http://www.mathematik.uni-karlsruhe.de/~semlv
Seminar LV, No. 26, 5 pp., *1.6.2006*

DRAZIN INVERTIBILITY OF PRODUCTS

CHRISTOPH SCHMOEGER

Abstract. If a and b are elements of an algebra, then we show that ab is Drazin invertible if and only if ba is Drazin invertible. With this result we investigate products of bounded linear operators on Banach spaces.

1. Drazin inverses

Throughout this section $\mathcal{A}$ is a real or complex algebra with identity $e \neq 0$. We denote the group of invertible elements of $\mathcal{A}$ by $\mathcal{A}^{-1}$. We call an element $a \in \mathcal{A}$ *relatively regular* if there is $b \in \mathcal{A}$ for which $a = aba$. In this case b is called a *generalized inverse* of a.

An element $a \in \mathcal{A}$ is said to be *Drazin invertible* if there is $c \in \mathcal{A}$ and $k \in \mathbb{N} \cup \{0\}$ such that

$$a^k ca = a^k, \quad cac = c \quad \text{and} \quad ac = ca\,.$$

In this case c is called a *Drazin inverse* of a and the least non-negative integer k satisfying $a^k ca = a^k$ is called the *Drazin index* $i(a)$ of a. We write $\mathcal{D}(\mathcal{A})$ for the set of all Drazin invertible elements of $\mathcal{A}$. With the convention $a^0 = e$ we have

$$a \in \mathcal{A}^{-1} \Longleftrightarrow a \in \mathcal{D}(\mathcal{A}) \quad \text{and} \quad i(a) = 0\,.$$

Proposition 1. *If $a \in \mathcal{D}(\mathcal{A})$, then a has a unique Drazin inverse.*

Proof. [3]. □

Proposition 2. *For $a \in \mathcal{A}$ the following assertions are equivalent:*
(1) *$a \in \mathcal{D}(\mathcal{A})$ and $i(a) \leq 1$;*
(2) *there is $b \in \mathcal{A}$ such that $aba = a$ and $e - ab - ba \in \mathcal{A}^{-1}$.*

Proof. [5, Proposition 3.9]. □

The main result of this section reads as follows:

Theorem 1. *Let $a, b \in \mathcal{A}$. Then:*

$$ab \in \mathcal{D}(\mathcal{A}) \Longleftrightarrow ba \in \mathcal{D}(\mathcal{A})\,.$$

In this case we have:
(1) $|i(ab) - i(ba)| \leq 1$*;*
(2) *if c is the Drazin inverse of ab, then bc^2a is the Drazin inverse of ba.*

2000 *Mathematics Subject Classification.* 47A10, 15A09.
Key words and phrases. Drazin inverse, Fredholm operator, poles of the resolvent.

Proof. Let c be the Drazin inverse of ab and let $k = i(ab)$, thus

$$(ab)^k c(ab) = (ab)^k, \quad c(ab)c = c \quad \text{and} \quad c(ab) = (ab)c\,.$$

Let $d = bc^2a$. Then

$$d(ba) = bc^2aba = b(cabc)c = bca$$

and

$$(ba)d = babc^2a = b(cabc)a = bca\,,$$

hence

$$d(ba) = (ba)d\,.$$

From $dba = bca$ it follows that

$$d(ba)d = bcabc^2a = b(cabc)ca = bc^2a = d$$

and

$$\begin{aligned}(ba)^{k+1}d(ba) &= (ba)^{k+1}bca = b(ab)^k abca \\ &= b[(ab)^k c(ab)]a = b(ab)^k a = (ba)^{k+1}\,.\end{aligned}$$

Therefore we have shown that $ba \in \mathcal{D}(\mathcal{A})$, $d = bc^2a$ is the Drazin inverse of ba and that $i(ba) \leq k+1$. Let $n = i(ba)$. Similar arguments as above show that $k = i(ab) \leq n+1$, thus $i(ba) = n \geq k-1$, hence $i(ba) \in \{k-1,\, k,\, k+1\}$. □

Example 1. Let $\mathcal{A} = \mathbb{C}^{2\times 2}$,

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $ab = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $ba = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Since $(ab)^2 = 0$, it is easy to see that ab is Drazin invertible with Drazin inverse $c = 0$ and $i(ab) = 2$. From $(ba)0(ba) = ba,\ 0(ba)0 = 0$ and $(ba)0 = 0(ba)$ we derive $i(ba) = 1$.

2. Fredholm and generalized Fredholm operators

Let X be a real or complex Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. By $\mathcal{F}(X)$ we denote the ideal of all finite-dimensional operators in $\mathcal{L}(X)$, by $\widehat{\mathcal{L}}$ we denote the quotient algebra $\mathcal{L}(X)/\mathcal{F}(X)$ and by $\widehat{A}$ we denote the equivalence class of $A \in \mathcal{L}(X)$ in $\widehat{\mathcal{L}}$, i. e. $\widehat{A} = A + \mathcal{F}(X)$. Moreover, by $N(A)$ and $A(X)$ we denote the kernel and the range of A, respectively.

As usual, $A \in \mathcal{L}(X)$ is called *Fredholm operator* if $\dim N(A)$ and $\operatorname{codim} A(X)$ are both finite. It is well known that

$$A \text{ is Fredholm} \iff \widehat{A} \in \widehat{\mathcal{L}}^{-1}$$

(see [4, Satz 81.1]). If $A \in \mathcal{L}(X)$ is Fredholm, then $A(X)$ is closed and A is relatively regular (see [4, § 74]). The set of all Fredholm operators in $\mathcal{L}(X)$ is denoted by $\Phi(X)$.

In [2, Chapter 3] Caradus shows the following:

Proposition 3. *Let $A \in \Phi(X)$ and let B any generalized inverse of A. Then $I - AB - BA \in \Phi(X)$.*

This suggests the following definition due to Caradus [2]:

$A \in \mathcal{L}(X)$ is called a *generalized Fredholm operator* if A is relatively regular and $I - AB - BA \in \Phi(X)$ for some generalized inverse B of A. By $\Phi_g(X)$ we denote the class of all generalized Fredholm operators on X.

Proposition 4.
(1) $\mathcal{F}(X) \subseteq \Phi_g(X)$.
(2) $\Phi(X) \subseteq \Phi_g(X)$.
(3) $\Phi_g(X) + \mathcal{F}(X) \subseteq \Phi_g(X)$.
(4) $\Phi_g(X) \subseteq \overline{\Phi(X)}$ *(where the bar denotes closure).*
(5) $A \in \Phi_g(X) \Longleftrightarrow \widehat{A} \in \mathcal{D}(\widehat{\mathcal{L}})$ *and* $i(\widehat{A}) \leq 1$.

Proof. (1) is shown in [5, Proposition 1.3], (2) and (4) are due to Caradus [2], (5) follows from [5, Theorem 2.3] and Proposition 2, (3) is a consequence of (5). □

Remark. By Proposition 4(5) we have for $A \in \Phi_g(X)$:

$$A \in \Phi(X) \Longleftrightarrow i(\widehat{A}) = 0.$$

As an immediate consequence of Theorem 1 we get the following result (see also [6]):

Theorem 2. *If $A, B \in \mathcal{L}(X)$ and $AB \in \Phi(X)$, then $BA \in \Phi_g(X)$.*

Example 2. Let $X = l^2$ with the usual orthonormal basis $(u_k)_{k=1}^{\infty}$. Define $A, B \in \mathcal{L}(X)$ by

$$Au_{2k} = u_k, \quad Au_{2k+1} = 0 \qquad (k \in \mathbb{N})$$

and

$$Bu_k = u_{2k} \qquad (k \in \mathbb{N}).$$

Then $AB = I$, hence $AB \in \Phi(X)$, but $BA \notin \Phi(X)$. From Theorem 2 we get $BA \in \Phi_g(X)$, hence

$$i(\widehat{A}\widehat{B}) = 0 \quad \text{and} \quad i(\widehat{B}\widehat{A}) = 1.$$

3. Poles of the resolvent

Let X be a complex Banach space and $A \in \mathcal{L}(X)$. We write $\sigma(A), \rho(A)$ and $r(A)$ for the spectrum, the resolvent set and the spectral radius of A, respectively. For $\lambda \in \rho(A)$ we denote the resolvent $(\lambda I - A)^{-1}$ by $R_\lambda(A)$.

Let $A, B \in \mathcal{L}(X)$. In [1, Theorem 9] Barnes shows the following:

An isolated point $\lambda_0 \neq 0$ of $\sigma(AB)$ is a pole of order p of $R_\lambda(AB)$ if and only if λ_0 is a pole of order p of $R_\lambda(BA)$.

In this section we treat the case $\lambda_0 = 0$. To this end let $A \in \mathcal{L}(X)$ and suppose that 0 is an isolated point in $\sigma(A)$. Define the operator $P_0 \in \mathcal{L}(X)$ by

$$P_0 = \frac{1}{2\pi i} \int_\gamma R_\lambda(A) d\lambda,$$

where γ is a small circle surrounding $0 \in \mathbb{C}$ and separating 0 from $\sigma(A) \setminus \{0\}$. Then we have $P_0^2 = P_0$. P_0 is called the *spectral projection* of A associated with 0. Occasionally we shall denote it more precisely by $P_0(A)$.

In [2] the following is shown:

Proposition 5. *Let $A \in \mathcal{L}(X)$ and $p \geq 1$. Then $R_\lambda(A)$ has a pole of order p at $\lambda = 0$ if and only if $A \in \mathcal{D}(\mathcal{L}(X))$ and $i(A) = p$. In this case we have:*

(1)

$$R_\lambda(A) = \left(\frac{A^{p-1}}{\lambda^p} + \frac{A^{p-2}}{\lambda^{p-1}} + \cdots + \frac{I}{\lambda} \right) (I - AC) - \sum_{n=0}^{\infty} \lambda^n C^{n+1},$$

in the region $0 < |\lambda| < r(C)^{-1}$, where C is the Drazin inverse of A;

(2) $P_0(A) = I - AC$*;*

(3) *if $\sigma(A) \neq \{0\}$, then* $\operatorname{dist}(0, \sigma(A) \setminus \{0\}) = r(C)^{-1}$.

Example 3. Suppose that H is a complex Hilbert space, $A \in \mathcal{L}(H)$, $A^2 = A$, $0 \neq A \neq I$ and $B = A^*$. Let $T = AB$ and $S = BA$. Then T and S are selfadjoint, $TST = T^2$ and $STS = S^2$. We have $0 \in \sigma(T)$ and $0 \in \sigma(S)$ (indeed, if $0 \in \rho(T)$ or $0 \in \rho(S)$, then $T = S = I$, hence $AA^* = A^*A = I$, thus A is unitary, therefore $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, but since $\sigma(A) = \{0, 1\}$, we get $\sigma(A) = \{1\}$, hence $A = I$, a contradiction). From

$$\begin{aligned} T(S-I)^2T &= TS^2T - 2TST + T^2 = TSTST - T^2 \\ &= T^3 - T, \end{aligned}$$

we see that $T^3 - T = T(S-I)(T(S-I))^* \geq 0$. Hence the spectral mapping theorem gives $\sigma(T) \subseteq \{0\} \cup [1, \infty)$. This shows that 0 is an isolated point of $\sigma(T)$. It follows from [4, Satz 112] that 0 is a simple pole of $R_\lambda(T)$. The same arguments show that 0 is a simple pole of $R_\lambda(S)$. Hence, by Proposition 5, $AB,\ BA \in \mathcal{D}(\mathcal{L}(H))$ and

$$i(AB) = i(BA).$$

An application of Theorem 1 and Proposition 5 gives our next result:

Theorem 3. *Suppose that $A, B \in \mathcal{L}(X)$, $p \geq 1$ and that $R_\lambda(AB)$ has a pole of order p at $\lambda = 0$. Then:*

$$0 \in \rho(BA)$$

or

$$R_\lambda(BA) \ \textit{ has a pole of order } \ q \text{ at } \lambda = 0,$$

where $q \in \{p-1, p, p+1\}$.

Corollary 1. *Let A, B and p as in Theorem 3. Suppose that $0 \in \sigma(BA)$, C is the Drazin inverse of AB and that D is the Drazin inverse of BA. Then $D = BC^2A$,*

$$P_0(A) = I - ABC, \ P_0(BA) = I - BCA$$

and

$$\sigma(C) = \sigma(D).$$

Proof. From Theorem 1 we obtain $D = BC^2A$. Use Proposition 5 to derive $P_0(AB) = I - ABC$ and $P_0(BA) = I - BCA$. Since $0 \in \sigma(AB)$, we have $0 \in \sigma(C)$. Similar $0 \in \sigma(D)$. To obtain $\sigma(D) = \sigma(C)$ observe that

$$\begin{aligned}\sigma(D) \setminus \{0\} &= \sigma((BC^2)A) \setminus \{0\} = \sigma(A(BC^2)) \setminus \{0\} \\ &= \sigma(CABC) \setminus \{0\} = \sigma(C) \setminus \{0\}.\end{aligned}$$

□

Corollary 2. *Let A, B as in Theorem 3.*
(1) *If $R_\lambda(BA)$ has a pole of order p at $\lambda = 0$, then*

$$BR_\lambda(AB)A = BAR_\lambda(BA) \quad for \quad 0 < |\lambda| < r(C)^{-1}.$$

(2) *If $R_\lambda(BA)$ has a pole of order $p+1$ at $\lambda = 0$, then*

$$R_\lambda(BA) = \frac{1}{\lambda}(I + BR_\lambda(AB)A) \quad for \ 0 < |\lambda| < r(C)^{-1}.$$

Proof. Observe that $D^{n+1} = (BC^2A)^{n+1} = BC^{n+1}A$ for $n \geq 0$ and that $r(C) = r(D)$ (Corollary 1). The results follow by easy computations, use Proposition 5(1).

□

References

[1] B. A. Barnes, *Common operator properties of the linear operators RS and SR*, Proc. Amer. Math. Soc. **126** (1998), 1055–1061.

[2] S. R. Caradus, *Operator theory of the pseudo-inverse*, Queen's papers in pure and appl. math., No. 38 (1974).

[3] M. P. Drazin, *Pseudoinverse in associative rings and semigroups*, Amer. Math. Monthly **65** (1958), 506–514.

[4] H. Heuser, *Funktionalanalysis*, 2nd ed., Teubner (1986).

[5] Ch. Schmoeger, *On a class of generalized Fredholm operators, I*, Demonstratio Math. **30** (1997), 829–842.

[6] Ch. Schmoeger, *On Fredholm properties of operator products*, Math. Proc. Royal Irish Acad. **103 A** (2003), 203–208.

Christoph Schmoeger
Mathematisches Institut I
Universität Karlsruhe (TH)
Englerstrasse 2
76128 Karlsruhe
Germany
E-mail address: christoph.schmoeger@math.uni-karlsruhe.de