Operator Exponentials on Hilbert Spaces

Christoph Schmoeger

Abstract

Let $H$ be a complex Hilbert space and let $L(H)$ be the Banach algebra of all bounded linear operators on $H$. In this paper we consider the following class of operators:

$$
\hat{\Sigma}(H) = \{ S \in L(H) : S \text{ is a scalar type operator and } \\
\sigma(S) \cap \sigma(S + 2k\pi i) \subseteq \{ k\pi i \} \text{ for } k = 1, 2, \ldots \}.
$$

The main results of this paper read as follows:

1. If $T, S \in \hat{\Sigma}(H)$ and $e^Te^S = e^Se^T$ then $T^2S^2 = S^2T^2$.
2. If $S \in \hat{\Sigma}(H), T \in L(H)$ and $e^T = e^S$ then $TS^2 = S^2T$.

Math. Subject Classification: 47A10, 47A60

Key words and phrases: scalar type operators, exponentials

1 Terminology and results

Throughout this paper let $H$ denote a complex Hilbert space and $L(H)$ the Banach algebra of all bounded linear operators on $H$. For $A \in L(H)$ the spectrum and the spectral radius of $A$ are denoted by $\sigma(A)$ and $r(A)$, respectively. The set of eigenvalues of $A$ is denoted by $\sigma_p(A)$. For the resolvent set of $A$ we write $\rho(A)$. We use $N(A)$ and $A(H)$ to denote the kernel and the range of $A$, respectively.

An operator $S \in L(H)$ is called a scalar type operator if $S$ admits a representation

$$S = \int_{\sigma(S)} \lambda E(d\lambda),$$

where $E(d\lambda)$ denotes integration with respect to a spectral measure $E(\cdot)$ on $H$. See [1], [2] and [14] for properties of spectral measures and scalar type operators.

If $A \in L(H)$ is normal ($AA^* = A^*A$) then $A$ is a scalar type operator and the values of the spectral measure of $A$ are selfadjoint projections (see [1], Theorem 7.18).

J. Wermer [14] has shown that the scalar type operators on $H$ are those operators which are similar to normal operators. More precisely, Wermer has shown that for every finite set $S_1, \ldots, S_n$ of commuting scalar type operators on $H$ there is a selfadjoint operator
$B \in \mathcal{L}(\mathcal{H})$ with a bounded everywhere defined inverse such that the operators $BS_i B^{-1}$, $i = 1, \ldots, n$, are all normal.

We write $\Sigma(\mathcal{H})$ for the class of all scalar type operators on $\mathcal{H}$. In the present paper we consider the following class of operators:

$$\tilde{\Sigma}(\mathcal{H}) = \{ S \in \Sigma(\mathcal{H}) : \sigma(S) \cap \sigma(S + 2k\pi i) \subseteq \{k\pi i\} \text{ for } k = 1, 2, \ldots \}.$$  

Now we state the main results. Proofs will be given in Section 3, in Section 4 we present some corollaries.

**Theorem 1.1** If $T \in \tilde{\Sigma}(\mathcal{H})$, $S \in \mathcal{L}(\mathcal{H})$ and $e^T e^S = e^S e^T$ then $e^ST^2 = T^2e^S$. If in addition $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \ldots \} = \emptyset$ then $e^ST = Te^S$.

**Theorem 1.2** If $T, S \in \tilde{\Sigma}(\mathcal{H})$ and $e^T e^S = e^S e^T$ then $T^2S^2 = S^2T^2$.

**Theorem 1.3** Suppose that $T, S \in \tilde{\Sigma}(\mathcal{H})$ and that $e^T e^S = e^S e^T$.

(a) If $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \ldots \} = \emptyset$ then $TS^2 = S^2T$.

(b) If $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \ldots \} = \sigma_p(S) \cap \{k\pi i : k = 1, 2, \ldots \} = \emptyset$ then $TS = ST$.

For related results concerning the equation $e^A e^B = e^B e^A$ see [10], [11], [12] and [15].

**Theorem 1.4** Suppose that $T, S \in \mathcal{L}(\mathcal{H})$, $T + S \in \tilde{\Sigma}(\mathcal{H})$ and that $e^{T+S} = e^T e^S = e^S e^T$.

If $\sigma_p(T + S) \cap \{k\pi i : k = 1, 2, \ldots \} = \emptyset$ then $TS = ST$.

**Theorem 1.5** If $S \in \tilde{\Sigma}(\mathcal{H})$, $T \in \mathcal{L}(\mathcal{H})$ and $e^T = e^S$ then $TS^2 = S^2T$.

If in addition $\sigma_p(S) \cap \{k\pi i : k = 1, 2, \ldots \} = \emptyset$ then $TS = ST$.

For related results concerning the equation $e^A = e^B$ see [3], [9] and [11].

**2 Preparations**

In this section we collect some results which we need for the proofs of the theorems in Section 1.

**Proposition 2.1** Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal.

(a) If $\mu \in \mathbb{C}$ then $(A - \mu)(\mathcal{H}) = (A^* - \overline{\mu})(\mathcal{H})$. 

2
If $B \in \mathcal{L}(\mathcal{H})$ then

$$E(\sigma(A) \cap \sigma(B))(\mathcal{H}) = \bigcap_{\lambda \in \rho(B)} (A - \lambda)(\mathcal{H}),$$

where $E(\cdot)$ denotes the spectral measure of $A$.

Proof. (a) Since $A$ is normal, $A - \mu$ is normal. Exercise 12.36 in [8] gives the result. (b) is shown in [7, Theorem 1], see also [6].

Let $A \in \mathcal{L}(\mathcal{H})$. The map $\delta_A : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}),$ defined by

$$\delta_A(C) = CA - AC \quad (C \in \mathcal{L}(\mathcal{H}))$$

is called the inner derivation determined by $A$. It is clear that $\delta_A$ is a bounded linear operator on $\mathcal{L}(\mathcal{H})$ with $\|\delta_A\| \leq 2\|A\|$.

Throughout this paper let $f$ denote the entire function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} z^{-1}(e^z - 1), & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases}$$

Let $M_A = \{\lambda \in \sigma(\delta_A) : f(\lambda) = 0\}$.

**Proposition 2.2** Let $A \in \mathcal{L}(\mathcal{H})$.

(a) If $M_A = \emptyset$, then $f(\delta_A)$ is an invertible operator on $\mathcal{L}(\mathcal{H})$.

(b) If $\lambda \in M_A$ then $\lambda$ is a simple zero of $f$ and there is $j \in \mathbb{Z} \setminus \{0\}$ with $\lambda = 2j\pi i$.

(c) $M_A$ is a finite set, $M_A \subseteq \{\pm 2\pi i, \pm 4\pi i, \ldots\}$.

(d) If $M_A \neq \emptyset$ and $M_A = \{\lambda_1, \ldots, \lambda_p\}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$ then

$$N(f(\delta_A)) = N(\delta_A - \lambda_1) \oplus \ldots \oplus N(\delta_A - \lambda_p).$$

(e) $\sigma(\delta_A) = \{\lambda - \mu : \lambda, \mu \in \sigma(A)\}$.

(f) $e^{\delta_A}(C) = e^{-A}Ce^A$ for all $C \in \mathcal{L}(\mathcal{H})$.

(g) $f(\delta_A)(\delta_A(C)) = e^{-A}Ce^A - C$ for all $C \in \mathcal{L}(\mathcal{H})$.

Proof. (a) If $M_A = \emptyset$, then $f(\lambda) \neq 0$ for all $\lambda \in \sigma(\delta_A)$, thus $f(\delta_A)$ is invertible.

(b), (c) and (d) are shown in [11].

(e) follows from [4], and Proposition 6.4.8 in [5] shows that (f) holds.

(g) follows from (f) and $zf(z) = f(z)z = e^z - 1$.  

3
Proposition 2.3 Let $A$ be a normal operator in $L(H)$ and let $E(\cdot)$ be its spectral measure. If $\lambda_0 \in \mathbb{C}$, $C \in N(\delta_A - \lambda_0)$, $D \in N(\delta_A + \lambda_0)$ then

$$C(H) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(H)$$

and

$$D^*(H) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(H).$$

Proof. From $CA - AC = \lambda_0 C$ we get $AC = C(A - \lambda_0)$. Put $B = A - \lambda_0$. Now take $\mu \in \rho(B)$. Then

$$(A - \mu)C(B - \mu)^{-1} = AC(B - \mu)^{-1} - \mu C(B - \mu)^{-1} = CB(B - \mu)^{-1} - \mu C(B - \mu)^{-1} = C(B - \mu)(B - \mu)^{-1} = C,$$

thus $C(H) \subseteq (A - \mu)(H)$. Since $\mu \in \rho(B)$ was arbitrary, we derive

$$C(H) \subseteq \bigcap_{\mu \in \rho(B)} (A - \mu)(H).$$

Proposition 2.1(b) implies now that

$$C(H) \subseteq E(\sigma(A) \cap \sigma(B))(H) = E(\sigma(A) \cap \sigma(A - \lambda_0))(H).$$

Now suppose that $D \in N(\delta_A + \lambda_0)$, hence $DA = (A - \lambda_0)D = BD$. Therefore $D^*B^* = A^*D^*$. A similar computation as above shows that for $\mu \in \rho(B^*)$ we have

$$(A^* - \mu)D^*(B^* - \mu)^{-1} = D^*,$$

thus

$$D^*(H) \subseteq \bigcap_{\mu \in \rho(B^*)} (A^* - \mu)(H).$$

Since $\rho(B^*) = \{\overline{\lambda} : \lambda \in \rho(B)\}$, we get from Proposition 2.1 that

$$D^*(H) \subseteq \bigcap_{\lambda \in \rho(B)} (A - \lambda)^*(H) = \bigcap_{\lambda \in \rho(B)} (A - \lambda)(H) = E(\sigma(A) \cap \sigma(B))(H) = E(\sigma(A) \cap \sigma(A - \lambda_0))(H).$$

The following propositions are of central importance for our investigations.

\[4\]
Proposition 2.4 Let $A$ be a normal operator in $L(H)$ and suppose that
\[
\sigma(A) \cap \sigma(A + 2k\pi i) \subseteq \{k\pi i\} \quad \text{for} \quad k = 1, 2, \ldots
\]
If $k \in \mathbb{N} \setminus \{0\}$, $C \in N(\delta_A + 2k\pi i)$ and $D \in N(\delta_A - 2k\pi i)$ then
\[
AC = k\pi iC = -CA
\]
and
\[
DA = k\pi iD = -AD.
\]

Proof. Put $\lambda_0 = -2k\pi i$. From $C \in N(\delta_A - \lambda_0)$ we get from Proposition 2.3 that
\[
C(H) \subseteq E(\sigma(A) \cap \sigma(A + 2k\pi i))(H).
\]
Since $\sigma(A) \cap \sigma(A + 2k\pi i) \subseteq \{k\pi i\}$,
\[
E(\sigma(A) \cap \sigma(A + 2k\pi i))(H) \subseteq E(\{k\pi i\}).
\]
From Theorem 12.29 in [8] it follows that $E(\{k\pi i\}) = N(A - k\pi i)$. Thus
\[
C(H) \subseteq N(A - k\pi i),
\]
hence $AC = k\pi iC$. From $CA - AC = -2k\pi iC$ we conclude that $CA = -k\pi iC = -AC$.
For $D \in N(\delta_A - 2k\pi i) = N(\delta_A + \lambda_0)$ we get from Proposition 2.3 that
\[
D^*(H) \subseteq E(\sigma(A) \cap \sigma(A + 2k\pi i))(H) \subseteq N(A - k\pi i).
\]
Thus $AD^* = k\pi iD^*$. Therefore $AD^*x = k\pi iD^*x$ for each $x \in H$. The normality of $A$ gives
\[
A^*D^*x = -k\pi iD^*x,
\]
hence $A^*D^* = -k\pi iD^*$, thus $DA = k\pi iD$. From $DA - AD = 2k\pi i$ we derive
\[
AD = DA - 2k\pi i = -k\pi iD = -DA.
\]

Proposition 2.5 Suppose that $S \in \dot{\Sigma}(H)$ and $k \in \mathbb{N} \setminus \{0\}$.

(a) If $C \in N(\delta_S + 2k\pi i)$ then $SC = k\pi iC = -CS$.

(b) If $D \in N(\delta_S - 2k\pi i)$ then $DS = k\pi iD = -SD$.

(c) If $U \in N(f(\delta_S))$ then $SU + US = 0$.

(d) If $\sigma_p(S) \cap \{n\pi i : n = 1, 2, \ldots\} = \emptyset$ then $N(f(\delta_S)) = \{0\}$.
Proof. We know that there are operators $X$ and $A$ in $\mathcal{L}(\mathcal{H})$ such that $X$ is invertible in $\mathcal{L}(\mathcal{H})$, $A$ is normal and
\[ S = X^{-1}AX. \]
Therefore we have $S - \lambda = X^{-1}(A - \lambda)X$ for each $\lambda \in \mathbb{C}$ and $\sigma(S) = \sigma(A)$ and $\sigma(S - \lambda) = \sigma(A - \lambda)$. Since $S \in \hat{\Sigma}(\mathcal{H})$, we derive that
\[ (*) \quad \sigma(A) \cap \sigma(A + 2n\pi i) \subseteq \{n\pi i\} \]
for $n = 1, 2, \ldots$. (a) From $CS - SC = -2k\pi iC$, we get
\[ CX^{-1}AX - X^{-1}AXC = -2k\pi iC, \]
therefore $(XCX^{-1})A - A(XCX^{-1}) = -2k\pi i(XCX^{-1})$. This shows that $XCX^{-1} \in N(\delta_A + 2k\pi i)$. From $(*)$ and Proposition 2.4 we see that
\[ AXCX^{-1} = k\pi iX^{-1} = -X^{-1}A, \]
hence $SC = k\pi iC = -CS$. (b) Similar. (c) Follows from (a), (b) and Proposition 2.2(d). (d) Let $n \in \mathbb{N}\setminus\{0\}$. Since $n\pi i \not\in \sigma_p(S)$, we see from (a) that $N(\delta_S + 2n\pi i) = \{0\}$. In view of Proposition 2.2(d) it remains to show that $N(\delta_S - 2n\pi i) = \{0\}$. Take $D \in N(\delta_S - 2n\pi i)$ and put $\tilde{D} = XDX^{-1}$. As in the proof of (a) we see that $\tilde{D} \in N(\delta_A - 2n\pi i)$. From Proposition 2.3 it follows that
\[ \tilde{D}^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A + 2n\pi i))(\mathcal{H}). \]
By $(*)$ we get $\tilde{D}^*(\mathcal{H}) \subseteq E(\{n\pi i\}) = N(A - n\pi i)$. Since $\sigma_p(A) = \sigma_p(S)$ and $n\pi i \not\in \sigma_p(S)$, it follows that $N(A - n\pi i) = \{0\}$. Thus $\tilde{D}^* = 0$, hence $D = 0$.  

3 Proofs 

Proof of Theorem 1.1. Use Proposition 2.2(g) to see that
\[ f(\delta_T)(\delta_T(e^S)) = e^{-T}e^Se^T - e^S = 0, \]
hence $V = \delta_T(e^S) = e^ST - Te^S \in N(f(\delta_T))$. Proposition 2.5(c) shows that
\[ 0 = TV + VT = Te^ST - T^2e^S + e^ST^2 - Te^S = e^ST^2 - T^2e^S. \]
If $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \ldots\} = \emptyset$, then by Proposition 2.5(d), $V = 0$, thus $e^ST = Te^S$. 

6
Proof of Theorem 1.2. It follows from Theorem 1.1 that $T^2e^S = e^ST^2$. By Proposition 2.2(g) we derive

$$f(\delta_S)(\delta_S(T^2)) = e^{-S}T^2e^S - T^2 = 0,$$

hence $U = \delta_S(T^2) = T^2S - ST^2 \in N(f(\delta_S))$. Proposition 2.5(c) gives now

$$0 = SU + US = ST^2 - S^2T^2 + T^2S^2 - ST^2 = T^2S^2 - S^2T^2.$$

\[ \square \]

Proof of Theorem 1.3.

(a) We know from Theorem 1.1 that $e^ST = Te^S$, thus

$$f(\delta_S)(\delta_S(T)) = e^{-S}Te^S - T = 0,$$

therefore $TS - ST \in N(f(\delta_S))$. Use again Proposition 2.5(c) to see that

$$0 = S(TS - ST) + (TS - ST)S = TS^2 - S^2T$$

(b) Proposition 2.5(d) gives $N(f(\delta_S)) = \{0\}$. Hence $TS = ST$. \[ \square \]

Proof of Theorem 1.4. Proposition 2.2(g) shows that

$$f(\delta_{T+S})(\delta_{T+S}(e^T)) = e^{-(T+S)}e^Te^{T+S} - e^T$$

therefore $U = e^T(T + S) - (T + S)e^T = e^TS - Se^T \in N(f(\delta_{T+S}))$. Since $N(f(\delta_{T+S})) = \{0\}$ (Proposition 2.5(d)), it follows that $U = 0$, hence $e^TS = Se^T$, therefore

$$f(\delta_{T+S})(\delta_{T+S}(S)) = e^{-(T+S)}Se^{T+S} - S$$

$$= e^{-S}e^{-T}Se^Te^S - S$$

$$= 0.$$ 

Hence we see that $S(T + S) - (T + S)S = ST - TS \in N(f(\delta_{T+S})) = \{0\}$. \[ \square \]

Proof of Theorem 1.5. Since

$$f(\delta_S)(\delta_S(T)) = e^{-S}Te^S - T = e^{-T}Te^T - T = 0,$$

we have $TS - ST \in N(f(\delta_S))$, thus, by Proposition 2.5(c)

$$0 = S(TS - ST) + (TS - ST)S = TS^2 - S^2T,$$

hence $TS^2 = S^2T$.

If $\sigma_p(S) \cap \{k\pi i : k = 1, 2, \ldots\} = \emptyset$, we see from Proposition 2.5(d) that $N(f(\delta_S)) = \{0\}$, thus $TS = ST$. \[ \square \]
4 Corollaries

Corollary 4.1 If $A \in \mathcal{L}(\mathcal{H})$ then

$$A \text{ is normal } \iff e^A e^{A^*} = e^{A+A^*} = e^{A^*} e^A.$$ 

Proof. The implication $\Rightarrow$ is clear.

$\Leftarrow$: Since $A + A^*$ is selfadjoint, $\sigma(A + A^*) \subseteq \mathbb{R}$. Thus $A + A^* \in \bar{\Sigma}(\mathcal{H})$ and $\sigma_p(A + A^*) \cap \{k\pi i : k = 1, 2, \ldots\} = \emptyset$. Theorem 1.4 shows now that $AA^* = A^*A$. \hfill \blacksquare

Corollary 4.2 If $A, B \in \mathcal{L}(\mathcal{H})$ are selfadjoint then

$$A = B \iff e^A = e^B.$$ 

Proof. The implication $\Rightarrow$ is clear.

$\Leftarrow$: Since $A \in \bar{\Sigma}(\mathcal{H})$ and $\sigma_p(A) \cap \{k\pi i : k = 1, 2, \ldots\}$ we see from Theorem 1.5 that $AB = BA$. Thus $A - B$ is selfadjoint and $e^{A-B} = I$. Take $\lambda \in \sigma(A - B)$. Thus $\lambda \in \mathbb{R}$ and $e^\lambda = 1$, hence $\lambda = 0$. This gives $\sigma(A - B) = \{0\}$. From $\|A - B\| = r(A - B) = 0$ we get $A = B$. \hfill \blacksquare

Corollary 4.3 Suppose that $A$ and $B$ are normal operators in $\mathcal{L}(\mathcal{H})$ and that $e^A = e^B$. Then

$$A + A^* = B + B^*.$$ 

Proof. Use Corollary 4.1 to see that $e^{A+A^*} = e^{B+B^*}$. By Corollary 4.2, $A + A^* = B + B^*$. \hfill \blacksquare

Corollary 4.4 If $A \in \mathcal{L}(\mathcal{H})$ is normal then

$$A = -A^* \iff e^A \text{ is unitary.}$$ 

Proof. The implication $\Rightarrow$ is clear.

$\Leftarrow$: Since $A$ is normal,

$$e^{A+A^*} = e^A e^{A^*} = e^A (e^{A^*})^* = I = e^0.$$ 

Now use Corollary 4.2 to derive $A + A^* = 0$. \hfill \blacksquare

For our next result we need the following lemma (see also [8, Theorem 12.37]).
Lemma 4.1 If \( T \in \mathcal{L}(\mathcal{H}) \) is invertible then there are selfadjoint operators \( A \) and \( B \) in \( \mathcal{L}(\mathcal{H}) \) such that

\[
T = e^{iA}e^{B}, \quad \sigma(A) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(A).
\]

Proof. If \( T \) is invertible, so are \( T^* \) and \( T^*T \). Theorem 12.33 in [8] shows that the positive square root \((T^*T)^{1/2}\) is also invertible. By [8, Theorem 12.35] there is a unitary \( U \in \mathcal{L}(\mathcal{H}) \) with \( T = U(T^*T)^{1/2} \). Since \( \sigma((T^*T)^{1/2}) \subseteq (0, \infty) \), log is a continuous real function on \( \sigma((T^*T)^{1/2}) \). Thus the symbolic calculus for selfadjoint operators shows that there is a selfadjoint \( B \in \mathcal{L}(\mathcal{H}) \) such that \((T^*T)^{1/2} = e^B\). A. Wintner has shown in [16] that there is a selfadjoint \( A \in \mathcal{L}(\mathcal{H}) \) such that 

\[
U = e^{iA}, \quad \sigma(A) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(A).
\]

Remarks.

1. It is shown in [13] that if \( U \in \mathcal{L}(\mathcal{H}) \) is unitary then there is a unique selfadjoint operator \( A \in \mathcal{L}(\mathcal{H}) \) such that 

\[
U = e^{iA}, \quad \sigma(A) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(A).
\]

For related results see [9].

2. Lemma 4.1 shows that an invertible operator in \( \mathcal{L}(\mathcal{H}) \) is the product of two exponentials. It is natural to ask whether every invertible operator is an exponential, rather than merely the product of two exponentials. The answer is affirmative if \( \dim \mathcal{H} < \infty \), as a consequence of [8, Theorem 10.30]. But in general the answer is negative, as one can see from [8, Theorem 12.38]. For normal and invertible operators we have the following results.

Corollary 4.5 Suppose that \( T \in \mathcal{L}(\mathcal{H}) \) is invertible. The following assertions are equivalent:

(a) \( T \) is normal.

(b) There is some normal \( S \in \mathcal{L}(\mathcal{H}) \) such that \( T = e^S \).

Proof. (b) \( \Rightarrow \) (a): Clear.

(a) \( \Rightarrow \) (b): By Lemma 4.1 there are selfadjoint operators \( A, B \in \mathcal{L}(\mathcal{H}) \) such that

\[
T = e^{iA}e^{B}
\]

and

1. \( \sigma(A) \subseteq [-\pi, \pi] \) and \( \pi \notin \sigma_p(A) \).

From \( T^* = e^{B}e^{-iA} \) and the normality of \( T \) we see that 

\[
e^{2B} = T^*T = TT^* = e^{iA}e^{2B}e^{-iA},
\]

thus
Use (1) to get
\( iA \in \hat{\Sigma}(H) \) and \( \sigma_p(iA) \cap \{ k\pi i : k = 1, 2, \ldots \} = \emptyset. \)

Since \( 2B \) is selfadjoint, we have
\( 2B \in \hat{\Sigma}(H) \) and \( \sigma_p(2B) \cap \{ k\pi i : k = 1, 2, \ldots \} = \emptyset. \)

Therefore it follows from (2), (3), (4) and Theorem 1.3(b) that \( AB = BA. \) Thus \( T = e^{iA+B}. \) Put \( S = iA + B. \) Then \( T = e^S \) and \( S \) is normal.

\[ e^{2B}e^{iA} = e^{iA}e^{2B}. \]

\textbf{Corollary 4.6} Suppose that \( T \in \mathcal{L}(H) \) is invertible and normal. Then there is a unique normal operator \( S \in \mathcal{L}(H) \) such that
\[ T = e^S, \quad r(S - S^*) \leq 2\pi \quad \text{and} \quad 2\pi i \not\in \sigma_p(S - S^*). \]

\textit{Proof.} The proof of Corollary 4.5 shows that there is a normal \( S \in \mathcal{L}(H) \) with \( T = e^S, \)
\( S = iA + B, \) where \( A \) and \( B \) are selfadjoint, \( AB = BA, \) \( \sigma(A) \subseteq [-\pi, \pi] \) and \( \pi \not\in \sigma_p(A). \)
Since \( S - S^* = 2iA, \) we get \( r(S - S^*) \leq 2\pi \) and \( 2\pi i \not\in \sigma_p(S - S^*). \) Now suppose that \( R \in \mathcal{L}(H) \) is normal, \( T = e^R, \) \( r(R - R^*) \leq 2\pi \) \( \) and \( 2\pi i \not\in \sigma_p(R - R^*). \) Then there are selfadjoint operators \( C, D \in \mathcal{L}(H) \) with
\[ R = iC + D \quad \text{and} \quad CD = DC. \]

From \( R - R^* = 2iC \) we see that
\[ \sigma(C) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \not\in \sigma_p(C). \]

It follows from \( e^S = e^R \) that \( T^* = e^B e^{-iA} = e^D e^{-iC}, \) thus \( e^{2B} = T^* T = e^{2D}. \) Now use Corollary 4.2 to derive \( B = D. \) From \( e^{iA} e^B = e^C e^D \) we see that
\[ e^{iA} = e^{iC}. \]

It is shown in [13] that then \( A = C \) (see Remark (1)). Hence \( S = T. \)

Our final result reads as follows:

\textbf{Corollary 4.7} For \( P \in \mathcal{L}(H) \) the following assertions are equivalent:
\[ \text{(a) } e^{T+P} = e^T \text{ for all } T \in \mathcal{L}(H). \]
\[ \text{(b) There is some } k \in \mathbb{Z} \text{ such that } P = 2k\pi i I. \]
Proof. (b) ⇒ (a): Clear.

(a) ⇒ (b): Take $T \in \mathcal{L}(\mathcal{H})$ with $r(T) < \pi$. Proposition 2.2(e) shows that $r(\delta_T) < 2\pi$. Thus, by Proposition 2.2(c), $M_T = \emptyset$, hence $N(f(\delta_T)) = \{0\}$ (Proposition 2.2(a)). From

$$f(\delta_T)(\delta_T(T + P)) = e^{-T}(T + P)e^T - (T + P) = e^{-(T+P)}(T + P)e^{T+P} - (T + P) = 0$$

we see that $(T + P)T = T(T + P)$, hence $TP = PT$. Therefore we have shown that $TP = PT$ for each $T \in \mathcal{L}(\mathcal{H})$ with $r(T) < \pi$.

Now take $T \in \mathcal{L}(\mathcal{H})$ with $r(T) \geq \pi$ and put $T_0 = \frac{\pi}{2r(T)} T$. Then $r(T_0) = \frac{\pi}{2}$. (5) shows that $T_0 P = PT_0$. Therefore we have that $TP = PT$ for all $T \in \mathcal{L}(\mathcal{H})$. Thus $P = \alpha I$ for some $\alpha \in \mathbb{C}$. Since $e^P = I$, $I = e^\alpha I$, hence $e^\alpha = 1$.  \[\square\]

References


Christoph Schmoeger
Mathematisches Institut I
Universität Karlsruhe
D-76128 Karlsruhe
Germany
E-mail: christoph.schmoeger@math.uni-karlsruhe.de