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A NOTE ON COMMUTING POWERS IN BANACH ALGEBRAS

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Throughout $\mathcal{A}$ is a complex unital Banach algebra with unit $\mathbf{1}$. For $a \in \mathcal{A}$ the spectrum and the spectral radius of a are denoted by $\sigma(a)$ and $r(a)$, respectively.

Let m be a positive integer and $a \in \mathcal{A}$. We say that $\sigma(a)$ is *irrotational* $(\bmod\, 2\pi/m)$ (see [1]) if $\lambda, \mu \in \sigma(a)$ and $\lambda^m = \mu^m$ imply that $\lambda = \mu$.

The main result of this paper reads as follows:

Theorem. *Let $a, b \in \mathcal{A}$ be invertible and let m be a positive integer.*
(1) *If $a^m b^m = b^m a^m$ and if $\sigma(a)$ is irrotational $(\bmod\, 2\pi/m)$, then $ab^m = b^m a$.*
(2) *If $a^m b^m = b^m a^m$ and if $\sigma(a)$ and $\sigma(b)$ are irrotational $(\bmod\, 2\pi/m)$, then $ab = ba$.*
(3) *If $a^m b^m = (ab)^m = b^m a^m$ and if $\sigma(ab)$ is irrotational $(\bmod\, 2\pi/m)$, then $ab = ba$.*

For the proof of the above result we need some preparations.

If m is a positive integer, let $\epsilon_k = e^{2k\pi i/m}$ for $k = 1, \dots, m$. Then

$$\epsilon_k^m = 1 \quad (k = 1, \dots, m),\ \epsilon_k \neq 1 \quad (k = 1, \dots, m-1)$$
$$\text{and } \epsilon_m = 1\,.$$

If $a \in \mathcal{A}$ is invertible, define the bounded linear operator $T_a : \mathcal{A} \to \mathcal{A}$ by

$$T_a x = a^{-1} x a \quad (x \in \mathcal{A})\,.$$

Proposition 1. *Suppose that $a \in \mathcal{A}$ is invertible, m is a positive integer and that $\sigma(a)$ is irrotational $(\bmod\, 2\pi/m)$. Let the bounded linear operator $S : \mathcal{A} \to \mathcal{A}$ be given by*

$$S = \prod_{k=1}^{m-1} (T_a - \epsilon_k I)\,,$$

where I denotes the identity operator on $\mathcal{A}$. Then:
(1) *S is invertible;*
(2) *$T_a^m - I = (T_a - I)S = S(T_a - I)$.*

Proof. (1) We show that $T_a - \epsilon_k I$ is invertible for $k = 1, \dots, m-1$. To this end suppose that $T_a - \epsilon_k I$ is not invertible for some $k \in \{1, \dots, m-1\}$. It follows from [1, Proposition 18.9] that there are $\lambda, \mu \in \sigma(a)$ such that $\lambda = \epsilon_k \mu$, hence $\lambda^m = \epsilon_k^m \mu^m = \mu^m$. Consequently $\lambda = \mu$ and therefore $\epsilon_k = 1$, a contradiction.
(2) follows from the identity

$$\lambda^m - 1 = (\lambda - 1) \prod_{k=1}^{m-1} (\lambda - \epsilon_k) \quad (\lambda \in \mathbb{C})\,.$$

□

Proposition 2. *Let a and m be as in Proposition 1. If $x \in \mathcal{A}$ and $a^m x = x a^m$, then $ax = xa$.*

Proof. From $a^m x = x a^m$ we get $(T_a^m - I)x = 0$. By Proposition 1 we have $S(T_a - I)x = 0$. Since S is invertible, $T_a x = x$ and so $ax = xa$. □

Proof of the Theorem.
(1) is a consequence of Proposition 2.
(2) By (1), $b^m a = ab^m$. Now apply Proposition 2 to b.
(3) We have

$$a^m(ab)^m = a^m b^m a^m = b^m a^m a^m = (ab)^m a^m\,.$$

Thus, by Proposition 2, $aba^m = a^m ab$, therefore

$$ba^m = a^{-1}aba^m = a^{-1}a^m ab = a^m b\,,$$

hence

$$(ab)^m b = a^m b^m b = ba^m b^m = b(ab)^m\,.$$

Now use Proposition 2 to see that $abb = bab$. Since b is invertible we derive $ab = ba$. □

Proposition 3. *Let $a, b \in \mathcal{A}$ and m a positive integer.*
(1) *If $\sigma(a) \subseteq [0, \infty)$, then $\sigma(a)$ is irrotational* $(\mathrm{mod}\, 2\pi/m)$.
(2) *If $m \geq 2$ and $(1 + r(a))^{m-1} < 2$, then $\mathbf{1} - a$ is invertible and $\sigma(\mathbf{1} - a)$ is irrotational* $(\mathrm{mod}\, 2\pi/m)$.
(3) *Suppose that $b \in \mathcal{A}$ is invertible, a is invertible, $\sigma(a)$ is irrotational* $(\mathrm{mod}\, 2\pi/m)$ *and that $a^m = b^m$. Then $ab = ba$.*

Proof. (1) Clear.

(2) We have $r(a) < 1$, hence $\mathbf{1} - a$ is invertible. Now let $\lambda, \mu \in \sigma(\mathbf{1} - a)$ and $\lambda^m = \mu^m$. There are $\alpha, \beta \in \sigma(a)$ such that $\lambda = 1 - \alpha$ and $\mu = 1 - \beta$. Then

$$\begin{aligned} 0 = (1-\alpha)^m - (1-\beta)^m &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k (\alpha^k - \beta^k) \\ &= -m(\alpha - \beta) + \sum_{k=2}^{m} \binom{m}{k} (\alpha - \beta) h_k(\alpha, \beta) \end{aligned}$$

where $h_k(\alpha, \beta) = (-1)^k(\alpha^{k-1} + \alpha^{k-2}\beta + \cdots + \alpha\beta^{k-2} + \beta^{k-1})$.
Hence $|h_k(\alpha, \beta)| \leq kr(a)^{k-1}$. Therefore

$$m|\alpha - \beta| \leq |\alpha - \beta| \sum_{k=2}^{m} \binom{m}{k} kr(a)^{k-1} = |\alpha - \beta| \left(\sum_{k=1}^{m} \binom{m}{k} kr(a)^{k-1} - m \right).$$

If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = (1 + x)^m$, then $f(x) = \sum_{k=0}^{m} \binom{m}{k} x^k$, thus $f'(x) = \sum_{k=1}^{m} \binom{m}{k} kx^{k-1}$, hence $\sum_{k=1}^{m} \binom{m}{k} kx^{k-1} = m(1 + x)^{m-1}$.

It follows that

$$m|\alpha - \beta| \leq |\alpha - \beta|(m(1 + r(a))^{m-1} - m)$$

and so

$$|\alpha - \beta| \leq |\alpha - \beta|((1 + r(a))^{m-1} - 1)$$

Now suppose that $\alpha \neq \beta$. Then

$$1 \leq (1 + r(a))^{m-1} - 1 < 2 - 1 = 1\,,$$

a contradiction. This gives $\lambda = \mu$.

(3) We have

$$a^m b = b^m b = bb^m = ba^m\,.$$

Now use Proposition 2. $\square$

Examples.
(1) *If* $r(a) < 1$, *then* $\sigma(\mathbf{1} - a)$ *is irrotational* $(\operatorname{mod} 2\pi/2)$.
(2) *If* $r(a) < \sqrt{2} - 1$, *then* $\sigma(\mathbf{1} - a)$ *is irrotational* $(\operatorname{mod} 2\pi/3)$.

Corollary 1. *Suppose that* $a, b \in \mathcal{A}$, $\sigma(a) \subseteq (0, \infty)$, $\sigma(b) \subseteq (0, \infty)$ *and that* $a^m b^m = b^m a^m$ *for some positive integer* m. *Then* $ab = ba$.

Proof. Proposition 3 (1) and Theorem (2). $\square$

Corollary 2. *Let* $\mathcal{A}$ *be the Banach algebra of all bounded linear operators on a complex Hilbert space, let* $A \in \mathcal{A}$ *be invertible and let* m *be a positive integer.*
(1) *If* $\sigma(A)$ *is irrotational* $(\operatorname{mod} 2\pi/m)$ *and if* A^m *is normal, then* A *is normal.*
(2) *If* $A^m (A^*)^m = (AA^*)^m = (A^*)^m A^m$, *then* A *is normal.*

Proof. (1) A^* is invertible and $\sigma(A^*) = \{\lambda \in \mathbb{C},\ \bar{\lambda} \in \sigma(A)\}$, thus $\sigma(A^*)$ is irrotational $(\operatorname{mod} 2\pi/m)$. Now use part (2) of the Theorem.
(2) Since $\sigma(AA^*) \subseteq (0, \infty)$, the result follows from Proposition 3 (1) and part (3) of the Theorem. $\square$

Corollary 3. *Suppose that* $a, b \in \mathcal{A}$, $r(a) < 1$, $r(b) < 1$ *and*

$$(\mathbf{1} - a)^2 (\mathbf{1} - b)^2 = (\mathbf{1} - b)^2 (\mathbf{1} - a)^2 .$$

Then $ab = ba$.

Proof. Example (1) and part (1) of the Theorem. $\square$

The *quasi-product* $x \circ y$ of x, $y \in \mathcal{A}$ is defined by

$$x \circ y = x + y - xy .$$

Given $z \in \mathcal{A}$, a *quasi-square-root* of z is an element $x \in \mathcal{A}$ with

$$x \circ x = z .$$

Corollary 4. *Let* a, $b \in \mathcal{A}$ *and* $a \circ a = b \circ b$.
(1) *If* $r(a) < 1$, *then* $ab = ba$.
(2) *If* $r(a) < 1$ *and* $r(b) < 1$, *then* $a = b$.

Proof. (1) Since $a \circ a = b \circ b$, we have $(\mathbf{1} - a)^2 = (\mathbf{1} - b)^2$. Now $\mathbf{1} - a$ is invertible, hence $\mathbf{1} - b$ is invertible. The result follows from Example (1) and Proposition 3 (3).
(2) By (1), $ab = ba$. From $a \circ a = b \circ b$ we see that

$$(a - b)(a + b) = -2(a - b) ,$$

hence

$$(a - b)(a + b + 2\mathbf{1}) = 0 .$$

Corollary 4.3 in [1] gives $r(a + b) \leq r(a) + r(b) < 2$, thus $-2 \notin \sigma(a + b)$ and so $a = b$. $\square$

Corollary 5. *Let* a, $b \in \mathcal{A}$ *and* $a^2 = b^2$.
(1) *If* $r(\mathbf{1} - a) < 1$, *then* $ab = ba$.
(2) *If* $r(\mathbf{1} - a) < 1$ *and* $r(\mathbf{1} - b) < 1$, *then* $a = b$.

Proof. Let $\tilde{a} = \mathbf{1} - a$, $\tilde{b} = \mathbf{1} - b$. Then

$$\tilde{a} \circ \tilde{a} = \mathbf{1} - a^2 \quad \text{and} \quad \tilde{b} \circ \tilde{b} = 1 - b^2 \,,$$

thus $\tilde{a} \circ \tilde{a} = \tilde{b} \circ \tilde{b}$. Now use Corollary 4. □

Corollary 6. *Suppose that $a, b \in \mathcal{A}$, $\sigma(a) \subseteq (0, \infty)$, $\sigma(b) \subseteq (0, \infty)$ and $a^m = b^m$ for some positive integer m. Then $a = b$.*

Proof. By Proposition 3 (3), $ab = ba$. Let $c = ab^{-1}$. Then $c^m = \mathbf{1}$. Let $\lambda \in \sigma(c)$. Corollary 4.3 in [1] gives $\lambda = \alpha/\beta$ with $\alpha \in \sigma(a)$ and $\beta \in \sigma(b)$, hence $\lambda > 0$. Since $\lambda^m = 1$, it follows that $\lambda = 1$. Thus $\sigma(c) = \{1\}$. We have

$$\lambda^m - 1 = (\lambda - 1)h(\lambda)$$

with some entire function h such that $h(1) \neq 0$. Therefore

$$0 = c^m - \mathbf{1} = (c - \mathbf{1})h(c)$$

and $h(c)$ is invertible. Hence $c = \mathbf{1}$ and so $a = b$. □

References

[1] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer (1973).

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