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A characterization of quasimonotone increasing functions

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Abstract: We give an equivalent characterization of quasimonotone functions in certain ordered Banach spaces, in terms of directional derivatives of the norm.

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Let $(E, ||\cdot||)$ be a real Banach space, ordered by a cone K. A cone K is a closed convex subset of E with $\lambda K \subseteq K$ $(\lambda \geq 0)$, and $K \cap (-K) = \{0\}$. As usual $x \leq y :\iff y - x \in K$. Let $(E^*, ||\cdot||)$ denote the topological dual space of E, and let

$$K^* = \{\varphi \in E^* : \ \varphi(x) \geq 0 \ (x \geq 0)\}$$

denote the dual wedge.

Let $D \subseteq E$. A function $f : D \to E$ is *quasimonotone increasing*, in the sense of Volkmann [3], if

$$x, y \in D, \ x \leq y, \ \varphi \in K^*, \ \varphi(x) = \varphi(y) \implies \varphi(f(x)) \leq \varphi(f(y)).$$

We assume that K is reproducing, that is $K - K = E$, and that there exists $\Psi \in E^*$, $||\Psi|| = 1$ such that

$$(1) \qquad ||x|| = \inf\{\Psi(p) : -p \leq x \leq p\} \quad (x \in E).$$

Examples are $E = \mathbb{R}^n$ or $E = l^1(\mathbb{N})$ with $K = \{x : x_k \geq 0\}$, $||x|| = \sum_k |x_k|$, and $\Psi(x) = \sum_k x_k$. Note also that in some cases an equivalent norm can be defined by (1), for example in case $\dim E < \infty$ and if $\Psi \in K^*$ is such that $x \geq 0$, $\Psi(x) = 0 \Rightarrow x = 0$.

Next, let $m_\pm : E \times E \to \mathbb{R}$ denote the one-sided directional derivatives of the norm:

$$m_\pm[x, y] = \lim_{h \to 0\pm} \frac{||x + hy|| - ||x||}{h}.$$

We will prove:

Theorem: *Let $D \subseteq E$ and $f : D \to E$. Equivalent are*

1. *f is quasimonotone increasing;*
2. *$m_+[y-x, f(y)-f(x)] = \Psi(f(y)-f(x))$ $(x, y \in D,\ x \leq y)$.*

We first prove

$$K = \{x \in E : \Psi(x) = ||x||\}.$$

If $x \in K$ then obviously $\Psi(x) = ||x||$. On the other hand, let $\Psi(x) = ||x||$. To each $n \in \mathbb{N}$ there exists $p_n \in K$ such that

$$\Psi(p_n) \leq ||x|| + \frac{1}{n}, \quad -p_n \leq x \leq p_n.$$

Thus, $||p_n - x|| = \Psi(p_n - x) = \Psi(p_n) - ||x|| \leq 1/n$. Hence $x = \lim_{n\to\infty} p_n \geq 0$.

Next, we prove the following representation of K^*: Let $\varphi \in E^* \setminus \{0\}$. Then

$$\varphi \in K^* \iff ||\Psi - \frac{\varphi}{||\varphi||}|| \leq 1.$$

Set $\eta = \Psi - \varphi/||\varphi||$. If $||\eta|| \leq 1$ then

$$\varphi(x) = ||\varphi||(||x|| - \eta(x)) \geq 0 \quad (x \in K),$$

hence $\varphi \in K^*$. On the other hand, if $\varphi \in K^*$, then

$$0 \leq \eta(x) = ||x|| - \frac{\varphi(x)}{||\varphi||} \leq ||x|| \quad (x \in K).$$

Fix $x \in E$, and let $\varepsilon > 0$. Choose p_0 such that

$$\Psi(p_0) \leq ||x|| + 2\varepsilon, \quad -p_0 \leq x \leq p_0.$$

Set

$$x_1 = \frac{p_0 + x}{2}, \quad x_2 = \frac{p_0 - x}{2}.$$

Then $x = x_1 - x_2$, $x_1, x_2 \in K$,

$$||x_1|| = \Psi(x_1) = \frac{1}{2}(\Psi(x) + \Psi(p_0)) \leq ||x|| + \varepsilon,$$

and analogously $||x_2|| \leq ||x|| + \varepsilon$.
Therefore

$$-||x|| - \varepsilon \leq -||x_2|| \leq -\eta(x_2) \leq \eta(x_1 - x_2) \leq \eta(x_1) \leq ||x_1|| \leq ||x|| + \varepsilon,$$

that is $|\eta(x)| \leq ||x|| + \varepsilon$. For $\varepsilon \to 0+$ we obtain $|\eta(x)| \leq ||x||$. Hence $||\eta|| \leq 1$.

To prove the theorem we use Mazur's characterization of m_+, see [1], [2]:

$$(2) \qquad m_+[x,y] = \max\{\eta(y) : \eta \in E^*,\ ||\eta|| = 1,\ \eta(x) = ||x||\}.$$

Let $f : D \to E$ be quasimonotone increasing, let $x, y \in D$, $x \leq y$, and let

$$\eta \in E^*,\ ||\eta|| = 1,\ \eta(y-x) = ||y-x||.$$

Then $\varphi := \Psi - \eta \in K^*$, and

$$\varphi(y-x) = ||y-x|| - \eta(y-x) = 0.$$

Hence $\varphi(f(y) - f(x)) \geq 0$, that is

$$\eta(f(y) - f(x)) \leq \Psi(f(y) - f(x)).$$

By means of (2) we have $m_+[y-x, f(y)-f(x)] \leq \Psi(f(y)-f(x))$. Equality follows from

$$m_+[y-x, f(y)-f(x)] \geq \lim_{h \to 0+} \frac{\Psi(y-x+h(f(y)-f(x))) - \Psi(y-x)}{h},$$

since $||\Psi|| = 1$.

Now, let $m_+[y-x, f(y)-f(x)] \leq \Psi(f(y)-f(x))$ be valid for $x, y \in D$, $x \leq y$.

Let $x, y \in D$, $x \leq y$, and $\varphi \in K^* \setminus \{0\}$ with $\varphi(x) = \varphi(y)$. For $\eta = \Psi - \varphi/||\varphi||$ we know $||\eta|| \leq 1$, and $\eta(y-x) = ||y-x||$, in particular $||\eta|| = 1$. Equation (2) gives

$$\eta(f(y)-f(x)) \leq m_+[y-x, f(y)-f(x)] \leq \Psi(f(y)-f(x)),$$

that is

$$\varphi(f(y)-f(x)) = ||\varphi||(\Psi - \eta)(f(y)-f(x)) \geq 0.$$

Hence f is quasimonotone increasing.

Remarks:

1. From $m_+[x,-y] = -m_-[x,y]$ $(x, y \in E)$ we get: A function $f : D \to E$ is quasimonotone decreasing, that is $-f$ is quasimonotone increasing, if and only if

$$m_-[y-x, f(y)-f(x)] = \Psi(f(y)-f(x)) \quad (x, y \in D,\ x \leq y).$$

2. If $f : D \to E$ is increasing, then

$$m_+[y - x, f(y) - f(x)] = ||f(y) - f(x)|| \quad (x, y \in D,\ x \leq y),$$

and if $f : D \to E$ is decreasing, then

$$m_-[y - x, f(y) - f(x)] = -||f(y) - f(x)|| \quad (x, y \in D,\ x \leq y),$$

References

[1] Martin, R.H.: *Nonlinear Operators and Differential Equations in Banach spaces.* Robert E. Krieger Publ. Company, Malabar, 1987.

[2] Mazur, S.: *Über konvexe Mengen in linearen normierten Räumen*. Stud. Math. **4** (1933), 70-84.

[3] Volkmann, P.: *Gewöhnliche Differentialungleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen.* Math. Z. **127** (1972), 157-164.