http://www.mathematik.uni-karlsruhe.de/~semlv
Seminar LV, No. 14, 12 pp., *11.09.2002*

On functions close to homomorphisms between square symmetric structures

Karol Baron and Peter Volkmann

Abstract. Let $\circ : S \times S \to S$ and $* : E \times E \to E$ be binary operations. Suppose $f : S \to E$, $\varphi : E \times E \to [0, \infty)$, and numbers $\omega, \varepsilon > 0$ are given. We provide conditions for (P) $\Rightarrow$ (Q) and for (Q) $\Rightarrow$ (P) to hold, where (P), (Q) have the following meanings:

(P) *There is a homomorphism $h : S \to E$ such that*

$$\varphi(f(x), h(x)) \leq \varepsilon \quad (x \in S).$$

(Q) *There are real numbers δ, η such that*

$$\varphi(f(x) * f(y), f(x \circ y)) \leq \delta, \ \varphi(f(x)^{2^n}, f(x^{2^n})) \leq \omega^n \varepsilon + \eta \quad (x, y \in S; \ n \in \mathbb{N}).$$

The 2^n-th powers in (Q) concern the operations $*$ and $\circ$, respectively. For the more important implication (Q) $\Rightarrow$ (P) we suppose $\circ$ and $*$ to be square symmetric operations (i.e., $(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y)$ for $x, y \in S$, and similarly for $*$ in the set E). – We use our investigations to give a variant of a Forti's result on stability in the sense of Pólya, Szegő, Hyers, Ulam.

1. Introduction. By $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ we denote the system of natural numbers, integers, and reals, respectively; $\mathbb{N} = \{1, 2, 3, \dots\}$. Let $(S, \circ)$, $(E, *)$ be given sets with binary operations. A *homomorphism* $h : S \to E$ is a solution of the Cauchy functional equation

$$h(x \circ y) = h(x) * h(y) \quad (x, y \in S). \tag{1}$$

For $x \in S$ the powers x^{2^n} $(n \in \mathbb{N})$ are recursively defined by

$$x^2 = x \circ x, \ x^{2^{n+1}} = (x^{2^n})^2 \quad (n \geq 1), \tag{2}$$

and for $u \in E$ the powers u^{2^n} with respect to $*$ have a similar meaning. Then (1) implies

$$h(x^{2^n}) = h(x)^{2^n} \quad (x \in S, \ n \in \mathbb{N}). \tag{3}$$

Now let $f : S \to E,\ \varphi : E \times E \to [0, \infty)$ be given functions, let $\varepsilon > 0$, and consider the following requirement:

(P) *There is a homomorphism $h : S \to E$ such that*

$$\text{(4)} \qquad \varphi(f(x), h(x)) \le \varepsilon \ \ (x \in S).$$

(P) means that in some sense f is close to the homomorphism h. In the next paragraph we give conditions for the space $(E, *)$ and the function φ, in order to get from (P) the following properties (Q_1), (Q_2):

(Q_1) *There is a real number δ such that*

$$\text{(5)} \qquad \varphi(f(x) * f(y), f(x \circ y)) \le \delta \ \ (x, y \in S).$$

(Q_2) *There is a real number η such that*

$$\text{(6)} \qquad \varphi(f(x)^{2^n}, f(x^{2^n})) \le \omega^n \varepsilon + \eta \ \ (x \in S,\ n \in I\!N).$$

(Q_1) and (Q_2) together are sometimes simply called (Q), like in the abstract. In (6), ω is a given positive number, which later on will be linked to φ by the formula

$$\text{(A)} \qquad \varphi(u^2, v^2) = \omega \varphi(u, v) \ \ (u, v \in E).$$

To get (Q_2) from (P) we rather use

$$(\mathrm{A}_{\le}) \qquad \varphi(u^2, v^2) \le \omega \varphi(u, v) \ \ (u, v \in E).$$

The inverse inequality

$$(\mathrm{A}_{\ge}) \qquad \varphi(u^2, v^2) \ge \omega \varphi(u, v) \ \ (u, v \in E)$$

is used in the third paragraph to get (P) from (Q): We construct the function h occuring in (4). To do so, we equip E with a complete metric $\rho \le \varphi$, and we give conditions for obtaining h as the usual limit, which is known from Pólya and Szegő for $(S, \circ) = (I\!N, +), (E, *) = (I\!R, +)$ (cf. [10], Exercise I 99) and from Hyers [6] for Banach spaces S, E; cf. also Forti's survey paper [4]. To obtain the homomorphism property (1) for this function h, we suppose the operations $\circ$ in S and $*$ in E to be *square symmetric* (cf. [9]), i.e.

$$\text{(V)} \qquad (x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y) \ \ (x, y \in S),$$

(W) $\qquad (u{*}v){*}(u{*}v) = (u{*}u){*}(v{*}v) \;\; (u, v \in E).$

Of course, these formulas also can be written as $(x{\circ}y)^2 = x^2{\circ}y^2,\ (u{*}v)^2 = u^2{*}v^2$. From Forti's paper [2] it is already clear that square symmetric operations provide a natural setting for studying stability of Cauchy functional equations (cf. also [1] by Borelli and Forti; the first paper using square symmetry in this context is due to Rätz [11]; for more recent results cf. Páles [8]).

In the fourth paragraph we discuss uniqueness of the homomorphism h in (P), and we summarize the hypotheses for the equivalence between (P) and (Q).

The fifth paragraph is devoted to stability. Concerning conditions (P), (Q_1), (A) we are less general than Forti [2]: He allows variable $\varepsilon = \varepsilon(x),\ \delta = \delta(x, y)$, and instead of (A) he uses $\varphi(u^2, v^2) = k(\varphi(u, v))$, where $k : [0, \infty) \to [0, \infty)$ is an appropriate function. On the other hand, our function φ is not necessarily a metric on E, since $\varphi(v, u) = \varphi(u, v) \;\; (u, v \in E)$ will not be required. Examples in the concluding sixth paragraph show the advantage of this.

A special case of our considerations is a square symmetric structure $(S, \circ)$ (i.e., (V) holds) and $(E, *) = (E, +)$ with an arbitrary Banach space E, where $\rho(u, v) = \varphi(u, v) = \|u - v\| \;\; (u, v \in E)$ and $\omega = 2$. Then it is known from [16] (and it is easy to show) that (P), (Q) are equivalent; this result had been inspired by [5].

2. The implications (P) $\Rightarrow$ ($\mathbf{Q_1}$) and (P) $\Rightarrow$ ($\mathbf{Q_2}$). For the function $\varphi : E \times E \to [0, \infty)$ we deal with the following conditions:

(S) *There is a constant $a \geq 0$ such that*

$$\varphi(v, u) \leq a\varphi(u, v) \;\; (u, v \in E).$$

(T) *There are constants $b, c \geq 0$ such that*

$$\varphi(u, w) \leq b\varphi(u, v) + c\varphi(v, w) \;\; (u, v, w \in E).$$

(T_1) *There is a constant $c \geq 0$ such that*

$$\varphi(u, w) \leq \varphi(u, v) + c\varphi(v, w) \;\; (u, v, w \in E).$$

$$(\mathrm{T}_{11}) \qquad \varphi(u,w) \le \varphi(u,v) + \varphi(v,w) \ \ (u,v,w \in E).$$

Of course, $(\mathrm{T}_{11}) \Rightarrow (\mathrm{T}_1) \Rightarrow (\mathrm{T})$. The triangle inequality (T_{11}) will be used later, when discussing stability. At present we need a certain boundedness condition:

(B) *There is a real number β such that for $t,u,v,w \in E$ we have*

$$\varphi(t,v) \le \varepsilon, \ \varphi(u,w) \le \varepsilon \Rightarrow \varphi(t*u, v*w) \le \beta.$$

Proposition 1. *If* (S), (T), (B) *are satisfied, then* (P) $\Rightarrow$ (Q_1); *if* (S), (T_1), $(\mathrm{A}_\le)$ *hold, then* (P) $\Rightarrow$ (Q_2).

Proof. To get (Q_1) from (P), consider $x,y \in S$ and use (S), (T), (B), (P), and (1) as follows:

$$\begin{aligned}
&\varphi(f(x)*f(y), f(x\circ y)) \\
&\le b\varphi(f(x)*f(y), h(x)*h(y)) + c\varphi(h(x\circ y), f(x\circ y)) \\
&\le b\beta + ca\varphi(f(x\circ y), h(x\circ y)) \le b\beta + ca\varepsilon.
\end{aligned}$$

This proves (Q_1) with $\delta = b\beta + ca\varepsilon$. To get (Q_2) from (P) we use (3). Then (S), (T_1), $(\mathrm{A}_\le)$, (P) imply

$$\begin{aligned}
&\varphi(f(x)^{2^n}, f(x^{2^n})) \le \varphi(f(x)^{2^n}, h(x)^{2^n}) + c\varphi(h(x^{2^n}), f(x^{2^n})) \\
&\le \omega^n \varphi(f(x), h(x)) + ca\varphi(f(x^{2^n}), h(x^{2^n})) \le \omega^n\varepsilon + ca\varepsilon,
\end{aligned}$$

i.e., (Q_2) holds with $\eta = ca\varepsilon$.

3. The implication (Q) $\Rightarrow$ (P). Here we use the following property of $*$ in E:

(U) *To every $u \in E$ there is a unique $v \in E$ such that $v^2 = u$.*

We write $v = u^{1/2} = u^{2^{-1}}$, and we define recursively

$$u^{2^{-n-1}} = (u^{2^{-n}})^{2^{-1}} \ \ (u \in E, \ n \in \mathbb{N}).$$

Together with $u^{2^0} = u^1 = u$ and with the analogue of (2) for the operation $*$ in E, the powers u^{2^m} are defined for all $m \in \mathbb{Z}$, and the rule $(u^{2^m})^{2^n} = u^{2^{m+n}}$ for $u \in E$ and $m,n \in \mathbb{Z}$ can easily be verified.

As mentioned in the introduction, ρ will be a metric on E; we suppose:

(R) *(E, ρ) is a complete metric space, and $\rho \leq \varphi$.*

All further topological (and metric) notions in E are understood with respect to ρ. In particular the function $h : S \to E$ in (P) will be given by the limit

$$(7) \qquad h(x) = \lim_{n\to\infty} f(x^{2^n})^{2^{-n}} \quad (x \in S).$$

Proposition 2. *Suppose* (Q$_2$), (R), (U), (A$_\geq$), *and*

$$(\mathrm{E}) \qquad \omega > 1.$$

Then (7) *defines a function $h : S \to E$.*

Proof. We fix $x \in S$. Because of (U) the expressions $f(x^{2^n})^{2^{-n}}$ have a meaning, and because of (R) it is sufficient to show that they form a Cauchy sequence: We put

$$\delta_{m,m+n} = \rho(f(x^{2^m})^{2^{-m}}, f(x^{2^{m+n}})^{2^{-m-n}}) \quad (m, n \in \mathbb{N}).$$

By $\rho \leq \varphi$ and (A$_\geq$) we get

$$\delta_{m,m+n} \leq \frac{1}{\omega^{m+n}} \varphi(f(x^{2^m})^{2^n}, f((x^{2^m})^{2^n}))$$

((2) implies $x^{2^{m+n}} = (x^{2^m})^{2^n}$). Now (Q$_2$), (E) yield

$$\delta_{m,m+n} \leq \frac{1}{\omega^{m+n}}(\omega^n \varepsilon + \eta) \leq \frac{\varepsilon + |\eta|}{\omega^m},$$

and the last term tends to zero as $m \to \infty$.

The conditions (V), (W) will occur in the next proposition. From (V), (2) the formula $(x \circ y)^{2^n} = x^{2^n} \circ y^{2^n}$ $(x, y \in S;\ n \in \mathbb{N})$ easily follows. From (W) we get a similar formula for the operation $*$ in E, and if also (U) holds, then we have more generally $(u * v)^{2^m} = u^{2^m} * v^{2^m}$ $(u, v \in E;\ m \in \mathbb{Z})$. Two further conditions will be used:

(C) *$* : E \times E \to E$ is continuous.*

(D) *$\varphi : E \times E \to [0, \infty)$ is continuous with respect to the second variable.*

In the next proposition we use again the definition of $h : S \to E$ from Proposition 2.

Proposition 3. *Assume* (Q$_2$), (R), (U), (A$_\geq$), (E) *to hold and define $h : S \to E$ by* (7). *If* (D) *is satisfied, then* (4) *holds. If* (V), (W), (Q$_1$), (C) *are satisfied, then $h : S \to E$ is a homomorphism.*

Proof. Let (D) be satisfied: Dividing (6) by ω^n and using (A$_\geq$) yields

$$\varphi(f(x), f(x^{2^n})^{2^{-n}}) \leq \varepsilon + \frac{\eta}{\omega^n}.$$

By $n \to \infty$ we get (4).

Now let (V), (W), (Q$_1$), (C) be satisfied: For $x, y \in S$ and $n \in \mathbb{N}$ we get from (5) the inequality

$$\varphi(f(x^{2^n}) * f(y^{2^n}), f((x \circ y)^{2^n})) \leq \delta.$$

We divide by ω^n and we use (A$_\geq$) to obtain

$$\varphi(f(x^{2^n})^{2^{-n}} * f(y^{2^n})^{2^{-n}}, f((x \circ y)^{2^n})^{2^{-n}}) \leq \frac{\delta}{\omega^n}.$$

Because of $\rho \leq \varphi$ we can replace φ by ρ. Then, when using (C), $n \to \infty$ yields $h(x) * h(y) = h(x \circ y)$.

Observe that by the last reasoning we get $h(x) * h(x) = h(x \circ x)$, if (V), (W) are not required (cf. also Proposition 1 in Forti's paper [3]). But for this it is sufficient to have (5) only for $y = x$, and this point of view has been adopted in [18].

Observe furthermore that at the end of Proposition 3 we can replace (V) by a more general condition stemming from Józef Tabor [15] (cf. also [18]).

As an immediate consequence of Propositions 2, 3 we have:

Proposition 4. *Suppose* (R), (U), (V), (W), (A$_\geq$), (C), (D), (E) *to hold. Then* (Q) $\Rightarrow$ (P).

4. Uniqueness of the homomorphism h in (P) and the equivalence (P) $\Leftrightarrow$ (Q).

Proposition 5. *Assume* (S), (T), (A$_\geq$), (E), *and:*

(F) *For* $u, v \in E$, $\varphi(u, v) = 0$ *implies* $u = v$.

Then the homomorphism $h : S \to E$ *in* (P) *is unique.*

Proof. For homomorphisms $h_1, h_2 : S \to E$ satisfying

$$\varphi(f(x), h_1(x)) \leq \varepsilon, \quad \varphi(f(x), h_2(x)) \leq \varepsilon \quad (x \in S)$$

we have

$$\begin{aligned}&\varphi(h_1(x),h_2(x))\le b\varphi(h_1(x),f(x))+c\varphi(f(x),h_2(x))\\&\le ba\varphi(f(x),h_1(x))+c\varepsilon\le(ba+c)\varepsilon=:\gamma,\end{aligned}$$

hence, for $x\in S$ and $n\in\mathbb{N}$,

$$\begin{aligned}\varphi(h_1(x^{2^n}),h_2(x^{2^n}))&\le\gamma,\\ \varphi(h_1(x)^{2^n},h_2(x)^{2^n})&\le\gamma,\\ \omega^n\varphi(h_1(x),h_2(x))&\le\gamma,\\ \varphi(h_1(x),h_2(x))&\le\gamma/\omega^n\to0\quad(n\to\infty).\end{aligned}$$

Therefore, $\varphi(h_1(x),h_2(x))=0$ $(x\in S)$, and because of (F) we obtain $h_2=h_1$.

Since (F) is a consequence of (R), we get from Propositions 1, 4, 5 the result:

Theorem 1. *Assume* (R), (S), (T_1), (U), (V), (W), (A), (B), (C), (D), (E) *to hold. Then* (P) $\Leftrightarrow$ (Q), *and the homomorphism* $h:S\to E$ *in* (P) *is uniquely determined; it is given by the limit* (7).

5. Stability. $(S,\circ)$ and $(E,*)$ being given, we understand stability of equation (1) by means of the function $\varphi:E\times E\to[0,\infty)$ in the following way:

Definition. *The homomorphism equation* (1) *is stable, if for each* $\varepsilon>0$ *there exists a* $\delta>0$ *such that for functions* $f:S\to E$ *satisfying* (5) *also* (P) *holds.*

In view of Proposition 4 it is now of interest to get for each $\varepsilon>0$ some $\delta>0$ such that the inequality (5) in (Q_1) implies (Q_2): In such a case one has stability, if also the hypotheses of Proposition 4 are satisfied.

Proposition 6. *Assume* ($A_\le$), (E), *and the triangle inequality* (T_{11}) *to hold, and suppose* $0<\delta\le\varepsilon(\omega-1)$. *Then* (5) *implies* ($Q_2$).

Proof. We use (5) only for $y=x$, i.e.,

$$\varphi(f(x)^2,f(x^2))\le\delta\ \ (x\in S).\tag{8}$$

For $x\in S$ and $n\in\mathbb{N}$, (T_{11}) implies

$$\begin{aligned}&\varphi(f(x)^{2^n},f(x^{2^n}))\le\varphi(f(x)^{2^n},f(x^2)^{2^{n-1}})+\\&+\varphi(f(x^2)^{2^{n-1}},f(x^4)^{2^{n-2}})+\cdots+\varphi(f(x^{2^{n-1}})^2,f(x^{2^n})),\end{aligned}$$

and by ($A_{\leq}$), (8) we get

$$\begin{aligned}\varphi(f(x)^{2^n}, f(x^{2^n})) &\leq \omega^{n-1}\delta + \omega^{n-2}\delta + \cdots + \delta = \\ = \frac{\omega^n - 1}{\omega - 1}\delta &= \omega^n \frac{\delta}{\omega - 1} - \frac{\delta}{\omega - 1} \leq \omega^n \varepsilon - \frac{\delta}{\omega - 1},\end{aligned}$$

i.e., (6) holds with $\eta = -\delta/(\omega - 1)$.

As a consequence of Propositions 4, 6 we get:

Theorem 2. *Suppose* (R), (T_{11}), (U), (V), (W), (A), (C), (D), (E) *are fulfilled. Then equation* (1) *is stable: If* $\varepsilon > 0$ *is arbitrary and* $\delta = \varepsilon(\omega - 1)$*, then* (5) *implies* (P).

Remark. In the proof of Proposition 6 the inequality (5) was only needed for $y = x$. Therefore Theorem 2 can be strengthened in the following way: Suppose the hypotheses (R), ..., (E) of that theorem to hold. Let $\varepsilon > 0$ be given, suppose (5) to hold with some $\delta \geq 0$ (this δ not necessarily being linked to ε), and suppose

$$\varphi(f(x)^2, f(x^2)) \leq \varepsilon(\omega - 1) \quad (x \in S).$$

Then (P) is true.

In the simple case $(S, \circ) = (E, *) = (\mathbb{R}, +)$ (and $\varphi(x, y) = |x - y|$) this remark means that for $f : \mathbb{R} \to \mathbb{R}$ having the properties

$$|f(x) + f(y) - f(x + y)| \leq \delta, \quad |f(2x) - 2f(x)| \leq \varepsilon \quad (x, y \in \mathbb{R}),$$

there is an additive $h : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - h(x)| \leq \varepsilon$ $(x \in \mathbb{R})$.

6. Examples. 1. Let E be a Banach space. As square symmetric operation in this space we take the addition (and we write $+$, not $*$), as metric we take

$$\rho(u, v) = \alpha\|u - v\| \quad (u, v \in E), \tag{9}$$

where $\alpha > 0$ will be specified in a moment. Let V be a closed, convex, bounded subset of E, having zero in its interior, and let $\mu : E \to [0, \infty)$ be the Minkowski functional of this set (cf., e.g., Rudin [13]), in particular we have

$$V = \{u \mid u \in E,\ \mu(u) \leq 1\}. \tag{10}$$

We take

$$\varphi(u,v) = \mu(u-v) \ \ (u,v \in E), \tag{11}$$

and we choose α in (9) such that $\rho \leq \varphi$. Then E, φ, ρ meet all the conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2 and we have $\omega = 2$ for this case. In condition (B) the dependence of β upon ε is given by $\beta = 2\varepsilon$.

Moreover, let $(S, \circ)$ be an arbitrary square symmetric structure (i.e., also (V) holds true); by Theorem 2 we get stability with $\delta = \varepsilon$, and because of (10), (11) this means for $\varepsilon = 1$ the following: If $f : S \to E$ satisfies

$$f(x)+f(y)-f(x\circ y) \in V \ \ (x,y \in S), \tag{12}$$

then there is $h : S \to E$ such that

$$h(x\circ y) = h(x)+h(y), f(x)-h(x) \in V \ \ (x,y \in S). \tag{13}$$

This result is already known for the more general case of bounded subsets V of E, which are ideally convex in the sense of Lifšic [7]; the proof in [17] is the same as the former proof by Jacek Tabor [14] for commutative semigroups $(S, \circ)$.

2. Suppose $n \in \mathbb{N}$, $n \geq 2$, and $0 < p < 1$. We take $E = \mathbb{R}^n$ with its addition $+$ as square symmetric operation, and we equip $\mathbb{R}^n$ with the F-norm

$$\|u\| = \sum_{\nu=1}^{n} |u_\nu|^p \ \ (u = (u_1, \ldots, u_n) \in \mathbb{R}^n). \tag{14}$$

Then $\rho(u,v) = \|u - v\|$ $(u, v \in \mathbb{R}^n)$ defines a translation invariant metric, by which $\mathbb{R}^n$ becomes a complete metric linear space (cf. Rolewicz [12]). We take $\varphi = \rho$, and again E, φ, ρ meet all conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2; this time we have $\omega = 2^p$ in (A), hence $\omega < 2$.

In particular we get $\delta < \varepsilon$ in Theorem 2, and actually $\delta = \varepsilon$ is not possible: To see this, suppose the contrary and define

$$V = \{u \mid u \in \mathbb{R}^n, \ \|u\| \leq 1\}. \tag{15}$$

As in the previous example, if $(S, \circ)$ is a square symmetric structure, then to each function $f : S \to E$ satisfying (12), there is an $h : S \to E$ such that

(13) holds. If we take $(S,\circ)=(\mathbb{R},+)$, then a theorem of Jacek Tabor [14] forces V to be a convex subset of $\mathbb{R}^n$ (this space now being considered as a Banach space). But because of $0<p<1$ (and $n\geq 2$) in (14), the set (15) is not convex.

3. In the foregoing example φ is a metric ($\varphi=\rho$), and such cases are covered by the papers of Forti [2] and of Borelli and Forti [1]. Now we take $E=\mathbb{R}^2$, again with $+$ as operation, and we define

$$\mu(u)=\mu(u_1,u_2)=\begin{cases}\sqrt{2u_1}+\sqrt{|u_2|} & (u_1\geq 0)\\ \sqrt{-u_1}+\sqrt{|u_2|} & (u_1\leq 0)\end{cases}\qquad (u=(u_1,u_2)\in\mathbb{R}^2).$$

Then $\varphi(u,v)=\mu(u-v)$ $(u,v\in\mathbb{R}^2)$ is not symmetric, hence not a metric. Finally we put $\rho(u,v)=\|u-v\|$ $(u,v\in\mathbb{R}^2)$ where $\|\cdot\|$ is given by (14) with $n=2,\ p=\frac{1}{2}$. Then E,φ,ρ meet all the conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2; here we have $\omega=\sqrt{2}$.

Let $(S,\circ)$ be an arbitrary square symmetric structure, and let us look at Theorem 2: If

$$W=\{u\mid u\in\mathbb{R}^2,\ \mu(u)\leq 1\},$$

$\varepsilon>0$, and if $f:S\to E$ satisfies

$$f(x)+f(y)-f(x\circ y)\in\delta W$$

(where $\delta=\varepsilon(\sqrt{2}-1)^2=\varepsilon(3-2\sqrt{2})$), then there is $h:S\to E$ such that

$$h(x\circ y)=h(x)+h(y),\ f(x)-h(x)\in\varepsilon W\ \ (x,y\in S). \tag{16}$$

The square in $\delta=\varepsilon(\sqrt{2}-1)^2$ comes from the fact that for $r\geq 0$ we have $\mu(u)\leq r$ if and only if $u\in r^2W$.

As Jacek Tabor has pointed out (oral communication), such type of stability result can be reduced to our first example: Take $E=\mathbb{R}^2$ and choose $\delta_1\in(0,\varepsilon)$ according to

$$V:=\delta_1\cdot\operatorname{conv}W\subseteq\varepsilon W$$

(where $\operatorname{conv}W$ denotes the convex hull of W). Then, if a function $f:S\to E$ satisfies

$$f(x)+f(y)-f(x\circ y)\in\delta_1 W,$$

we get (12), hence also (13) for some $h:S\to E$, and therefore we have (16).

4. Let us conclude by an infinite-dimensional version of the foregoing example: We take the complete metric linear space

$$E = \{u \mid u = (u_1, u_2, \dots),\ \|u\| = \sum_{n=1}^{\infty} \sqrt{|u_n|} < \infty\}$$

with + as operation, and for $u = (u_1, u_2, \dots) \in E$ we define

$$\mu(u) = \begin{cases} \|(2u_1, u_2, u_3, u_4, \dots)\| & (u_1 \geq 0) \\ \|u\| & (u_1 \leq 0). \end{cases}$$

Again $\varphi(u, v) = \mu(u - v)$ $(u, v \in E)$ is not symmetric, hence not a metric, and again we take $\rho(u, v) = \|u - v\|$ $(u, v \in E)$.

Then E, φ, ρ meet all the conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2, where $\omega = \sqrt{2}$.

Acknowledgment. The research of both authors was supported by the Mathematics Department of the Silesian University at Katowice (program "Iterative Functional Equations and Real Analysis"). The research of the first author also was supported by the DFG (Deutsche Forschungsgemeinschaft).

References

[1] Costanza BORELLI and Gian Luigi FORTI: *On a general Hyers-Ulam stability result.* Internat. J. Math. Math. Sci. **18**, 229-236 (1995).

[2] Gian Luigi FORTI: *An existence and stability theorem for a class of functional equations.* Stochastica **4**, 23-30 (1980).

[3] —: *The stability of homomorphisms and amenability, with applications to functional equations.* Abh. Math. Sem. Univ. Hamburg **57**, 215-226 (1987).

[4] —: *Hyers-Ulam stability of functional equations in several variables.* Aequationes Math. **50**, 143-190 (1995).

[5] Roman GER and Peter VOLKMANN: *On sums of linear and bounded mappings.* Abh. Math. Sem. Univ. Hamburg **68**, 103-108 (1998).

[6] Donald H. HYERS: *On the stability of the linear functional equation.* Proc. Nat. Acad. Sci. U.S.A. **27**, 222-224 (1941).

[7] E. A. LIFŠIC: *Ideal'no vypuklye množestva.* Funkcional'. Analiz Priložen. **4**, No. 4, 76-77 (1970).

[8] Zsolt PÁLES: *Hyers-Ulam stability of the Cauchy functional equation on square-symmetric groupoids.* Publ. Math. Debrecen **58**, 651-666 (2001).

[9] —, Peter VOLKMANN, and R. Duncan LUCE: *Hyers-Ulam stability of functional equations with a square-symmetric operation.* Proc. Nat. Acad. Sci. U.S.A. **95**, 12772-12775 (1998).

[10] György PÓLYA and Gábor SZEGŐ: *Aufgaben und Lehrsätze aus der Analysis, I.* Springer Berlin 1925.

[11] Jürg RÄTZ: *On approximately additive mappings.* International Series of Numerical Mathematics 47, Birkhäuser Basel, 233-251 (1980).

[12] Stefan ROLEWICZ: *Metric linear spaces.* Second enlarged edition, PWN - Polish Scientific Publishers Warszawa 1984.

[13] Walter RUDIN: *Functional analysis.* McGraw-Hill New York 1973.

[14] Jacek TABOR: *Ideally convex sets and Hyers theorem.* Funkcialaj Ekvac. **43**, 121-125 (2000).

[15] Józef TABOR: *Remark 18* (at the International Symposium on Functional Equations Oberwolfach 1984), Aequationes Math. **29**, 96 (1985).

[16] Peter VOLKMANN: *On the stability of the Cauchy equation.* Proceedings of the Numbers, Functions, Equations '98 International Conference, edited by Zsolt Páles, Janus Pannonius Tudományegyetem Pécs, 150-151 (1998).

[17] —: *Zur Rolle der ideal konvexen Mengen bei der Stabilität der Cauchyschen Funktionalgleichung.* Sem. LV, http://www.mathematik.uni-karlsruhe.de/~semlv, No. 6, 6 pp. (1999).

[18] —: *O stabilności równań funkcyjnych o jednej zmiennej.* Ibid. No. 11, 6 pp. (2001).

Typescript: Marion Ewald.

Authors' addresses:

K. Baron, Instytut Matematyki, Uniwersytet Śląski, Bankowa 14, 40-007 Katowice, Poland.

P. Volkmann, Mathematisches Institut I, Universität, 76128 Karlsruhe, Germany.