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Recent results in the theory of functional equations in a single variable

Karol Baron

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Abstract. In 2001 *Aequationes Mathematicae* published the survey paper *Recent results on functional equations in a single variable, perspectives and open problems* written jointly with Witold Jarczyk in which we pay attention on papers published in the last decade. I continue this review and discuss mainly papers published or written in 2000 or later.

Contents

1. General remarks. Iterative functional equations methods are connected mainly with real, complex and functional analysis. But recently some linear functional equations in a single variable have been solved with the aid of probabilistic tools. Among recent papers developing this new method are: Derfel (1989), Ger and Sablik (1998), Pittenger and Ryff (1999), Deliu and Spruill (2000), Baron and Jarczyk (M). Some results from these papers are presented in the survey paper Baron and Jarczyk (2001). Later on I will present also some results of Kapica (A).

Until quite lately the Hyers–Ulam stability was considered almost exclusively for functional equations in several variables. It seems that this changes: a lot of recent papers deal with this type of stability also for iterative functional equations. I mention here Lee and Jun (1998), where the Hyers–Ulam stability of

$$\varphi(x+\alpha) = a\varphi(x)$$

for $a \neq 1$ is proved, Jung (1998) and Alzer (1999) on the stability of the gamma equation

$$\varphi(x+1) = x\varphi(x),$$

Kim and Lee (2000), Kim (2000) on the stability of

$$\varphi(x+\alpha) = g(x)\varphi(x),$$

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Volkmann (2001) on the stability of

$$\varphi(f(x)) = 2\varphi(x),$$

Trif (2002) on

$$\varphi(f(x)) = g(x)\varphi(x) + h(x),$$

Jun, Kim and Lee (2000) and Kim (2001) concerning equations of the form

$$\varphi(x + \alpha, y + \beta) = g(x, y)\varphi(x, y).$$

However it seems that Brydak (1970) was the first who touched the problem of Hyers–Ulam stability of functional equations in a single variable.

Another new subject which appeared in the theory of iterative functional equations is initiated and elaborated by Hans–Heinrich Kairies studies of the continuous automorphism F of the Banach space of all real bounded functions defined, for fixed $\alpha \in (0, 1)$ and $\beta \in (0, \infty)$, by

$$F(\varphi)(x) = \sum_{n=0}^{\infty} \alpha^n \varphi(\beta^n x).$$

Let me only call here the rich in results papers Kairies (2000a, 2001, 2002, 2002a, A) and Kairies and Volkmann (2002), and to express my opinion that a survey talk on this subject in a future is required.

2. Iterative roots and equations with superpositions of the unknown function. The Babbage equation

$$\varphi^N(x) = x, \tag{2.1}$$

where $N > 1$ is an integer, belongs to the oldest functional equations. In the case of a real interval its continuous and strictly decreasing solutions depend on an arbitrary function and every its monotonic solution is either the identity or a decreasing involution. Jarczyk (M) showed similar effects for self–mappings of the unit circle S^1. He described all the continuous solutions and proved the following:

Theorem 2.1. *Assume $\varphi : S^1 \to S^1$ is a continuous solution of* (2.1) *and has a fixed point. If φ preserves orientation, then it is the identity. If φ reverses orientation, then it is an involution.*

A homeomorphism of a topological space is called continuously reversible if it is a composition of two continuous involutions. Reversible interval homeomorphisms were studied by Jarczyk (2002, A). He described all of them and got the following:

Theorem 2.2. *Any homeomorphism of an open interval is a composition of two continuously reversible ones.*

Equation (2.1) is a special case of

$$\sum_{n=0}^{N} a_n \varphi^n(x) = 0. \tag{2.2}$$

The general solution of (2.2) on subsets of the positive or negative half–line was found by Jarczyk (1996) in the case where $a_0 = -1$ and $a_n \geq 0$ for $n \in \{1, \dots, N\}$. His result was generalized by Tabor and Tabor (1995). An interesting comparison of this two results is contained in Bézivin (2002). (It turns out that in fact the assumptions of the theorems of Witold Jarczyk and Jacek Tabor and Józef Tabor are equivalent.) Jean–Paul Bézivin studies there also the equation

$$\varphi^N(x) = F(x, \varphi(x), \dots, \varphi^{N-1}(x)). \tag{2.3}$$

Assuming some conditions on the rational function F he shows that the general solution φ of (2.3) defined on a subset of $[0, \infty)$ is necessarily of the form

$$\varphi(x) = \frac{ax}{1+bx}.$$

Some properties of solutions $\varphi : \mathbb{R} \to \mathbb{R}$ of (2.2) are established in Matkowski and Zhang (2000).

Greenfield and Nussbaum (2001) proved the uniqueness of continuous solutions $\varphi : (0, \infty) \to (0, \infty)$ of

$$\varphi^2(x) = \varphi(x) + x^2,$$

constructed this unique solution, showed that it is real analytic, described how to extend it to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ and proved that it has no holomorphic extension to any neighbourhood of zero.

Si and Zhang (2001) proved the existence of an analytic in a neighbourhood of zero and invertible solution of equations of the form

$$\varphi^2(x) = 2\varphi(x) - x + g(\varphi(x)) + g(x).$$

Kahlig and Smítal (2001) consider continuous solutions $\varphi : (0, \infty) \to J$ of

$$\varphi(x\varphi(x)) = g(\varphi(x)), \tag{2.4}$$

where g is a given increasing homeomorphism of an open subinterval J of $(0, \infty)$. Defining $G : (0, \infty) \times J \to (0, \infty) \times J$ by

$$G(x, y) = (xy, g(y)),$$

they proved among others the following:

Theorem 2.3. *Let $\varphi : (0, \infty) \to J$ be a continuous solution of* (2.4)*. Then*

$$G(\mathrm{graph}(\varphi)) = \mathrm{graph}(\varphi)$$

and if

$$\prod_{n=1}^{\infty} \frac{g^n(y)}{g^n(z)} = \infty \qquad \textit{for} \quad y > z \textit{ in } J \tag{2.5}$$

or

$$\prod_{n=1}^{\infty} \frac{g^{-n}(y)}{g^{-n}(z)} = \infty \qquad \textit{for} \quad y > z \ \textit{in} \ J, \tag{2.6}$$

then φ increases.

The authors describe also all the continuous monotone solutions of (2.4) and conjecture that if any continuous solution of (2.4) is monotone, then either (2.5) or (2.6) holds.

I finish this part reminding the following problem posed by Brillouët–Belluot (2001):

Problem. Given $\alpha > 0$ find all the continuous solutions $\varphi : \mathbb{R} \to (0, \infty)$ of

$$\varphi^2(x) = \varphi(x + \alpha) - x.$$

3. Some linear equations. Given a probability space $(\Omega, \mathcal{A}, P)$ consider the equation

$$\varphi(x) = \int_{\Omega} \varphi \circ f(x, \cdot) dP, \tag{3.1}$$

where $f : X \times \Omega \to X$ is a given function and X is a set. Put $f^1 = f$,

$$f^{n+1}(x, \omega_1, \dots, \omega_{n+1}) = f(f^n(x, \omega_1, \dots, \omega_n), \omega_{n+1})$$

for $x \in X$ and $\omega_1, \dots, \omega_{n+1} \in \Omega$ and extend f^n on $\Omega^{\mathbb{N}}$ accepting

$$f^n(x, \omega_1, \omega_2, \dots) = f^n(x, \omega_1, \dots, \omega_n).$$

Let P^∞ denote the product measure $P \times P \times \dots$ and let $\mathcal{B}$ stand for the σ–algebra of all Borel subsets of $[0, 1]$. The following is proved by Baron and Jarczyk (M):

Theorem 3.1. *Assume $f : [0, 1] \times \Omega \to [0, 1]$ is measurable with respect to the product σ–algebra $\mathcal{B} \times \mathcal{A}$. If the function*

$$x \mapsto \int_{\Omega} f(x, \cdot) dP, \qquad x \in [0, 1],$$

is continuous and has no fixed point in $(0, 1)$, then:

(i) *for every $x \in [0, 1]$ the sequence $(f^n(x, \cdot))_{n \in \mathbb{N}}$ converges a.e. with respect to P^∞ to a function which takes the values 0 and 1 only;*

(ii) *the function $p : [0, 1] \to [0, 1]$ defined by*

$$p(x) = P^\infty \left(\lim_{n \to \infty} f^n(x, \cdot) = 0 \right)$$

is a Borel solution of (3.1)*;*

(iii) *if* $\varphi : [0,1] \to \mathbb{R}$ *is a solution of* (3.1)*, Borel, bounded, and continuous at* 0 *and* 1*, then*

$$\varphi(x) = (\varphi(0) - \varphi(1))p(x) + \varphi(1) \qquad \textit{for} \quad x \in [0,1].$$

Equation (3.1) with linear f, i.e. the equation

$$\varphi(x) = \int_\Omega \varphi(L(\omega)x + M(\omega))P(d\omega), \tag{3.2}$$

was considered by Kapica (A) on normed spaces. Adopting an idea of Derfel' (1989) he was able to get what follows.

Theorem 3.2. *Assume X is a separable normed space, the functions $L : \Omega \to (0,\infty)$ and $M : \Omega \to X$ are measurable and*

$$0 < \int_\Omega \log L(\omega)P(d\omega) < \infty, \qquad \int_\Omega \log\max\left\{\frac{\|M(\omega)\|}{L(\omega)}, 1\right\} P(d\omega) < \infty.$$

Let also a given $x^* \in X^*$ be such that

$$P(x^*M + cL = c) < 1 \qquad \textit{for every} \quad c \in \mathbb{R}. \tag{3.3}$$

Then there exists a continuous probability distribution function F such that $F \circ x^*$ is a solution of (3.2)*.*

Since every $x^* \in X^*$ satisfying (3.3) is non–zero, we have

$$(F \circ x^*)(X) = F(\mathbb{R}) \supset (0,1)$$

for every continuous distribution function F; in particular $F \circ x^*$ is non–constant.

In the case where L is constant, i.e. in the case of

$$\varphi(x) = \int_\Omega \varphi(\alpha x + M(\omega))P(d\omega), \tag{3.4}$$

Theorem 3.2 implies the following:

Corollary 3.1. *Assume X is a separable normed space and $\alpha \in (1,\infty)$. If $M : \Omega \to X$ is measurable, not concentrated at a point and*

$$\int_\Omega \log\max\{\|M(\omega)\|, 1\}P(d\omega) < \infty,$$

then (3.4) *has a continuous, bounded and non–constant solution $\varphi : X \to \mathbb{R}$.*

To complete results on (3.2) consider also the case where M is constant, i.e. the equation

$$\varphi(x) = \int_\Omega \varphi(L(\omega)x + \beta)P(d\omega). \tag{3.5}$$

Corollary 3.2. *Assume X is a separable normed space and $\beta \in X \setminus \{0\}$. If $L : \Omega \to (0, \infty)$ is measurable, not concentrated at a point and*

$$0 < \int_\Omega \log L(\omega) P(d\omega) < \infty,$$

then (3.5) *has a continuous, bounded and non–constant solution $\varphi : X \to \mathbb{R}$.*

The two–scale difference equation

$$\varphi(x) = \sum_{n=0}^{N} c_n \varphi(2x - n), \tag{3.6}$$

where $c_0, c_1, \ldots, c_N$ are given real constants with

$$c_0 \cdot c_N \neq 0,$$

arises especially in the construction of wavelets and is considered in a lot od papers. Its continuous solutions are studied in Berg and Plonka (2000), where among others the following two theorems are proved:

Theorem 3.3. *If*

$$\sum_{n=0}^{N} c_n = 2^{-k} \tag{3.7}$$

with a positive integer k, then (3.6) *has exactly one polynomial solution φ of the form*

$$\varphi(x) = x^k + \alpha_{k-1} x^{k-1} + \ldots + \alpha_1 x + \alpha_0. \tag{3.8}$$

Theorem 3.4. *If* (3.7) *holds with a positive integer k and*

$$\sum_{n=0}^{N} |c_n| < 1,$$

then (3.6) *has exactly one continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ vanishing on $(-\infty, 0]$ and coinciding with the polynomial solution* (3.8) *on $[N, \infty)$.*

The case $N = 1$, i.e. the two–coefficient dilation equation

$$\varphi(x) = c_0 \varphi(2x) + c_1 \varphi(2x - 1) \tag{3.9}$$

was considered by Morawiec (A). He described the general compactly supported solution and proved the following:

Theorem 3.5. *Assume $c_0c_1 \neq 0$. If $c_0 \neq 1$ or $c_1 \neq 1$, then the zero function is the only compactly supported solution $\varphi : \mathbb{R} \to \mathbb{R}$ of* (3.9) *which is continuous at a point of* $[0,1]$.

Concerning

$$\varphi(x) = \frac{1}{2}\varphi\left(\frac{x}{2}\right) + \frac{1}{2}\varphi\left(\frac{x+1}{2}\right) \tag{3.10}$$

we have the following result of Hilberdink (2001):

Theorem 3.6. *If a solution $\varphi : (0,1) \to \mathbb{R}$ of* (3.10) *is Riemann integrable on* $\left[\frac{1}{4}, \frac{3}{4}\right]$ *and there exist the limits*

$$c = \lim_{x\to 0^+} x\varphi(x), \qquad d = \lim_{x\to 0^+} x\varphi(1-x),$$

then $d = -c$ and

$$\varphi(x) = c\pi\cot\pi x + \varphi\left(\frac{1}{2}\right) \qquad \text{for} \quad x \in (0,1).$$

Given positive real constants a, b, α, β, Davis and Ostaszewski (2000) investigate the behaviour at infinity of continuous solutions $\varphi : \mathbb{R} \to \mathbb{R}$ of

$$\varphi(x) = a\varphi(x+\alpha) + b\varphi(x-\beta). \tag{3.11}$$

Typical results read:

Theorem 3.7. *Equation* (3.11) *admits a non–oscillatory at infinity continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ if and only if*

$$a^\beta b^\alpha \leq \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta}}. \tag{3.12}$$

If (3.12) *holds, then there exists a real constant c such that*

$$e^{cx}, \qquad x \in \mathbb{R}, \tag{3.13}$$

satisfies (3.11).

Theorem 3.8. *If*

$$a + b < 1,$$

then (3.11) *admits a continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{x\to\infty} |\varphi(x)| = \infty. \tag{3.14}$$

If

$$a + b = 1,$$

then (3.11) *admits a continuous solution* $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfying* (3.14) *if and only if*

$$\alpha a \leq \beta b.$$

If

$$a + b > 1,$$

then (3.11) *admits a continuous solution* $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfying* (3.14) *if and only if*

$$\alpha a < \beta b$$

and (3.12) *holds.*

Assume (3.11) *admits a continuous solution* $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfying* (3.14). *If*

$$a + b \neq 1 \quad \text{or} \quad \alpha a \neq \beta b,$$

then there exists a positive constant c *such that* (3.13) *is a solution of* (3.11). *If*

$$a + b = 1 \quad \text{and} \quad \alpha a = \beta b,$$

then $\mathrm{id}_{\mathbb{R}}$ *is a solution.*

Growth of transcendental meromorphic solutions of

$$\sum_{n=0}^{N} P_n(z)\varphi(c^n z) = Q(z), \tag{3.15}$$

where $P_0, \dots, P_N, Q$ are polynomials and $0 < |c| < 1$, is studied in Bergweiler, Ishizaki and Yanagihara (2002). A theorem on asymptotic behaviour of entire solutions of

$$\varphi(2z) = (ae^{\alpha z} + P(z))\varphi(z) + be^{\beta z} + Q(z),$$

where P, Q are polynomials is contained in Derfel and Vogl (2000).

A lot of interesting results on meromorphic solutions of (3.15) can be found in the paper Heittokangas, Laine, Rieppo and Yang (2000), also in the case where the P's are complex constants:

$$\sum_{n=0}^{N} a_n\varphi(c^n z) = Q(z). \tag{3.16}$$

For instance the following three results are proved there assuming that $0 < |c| < 1$ and $a_0 a_N \neq 0$.

Theorem 3.9. *If* $Q = 0$, *then the vector space of meromorphic solutions of* (3.16) *is at most* N*–dimensional and power functions*

$$z^{k_1}, \dots, z^{k_d}$$

with $k_1, \dots, k_d \in \mathbb{Z}$ *form a base of this space.*

Theorem 3.10. *If Q is meromorphic and has exactly one non–zero pole, then every meromorphic solution of* (3.16) *has infinitely many poles.*

Theorem 3.11. *Suppose*

$$\sum_{n=0}^{N} a_n c^{nk} \neq 0 \qquad \text{for every} \quad k \in \mathbb{Z}.$$

If Q is meromorphic, then (3.16) *has exactly one meromorphic solution.*

4. Simultaneous equations. Consider first replicative functions, i.e. solutions of

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi\left(\frac{x+k}{n}\right) = a_n \varphi(x) \qquad (n \in \mathbb{N}), \tag{4.1}$$

where $(a_n)_{n\in\mathbb{N}}$ is a given number sequence. Connections between the degree of regularity of solutions φ and the magnitude of $(a_n)_{n\in\mathbb{N}}$ were studied in Kairies (2000). The following three theorems are proved there:

Theorem 4.1. *Assume $(a_n)_{n\in\mathbb{N}}$ is a non–constant real sequence satisfying*

$$a_{m\cdot n} = a_m \cdot a_n \qquad \text{for} \quad m, n \in \mathbb{N} \qquad \text{and} \qquad a_1 = 1, \tag{4.2}$$

and let $\varphi : [0,1] \to \mathbb{R}$ be a non–trivial solution of (4.1) *with $\varphi(0) = \varphi(1)$.*
If $\varphi \in L^1$, then $(a_n)_{n\in\mathbb{N}} \in c_0$.
If $p \in (1,2], \varphi \in L^p$ and $1/p + 1/q = 1$, then $(a_n)_{n\in\mathbb{N}} \in l^q$.
If φ is of finite variation, then $(n \cdot a_n)_{n\in\mathbb{N}}$ is bounded.
If φ is of finite variation and continuous, then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} |a_k| = 0.$$

If φ satisfies a Hölder condition of order $\alpha > \frac{1}{2}$, then $\left(n^{\alpha-\frac{1}{2}} \sum_{k=n}^{\infty} |a_k|\right)_{n\in\mathbb{N}}$ is bounded.

If $(a_n)_{n\in\mathbb{N}} \in l^1$ and (4.2) holds, then the functions $c, s : [0,1] \to \mathbb{R}$ defined by

$$c(x) = \sum_{n=1}^{\infty} a_n \cos 2\pi n x, \qquad s(x) = \sum_{n=1}^{\infty} a_n \sin 2\pi n x$$

are continuous solutions of (4.1).

Theorem 4.2. *Assume* (4.2).

If $(a_n)_{n\in\mathbb{N}} \in l^1$, *then every continuous solution* $\varphi : [0,1] \to \mathbb{R}$ *of* (4.1) *with* $\varphi(0) = \varphi(1)$ *is a linear combination of* c *and* s.

If $\alpha \in (0,1)$ *and* $(n^\alpha \sum_{k=n}^\infty |a_k|)_{n\in\mathbb{N}}$ *is bounded, then every continuous solution* $\varphi : [0,1] \to \mathbb{R}$ *of* (4.1) *with* $\varphi(0) = \varphi(1)$ *satisfies a Hölder condition of order* α.

If $\left(n^\beta \sum_{k=n}^\infty |a_k|\right)_{n\in\mathbb{N}}$ *is bounded for some* $\beta > 1$, *then every continuous solution* $\varphi : [0,1] \to \mathbb{R}$ *of* (4.1) *with* $\varphi(0) = \varphi(1)$ *satisfies a Lipschitz condition.*

Theorem 4.3. *Assume* (4.2), *fix* $p \in (1,2]$, *let* $(a_n)_{n\in\mathbb{N}} \in l^p$ *and* $1/p + 1/q = 1$.

(i) *If* $\varphi : [0,1] \to \mathbb{R}$ *is a solution of* (4.1) *in* L^q *and* $\varphi(0) = \varphi(1)$, *then there exist real constants* α, β *such that*

$$\varphi(x) = \alpha c(x) + \beta s(x) \qquad a.e.\ on\ [0,1]. \tag{4.3}$$

(ii) *For every real constants* α, β *there exists a solution* $\varphi : [0,1] \to \mathbb{R}$ *of* (4.1) *in* L^q *such that* $\varphi(0) = \varphi(1)$ *and* (4.3) *holds.*

Bézivin (2000) is interested in continuous solutions $\varphi : \mathbb{R} \to \mathbb{C}$ of

$$\sum_{m=0}^{M} P_m(x)\varphi(x + \alpha m) = f(x), \qquad \sum_{n=0}^{N} Q_n(x)\varphi(x + \beta n) = g(x),$$

where P's and Q's are polynomials with complex coefficients and $P_M \cdot Q_N \neq 0$, the reals α, β are linearly independent over $\mathbb{Q}$, and $f, g : \mathbb{R} \to \mathbb{C}$ are exponential polynomials. The solutions are exponential polynomials divided by a polynomial (just exponential polynomials if P's and Q's are constants).

Marteau (2000) considers the system

$$\sum_{m=1}^{M} P_m(x)\varphi(x + \alpha_m) = 0, \qquad \sum_{n=1}^{N} Q_n(x)\varphi(x + \beta_n) = 0, \tag{4.4}$$

its continuous solutions $\varphi : \mathbb{R} \to \mathbb{C}$ and entire solutions $\varphi : \mathbb{C} \to \mathbb{C}$. Among others he got the following:

Theorem 4.4. *Assume* $P_1, \ldots, P_M, Q_1, \ldots, Q_N$ *are non–zero polynomials with complex coefficients and* $\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_N$ *are complex numbers such that the intersection of the two additive subgroups of* $\mathbb{C}$ *generated by the* α*'s and by the* β*'s, respectively, contains only zero. If*

$$\begin{gathered} \mathrm{Re}\ \alpha_1 < \ldots < \mathrm{Re}\ \alpha_M, \quad \mathrm{Im}\ \alpha_1 < \ldots < \mathrm{Im}\ \alpha_M, \\ \mathrm{Re}\ \beta_1 < \ldots < \mathrm{Re}\ \beta_N, \quad \mathrm{Im}\ \beta_N < \ldots < \mathrm{Im}\ \beta_1, \end{gathered}$$

then every entire solution of (4.4) *is an exponential polynomial divided by a polynomial.*

Zdun (2001) investigates bounded solutions $\varphi : [0,1] \to X$ of

$$\varphi(f_n(x)) = g_n(\varphi(x)) \qquad (n \in \{1, \ldots, N\}), \tag{4.5}$$

where f's are continuous self–mappings of $[0,1]$ and g's are continuous self–mappings of a complete metric space X. The paper provides conditions which imply the existence, uniqueness and continuity of bounded solutions. In the case where $X = \mathbb{C}$ solutions of some particular systems of form (4.5) determine some peculiar curves generating some fractals. In the case where X is a compact interval the paper brings conditions under which the only bounded solution $\varphi : [0,1] \to X$ of (4.5) is continuous, monotonic, singular continuous, continuous and a.e. non–differentiable, respectively. Some of the results obtained in this paper were applied by the author in Zdun (2000) to prove a uniqueness theorem for the conjugacy equation on the circle.

According to Sklar (2001) many real functions can be characterized as solutions of

$$\varphi(2x) = f(\varphi(x)), \qquad \varphi(3x) = g(\varphi(x)). \tag{4.6}$$

In particular, if

$$f(x) = \frac{2x}{1+x^2}, \qquad g(x) = x\frac{3+x^2}{1+3x^2},$$

$\varphi : [0,\infty) \to \mathbb{R}$ is a solution of (4.6) and there exists a $c > 0$ such that $\varphi|_{[0,c]}$ is monotonic with $\varphi(c) = \frac{1}{2}$, then

$$\varphi(x) = \tanh\left(\frac{\log 3}{2c}x\right) \qquad \text{for} \quad x \geq 0.$$

I finish this part with an information that Ciepliński and Zdun (2002) investigate continuous and homeomorphic solutions $\varphi : S^1 \to S^1$ of

$$\varphi(f_\lambda(x)) = g_\lambda(\varphi(x)) \qquad (\lambda \in \Lambda),$$

where Λ is an arbitrary non–empty set and f_λ, g_λ are homeomorphisms of the unit circle S^1 such that

$$f_{\lambda_1} \circ f_{\lambda_2} = f_{\lambda_2} \circ f_{\lambda_1}, \qquad g_{\lambda_1} \circ g_{\lambda_2} = g_{\lambda_2} \circ g_{\lambda_1}$$

for $\lambda_1, \lambda_2 \in \Lambda$.

5. Schröder's and Böttcher's equations. The Schröder equation

$$\varphi(f(x)) = \lambda\varphi(x) \tag{5.1}$$

is the fundamental equation of linearization. A theorem on the existence of a solution $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ of (5.1) with $\lambda = f'(0)$, being a C^∞ diffeomorphism is given in Zajtz (2000).

Equation (5.1) and the inequality

$$\varphi(f(x)) \leq \lambda\varphi(x) \tag{5.2}$$

are used by Jachymski (2000) to simplify a proof of Bessaga converse of the Banach contraction principle. Concerning (5.1) and (5.2) the following is proved there:

Theorem 5.1. *If f is a self–mapping of a non–empty set X and $\lambda \in (0,1)$, then the following two conditions* (i) *and* (ii) *are equivalent:*

(i) *there exists a complete metric d on X such that*

$$d(f(x), f(z)) \leq \lambda d(x,z) \qquad \textit{for} \quad x, z \in X;$$

(ii) *there exists a solution $\varphi : X \to [0,\infty)$ of* (5.2) *such that $\varphi^{-1}(\{0\})$ is a singleton.*

Theorem 5.2. *If f is a self–mapping of a non–empty set X and $\lambda \in (0,1)$, then f has no periodic point if and only if* (5.1) *has a solution $\varphi : X \to (0,\infty)$.*

Jones (2002) studies the multivariate Böttcher equation

$$\varphi(f(x)) = \varphi(x)^p$$

for $f : \mathbb{R}^N \to \mathbb{R}^N$ being a polynomial with non–negative coefficients. A solution $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is constructed from the limit of

$$\frac{\log f^n(x)}{p^n},$$

where log is taken componentwise.

Brydak and Choczewski (2000) get a representation of positive solutions φ of

$$\varphi(f(x)) \leq \varphi(x)^p$$

such that

$$\lim_{x \to 0+} \frac{\varphi(x)}{\beta(x)} > 0$$

holds for a positive solution β of the Böttcher equation.

6. Miscellaneous results. Now I would like to present some results of more individual character.

According to the Bohr–Mollerup theorem the gamma function is the only normalized logarithmically convex solution $\varphi : (0,\infty) \to (0,\infty)$ of

$$\varphi(x+1) = x\varphi(x). \tag{6.1}$$

Hans–Heinrich Kairies proposed to investigate the set of all functions $g : (0,\infty) \to \mathbb{R}$ with the following property: If $\varphi : (0,\infty) \to (0,\infty)$ is a normalized solution of (6.1) such that $g \circ \varphi$ is convex on (α,∞) for some $\alpha \geq 0$, then $\varphi = \Gamma$. Wach–Michalik (2001) found some elements of this set and studied its property.

Domsta (2002) presents an extensive study of solutions φ regular in the sense of Karamata of a variety of equations of the form

$$\varphi(f(x)) = g(x)\varphi(x) + h(x);$$

applications to the regular iteration are also given there.

Derfel', Romanenko and Sharkovskiĭ (2000) study asymptotic behaviour of C^1 solutions $\varphi : [0, \infty) \to I$ of

$$\varphi(qx + 1) = g(\varphi(x)), \tag{6.2}$$

where I is a compact interval and g is a C^1 self–mapping of I. Among others the authors show that if there exists a positive integer m such that

$$q > \max_{x \in I} \left| \frac{d}{dx} g^m(x) \right|^{1/m},$$

then for any C^1 solution $\varphi : [0, \infty) \to I$ of (6.2) we have

$$\lim_{x \to \infty} \varphi'(x) = 0.$$

Gundersen, Heittokangas, Laine, Rieppo and Yang (2002) established the growth of meromorphic solutions of equations of the form

$$\varphi(cz) = \frac{\sum_{m=0}^{M} f_m(z)\varphi(z)^m}{\sum_{n=0}^{N} g_n(z)\varphi(z)^n}$$

with meromorphic coefficients f's and g's.

Kahlig, Matkowski and Sharkovsky (2000) revisited the Dido's equation

$$2\varphi(2x) = \varphi(x) + \sqrt{\varphi(x)^2 + \frac{1}{x^2}}. \tag{6.3}$$

Applying the method of invariants the authors proved the following:

Theorem 6.1. *If $\alpha \in (0, \infty)$, then $\varphi : (\alpha, \infty) \to (0, \infty)$ is a solution of* (6.3) *if and only if there exists a 1–periodic $p : \mathbb{R} \to (0, \infty)$ such that*

$$\varphi(x) = \frac{1}{x} \cot \left(\frac{1}{x} p \left(\frac{\log x}{\log 2} \right) \right) \qquad \text{for} \quad x \in (\alpha, \infty)$$

and

$$p(x) < \frac{\pi}{2} \cdot 2^x \qquad \text{for} \quad x \in \left(\frac{\log \alpha}{\log 2}, \frac{\log \alpha}{\log 2} + 1 \right].$$

Let me finish in calling your attention to the fact that the notion of quasimonotone increasing functions, introduced thirty years ago by Peter Volkmann and very useful in the theory of differential equations, appeared in the theory of iterative functional equations. Namely Volkmann (2002) uses it to get a comparison theorem and Herzog (M) uses it to obtain semicontinuous solutions of

$$F(x, \varphi(x), \varphi(f_1(x)), \ldots, \varphi(f_N(x))) = 0.$$

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Address of the author: Uniwersytet Śląski, Instytut Matematyki, 40–007 Katowice, Poland.
e-mail: baron@us.edu.pl