$ we mean the minimal cardinal number $\kappa$ such that $X$ does not have the $\kappa$-SVM property, and we denote it by $\tau(X)$.

**Remark.** Note that $\tau(X)$ is properly defined for every Banach space not enjoying the SVM property. (That is, if $X$ has the $\kappa$-SVM property for each cardinal number $\kappa$, then $X$ has the SVM property.)
Let \( \kappa \) be a cardinal number. We say that a Banach space \( X \) has the \( \kappa \)-SVM property if and only if there exists a constant \( v(\kappa, X) < \infty \) (depending only on \( \kappa \) and \( X \)) such that given any algebra \( \mathcal{F} \subset 2^{\Omega} \) of cardinality less than \( \kappa \), and any map \( \nu : \mathcal{F} \to X \) satisfying

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\|\nu(A \cup B) - \nu(A) - \nu(B)\| \leq 1 \quad \text{for} \ A, B \in \mathcal{F}, \ A \cap B = \emptyset,
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there exists a vector measure \( \mu : \mathcal{F} \to X \) such that

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\|\nu(A) - \mu(A)\| \leq v(\kappa, X) \quad \text{for} \ A \in \mathcal{F}.
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If \( X \) is a Banach space which does not have the SVM property, then by the SVM character of \( X \) we mean the minimal cardinal number \( \kappa \) such that \( X \) does not have the \( \kappa \)-SVM property, and we denote it by \( \tau(X) \).

By writing \( \tau(X) > \kappa \) we simply mean that \( X \) has the \( \kappa \)-SVM property.
Let us recall our basic assumption on a given function $\nu: \mathcal{F} \rightarrow X$:

\[ (*) \quad \|\nu(A \cup B) - \nu(A) - \nu(B)\| \leq 1 \quad \text{for } A, B \in \mathcal{F}, A \cap B = \emptyset \]
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- $\tau(X) \geq \omega$ for every Banach space $X$. 


Basic observations concerning the SVM character

Let us recall our basic assumption on a given function $\nu: \mathcal{F} \to X$:

\[(\ast) \quad \|\nu(A \cup B) - \nu(A) - \nu(B)\| \leq 1 \quad \text{for } A, B \in \mathcal{F}, A \cap B = \emptyset\]

- $\tau(X) \geq \omega$ for every Banach space $X$. [Proof] Let $\mathcal{F} \subset 2^\Omega$ be a finite algebra of sets and $\nu: \mathcal{F} \to X$ satisfy $(\ast)$. We may assume that $\mathcal{F} = 2^\Omega$ and let $n = |\Omega|$. By a simple induction we get the inequality

$$\left\|\nu(A) - \sum_{a \in A} \nu\{a\}\right\| \leq |A| - 1 \quad \text{for } A \in \mathcal{F},$$

thus the measure $\mu: \mathcal{F} \to X$, defined by $\mu\{a\} = \nu\{a\}$ for $a \in \Omega$, does the job. Consequently, for every Banach space $X$ we have $\tau(X) \geq \omega$ and $\nu(2^n, X) \leq n - 1$ for each $n \in \mathbb{N}$. 

Tomasz Kochanek (University of Silesia)  
Stability of vector measures...
Basic observations concerning the SVM character

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- $\tau(X) \geq \omega$ for every Banach space $X$.
- If $\tau(X) > \omega$ and $X$ is complemented in its bidual, then $X$ has the SVM property. Moreover, if there is a projection of $X^{**}$ onto $X$ with norm not exceeding $\lambda$, then $v(X) \leq \lambda v(\omega, X)$. 

Basic observations concerning the SVM character

Let us recall our basic assumption on a given function $\nu: \mathcal{F} \to X$:

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- $\tau(X) \geq \omega$ for every Banach space $X$.
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[Proof] Let $\mathcal{F}$ be an arbitrary algebra of sets and let $\Gamma$ be the set of all finite subalgebras of $\mathcal{F}$, directed by the inclusion. We use the assumption $\tau(X) > \omega$ and the compactness of the unit ball of $X^{**}$ with respect to the $w^*$-topology to produce an approximating measure with values in $X^{**}$. Next we just have to project it onto $X$. 

Tomasz Kochanek (University of Silesia) Stability of vector measures...
Basic observations concerning the SVM character

Let us recall our basic assumption on a given function \( \nu: \mathcal{F} \to X \):

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- If \( \tau(X) > \omega \) and \( X \) is complemented in its bidual, then \( X \) has the SVM property. Moreover, if there is a projection of \( X^{**} \) onto \( X \) with norm not exceeding \( \lambda \), then \( v(X) \leq \lambda v(\omega, X) \).
- \( \tau(c_0) > \omega \), i.e. \( c_0 \) satisfies the \( \omega \)-SVM property.
Let us recall our basic assumption on a given function \( \nu: F \to X \):

\((*)\) \[ \|\nu(A \cup B) - \nu(A) - \nu(B)\| \leq 1 \quad \text{for } A, B \in F, A \cap B = \emptyset \]

- \( \tau(X) \geq \omega \) for every Banach space \( X \).
- If \( \tau(X) > \omega \) and \( X \) is complemented in its bidual, then \( X \) has the SVM property. Moreover, if there is a projection of \( X^{**} \) onto \( X \) with norm not exceeding \( \lambda \), then \( v(X) \leq \lambda v(\omega, X) \).
- \( \tau(c_0) > \omega \), i.e. \( c_0 \) satisfies the \( \omega \)-SVM property. [Proof] Let \( F \) be a finite algebra. Choose any \( \varepsilon \in (0, 1) \) and pick an \( n \in \mathbb{N} \) such that \( |e_j^*(\nu(A))| < \varepsilon \) for each \( j > n \) and \( A \in F \). For each \( j \)th coordinate \( (1 \leq j \leq n) \) there is an additive set function \( \mu_j: F \to \mathbb{R} \) satisfying \( |e_j^*(\nu(A)) - \mu_j(A)| \leq K \) for \( A \in F \). Then the measure \( \mu: F \to c_0 \) defined by \( \mu(A) = (\mu_1(A), \ldots, \mu_n(A), 0, 0, \ldots) \) satisfies \( \|\nu(A) - \mu(A)\| \leq K \) for \( A \in F \). We get \( \nu(\omega, c_0) = K \).
Basic observations concerning the SVM character

Let us recall our basic assumption on a given function $\nu: \mathcal{F} \rightarrow X$:

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- $\tau(X) \geq \omega$ for every Banach space $X$.
- If $\tau(X) > \omega$ and $X$ is complemented in its bidual, then $X$ has the SVM property. Moreover, if there is a projection of $X^{**}$ onto $X$ with norm not exceeding $\lambda$, then $\nu(X) \leq \lambda \nu(\omega, X)$.
- $\tau(c_0) > \omega$, i.e. $c_0$ satisfies the $\omega$-SVM property.
- $\tau(C[0,1]) > \omega$, i.e. $C[0,1]$ satisfies the $\omega$-SVM property.
Basic observations concerning the SVM character

Let us recall our basic assumption on a given function $\nu: \mathcal{F} \to X$:

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- $\tau(c_0) > \omega$, i.e. $c_0$ satisfies the $\omega$-SVM property.
- $\tau(C[0,1]) > \omega$, i.e. $C[0,1]$ satisfies the $\omega$-SVM property.

[Proof] We use the uniform continuity of $\nu(A) \in C[0,1]$ (for $A \in \mathcal{F}$).
Basic observations concerning the SVM character

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- \( \tau(X) \geq \omega \) for every Banach space \( X \).
- If \( \tau(X) > \omega \) and \( X \) is complemented in its bidual, then \( X \) has the SVM property. Moreover, if there is a projection of \( X^{**} \) onto \( X \) with norm not exceeding \( \lambda \), then \( \nu(X) \leq \lambda \nu(\omega, X) \).
- \( \tau(c_0) > \omega \), i.e. \( c_0 \) satisfies the \( \omega \)-SVM property.
- \( \tau(C[0,1]) > \omega \), i.e. \( C[0,1] \) satisfies the \( \omega \)-SVM property.
- Is every cardinal number equal to the SVM character of some Banach space?
Basic observations concerning the SVM character

Let us recall our basic assumption on a given function
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- \( \tau(c_0) > \omega \), i.e. \( c_0 \) satisfies the \( \omega \)-SVM property.
- \( \tau(C[0,1]) > \omega \), i.e. \( C[0,1] \) satisfies the \( \omega \)-SVM property.
- Is every cardinal number equal to the SVM character of some Banach space? We shall give a partial answer to this question in what follows.
Twisted sums machinery

Exact sequences

Let $X$, $Y$, $Z$ be $F$-spaces. A short **exact sequence** is a diagram

\[(*) \quad 0 \longrightarrow Y \xrightarrow{i} Z \xrightarrow{q} X \longrightarrow 0,\]

where $i : Y \to Z$ is a one-to-one operator with a closed range (embedding) and $q : Z \to X$ is a surjective operator such that $\text{im}(i) = \text{ker}(q)$.
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In other words, $Z$ contains a closed subspace $Y_1 \simeq Y$ (isomorphically) such that the quotient space $Z/Y_1 \simeq X$. 
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In other words, $Z$ contains a closed subspace $Y_1 \simeq Y$ (isomorphically) such that the quotient space $Z/Y_1 \simeq X$. We then say that $Z$ is a **twisted sum** of $Y$ and $X$ (in this order!), or that $Z$ is an **extension** of $X$ by $Y$. 
In fact, twisted sums are identified via the following natural equivalence relation:
In fact, twisted sums are identified via the following natural equivalence relation:

We say that two exact sequences of $F$-spaces $0 \to Y \to Z_1 \to X \to 0$ and $0 \to Y \to Z_2 \to X \to 0$ are **equivalent**, if there exists an operator $T : Z_1 \to Z_2$ such that the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0 \\
| & & | & \downarrow{T} & | & & | & & | \\
0 & \longrightarrow & Y & \longrightarrow & Z_2 & \longrightarrow & X & \longrightarrow & 0
\end{array}
$$

is commutative.
For any two $F$-spaces $X$ and $Y$ we have always the trivial exact sequence:

$$(\oplus) \quad 0 \to Y \to Y \oplus X \to X \to 0$$

produced by the direct sum, jointly with the natural embedding and projection.
For any two $F$-spaces $X$ and $Y$ we have always the trivial exact sequence:

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We say that exact sequence (\ast) \textbf{splits} if and only if it is equivalent to (\oplus).
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We say that exact sequence $(\ast)$ \textbf{splits} if and only if it is equivalent to $(\oplus)$. Equivalently: the copy $i(Y)$ of $Y$, inside $Z$, is complemented in $Z$. 
For any two $F$-spaces $X$ and $Y$ we have always the trivial exact sequence:

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produced by the direct sum, jointly with the natural embedding and projection.

We say that exact sequence $(\ast)$ **splits** if and only if it is equivalent to $(\oplus)$. Equivalently: the copy $i(Y)$ of $Y$, inside $Z$, is complemented in $Z$. In such a case we must have $Z \cong X \oplus Y$. 
Now, we focus on the case where $X$ and $Y$ are Banach spaces.
Now, we focus on the case where $X$ and $Y$ are Banach spaces. The functor $\text{Ext}$ assigns, to every pair $(X, Y)$ of Banach spaces, the class of all \textbf{locally convex} twisted sums of $Y$ and $X$, modulo the equivalence relation defined earlier.
Twisted sums machinery
Functor ‘Ext’

Now, we focus on the case where $X$ and $Y$ are Banach spaces. The functor $\text{Ext}$ assigns, to every pair $(X, Y)$ of Banach spaces, the class of all \textit{locally convex} twisted sums of $Y$ and $X$, modulo the equivalence relation defined earlier.

In other words, $\text{Ext}(X, Y)$ is the class of all \textbf{Banach spaces} $Z$ (identified by the equivalence relation defined earlier) which produce an exact sequence of the form

$$0 \to Y \to Z \to X \to 0.$$
Twisted sums machinery

Functor ‘Ext’

Now, we focus on the case where $X$ and $Y$ are Banach spaces. The functor $\text{Ext}$ assigns, to every pair $(X, Y)$ of Banach spaces, the class of all \textbf{locally convex} twisted sums of $Y$ and $X$, modulo the equivalence relation defined earlier.

In other words, $\text{Ext}(X, Y)$ is the class of all \textbf{Banach spaces} $Z$ (identified by the equivalence relation defined earlier) which produce an exact sequence of the form

\[(*) \quad 0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0.\]

We write $\text{Ext}(X, Y) = 0$ if every exact sequence $(*)$, where $Z$ is a Banach space, splits.
Let $X$, $Y$ and $Z$ be $F$-spaces. Assume they form an exact sequence

$$0 \to Y \to Z \to X \to 0,$$
Let $X$, $Y$ and $Z$ be $F$-spaces. Assume they form an exact sequence

$$0 \to Y \to Z \to X \to 0,$$

and that $X$ and $Y$ satisfy some property (P). Does it imply that the “middle” space $Z$ also satisfies (P)?
History behind twisted sums
The three-space problem (3SP problem)

Let $X$, $Y$ and $Z$ be $F$-spaces. Assume they form an exact sequence

$$0 \to Y \to Z \to X \to 0,$$

and that $X$ and $Y$ satisfy some property (P). Does it imply that the “middle” space $Z$ also satisfies (P)?

In other words, we suppose that $Z$ contains a closed subspace $Y_1 \cong Y$, and such that the quotient space $Z/Y_1 \cong X$. 
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In other words, we suppose that $Z$ contains a closed subspace $Y_1 \cong Y$, and such that the quotient space $Z/Y_1 \cong X$. If both $Y_1$ and $Z/Y_1$ satisfies (P), does $Z$ also have to satisfy (P)?
History behind 3SP problem
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History behind 3SP problem
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\[ 0 \to Y \to Z \to X \to 0 \]


- \( X, Y \) are (super)reflexive \( \Rightarrow \) \( Z \) is (super)reflexive;
History behind 3SP problem
The three-space problem (3SP problem)

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- $X$, $Y$ are (super)reflexive $\Rightarrow$ $Z$ is (super)reflexive;
- $X$, $Y$ are Hilbert spaces $\not\Rightarrow$ $Z$ is (isomorphic to) a Hilbert space (!)
We say that an $F$-space $X$ is a $\mathcal{K}$-space if and only if for any other $F$-space $Z$ which gives an exact sequence $0 \to \mathbb{R} \to Z \to X \to 0$ such a sequence splits. In other words, $X$ cannot be represented as a quotient by $\mathbb{R}$ of any non-locally convex $F$-space.
We say that an $F$-space $X$ is a **$\mathcal{K}$-space** if and only if for any other $F$-space $Z$ which gives an exact sequence $0 \to \mathbb{R} \to Z \to X \to 0$ such a sequence splits. In other words, $X$ cannot be represented as a quotient by $\mathbb{R}$ of any non-locally convex $F$-space.

The question, which was of special interest in around 1977-78, was: is $\ell_1$ a $\mathcal{K}$-space, or in other words: given an exact sequence

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The negative answer was given independently by N.J. Kalton, M. Ribe and J.W. Roberts.
Let $X$, $Y$ be a quasi-normed spaces.
History behind twisted sums
Quasi-linear and zero-linear maps

Let $X$, $Y$ be a quasi-normed spaces. A homogeneous mapping $F : X \rightarrow Y$ is called quasi-linear, if it satisfies
Let $X$, $Y$ be a quasi-normed spaces. A **homogeneous** mapping $F: X \to Y$ is called **quasi-linear**, if it satisfies

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\|F(x + y) - F(x) - F(y)\| \leq c(\|x\| + \|y\|) \quad \text{for } x, y \in X,
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with some constant $c < +\infty$. It is called **zero-linear**, if it satisfies

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\left\|F\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n F(x_i)\right\| \leq C \sum_{i=1}^n \|x_i\| \quad \text{for } x_i \in X,
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History behind twisted sums
The map constructed by Kalton, Ribe and Roberts

We shall define a special quasi-linear map acting between $\ell_1$ and $\mathbb{R}$.

Put $f(t) = t \log |t|$ for $t \in \mathbb{R}$ (with the convention $0 \cdot \log 0 = 0$).

Then $|f(s + t) - f(s) - f(t)| \leq (|s| + |t|) \cdot \log 2$ for $s, t \in \mathbb{R}$.

Let $c_{00} = \{x \in \ell_1 : x_i \neq 0$ for finitely many $i \in \mathbb{N}\}$.

Define $F : c_{00} \to \mathbb{R}$ by $F(x) = \sum x_i \log |x_i| - (\sum x_i) \log \left| \sum x_i \right|$.

Then $F$ is homogeneous and for all $x, y \in c_{00}$ we have $|F(x + y) - F(x) - F(y)| \leq 2 \log 2 \cdot (\|x\| + \|y\|)$.
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Every quasi-linear map, acting on a dense subspace of some quasi-normed space, admits an extension to a quasi-linear map defined on the whole space!
History behind twisted sums
The map constructed by Kalton, Ribe and Roberts

So, we extend the quasi-linear map $F : c_{00} \to \mathbb{R}$ to a quasi-linear map $\tilde{F} : \ell_1 \to \mathbb{R}$.
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Define (algebraically):

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and equip that linear space with the quasi-norm

$$\| (t, x) \| := \| x \| + | t - \tilde{F}(x) |.$$
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This yields a quasi-normed space $Z$ which gives the exact sequence above. However, it may be checked that $Z$ would be locally convex if and only if $	ilde{F}$ was zero-linear.
So, we extend the quasi-linear map $F : c_{00} \to \mathbb{R}$ to a quasi-linear map $\tilde{F} : \ell_1 \to \mathbb{R}$.

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In fact, we have

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Therefore, we have constructed a \textbf{non-locally convex} quasi-normed space \( Z \) such that \( Z/\mathbb{R} \simeq \ell_1 \). This was possible due to the failure of the stability effect for quasi-linear maps acting between \( \ell_1 \) and \( \mathbb{R} \).
Let us recall that:

Any $F$-space $Z$ giving an exact sequence $0 \to Y \to Z \to X \to 0$ is called a **twisted sum** of $Y$ and $X$. By $\text{Ext}(X, Y)$ we denote the family of all **locally convex** twisted sums of $Y$ and $X$ (equipped with some natural equivalence relation). By $\text{Ext}(X, Y) = 0$ we mean that every locally convex twisted sum of $Y$ and $X$ is trivial, i.e. isomorphic to $Y \oplus X$. 

**Theorem (Kalton & Peck, 1979)** Every twisted sum is produced by some quasi-linear mapping. Every locally convex twisted sum is produced by some zero-linear mapping.
History behind twisted sums

The theorem of Kalton and Peck

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History behind twisted sums

Some examples

- $\ell_2 \oplus \ell_2$ does not split (Enflo, Lindenstrauss, Pisier, 1975).

- $(\ell_1, \mathbb{R})$ does not split, i.e. $\ell_1$ is not a $K$-space (Kalton, Ribe, Roberts, 1977-79).

- $\ell_p$ is a $K$-space for any $0 < p < \infty$, $p \neq 1$ (Kalton, 1977).

- $(\ell_p, \ell_p)$ always fails to split, for $0 < p < \infty$ (Kalton, Peck, 1979).

- $c_0$ and $\ell_\infty$, as well as all $L_\infty$-spaces, are $K$-spaces (Kalton, Roberts, 1983); this is why Kalton and Roberts proved their theorem on stability of nearly additive real-valued set functions!

- $\operatorname{Ext}(X, \ell_\infty) = 0$ for any Banach space $X$ (by the injectivity of $\ell_\infty$).

- $\operatorname{Ext}(X, c_0) = 0$ for every separable Banach space $X$ (Sobczyk’s theorem).
History behind twisted sums

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History behind twisted sums

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Theorem 1

If $X$ is a Banach space complemented in its bidual such that $\tau(X) > \omega$, then for every Banach space $Y$, which is an $\mathcal{L}_\infty$-space, the pair $(Y, X)$ splits.
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Theorem 2

Let \( \Gamma \) be a cardinal number. If \( X \) is a Banach space which has the \((2^{\Gamma})^+\)-SVM property (i.e. \( \tau(X) > (2^{\Gamma})^+ \)), then the pair \((\ell_\infty(\Gamma), X)\) splits;
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Corollary:

$\tau(C[0,1]) = \omega_1$.

[Proof] By a result of Cabello Sánchez, Castillo, Kalton and Yost, we have $\text{Ext}(c_0, C[0,1]) \neq 0$, so $\tau(C[0,1]) \leq \omega_1$. On the other hand, we have seen that $\tau(C[0,1]) > \omega_1$. Similarly, $\tau(C[0,\omega]) = \omega_1$. 

Tomasz Kochanek (University of Silesia)
SVM property
Necessary conditions

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Theorem 3

Let $X$ be a Banach space complemented in its bidual. Then the following assertions are equivalent:

(i) $X$ has the SVM property;
(ii) $\text{Ext}(X^*, \ell_1) = 0$;
(iii) $\text{Ext}(\ell_\infty, X^{**}) = 0$;
(iv) $\text{Ext}(c_0, X) = 0$.

Corollary 1: Commutative von Neumann algebras and order-complete $C(K)$-spaces have the SVM property (they are complemented in their biduals and they are $L_1$-preduals; by a theorem of Lindenstrauss, $\text{Ext}(L_1(\mu), \ell_1) = 0$, so condition (ii) is satisfied).
SVM property
Characterisation of the SVM property for $X \hookrightarrow X^{**}$

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[Proof] If \( \ell_p \) had SVM character greater than \( \omega \) then, as it is complemented in its bidual, it would have the SVM property. Hence, each of conditions (ii)-(iv) would follow. However, according to Cabello Sánchez and Castillo, we have \( \text{Ext}(c_0, \ell_1) \neq 0 \) and \( \text{Ext}(\ell_p, \ell_1) \neq 0 \) for each \( 1 < p < \infty \).
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Therefore, by our necessary condition, we infer that $\tau(m_0(\Gamma)) \leq \Gamma^{++}$. 
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On the other hand, using a generalisation of Sobczyk’s theorem, due to Hasanov, we may prove the following result.

Consequently, \( \tau(m_0(\Gamma)) \leq \Gamma^{++} \leq \tau(m_0(\Gamma)) \leq \Gamma^{++} \). In particular, if \( \Gamma \) is a regular cardinal, then we have \( \tau(m_0(\Gamma)) = \Gamma^{++} \).

Corollary

For every infinite cardinal \( \Gamma \) we have \( \tau(c_0(\Gamma)) = \omega_2 \).
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**Theorem 4**

Let \( \Gamma \) be an infinite cardinal number. Then the space \( m_0(\Gamma) \) has the \( \text{cf}(\Gamma)^+ \)-SVM property with \( \nu(\text{cf}(\Gamma)^+, m_0(\Gamma)) \leq 16K < 720 \).
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