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On a Cauchy equation in norm

by

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Abstract. We deal with the exponential counterpart

(C) $$\|f(x+y)\| = \|f(x)f(y)\|$$

of the widely investigated Cauchy type functional equation

$$\|f(x+y)\| = \|f(x)+f(y)\|\,.$$

In contrast to the behaviour of solutions to the latter equation even good geometry of the target space fails to force the solutions of (C) to be exponential.

We exhibit the analytic form of these solutions in the case where their values are assumed in some special Banach algebras.

1. Introduction. The additive Cauchy equation in norm

$$\|f(x+y)\| = \|f(x)+f(y)\|$$

has extensively been studied through the last three decades and its background (the theory of so called alternative equations) goes back even to early sixties. The other reason why this functional equation arose so much interest lies in its close links with the theory of isometries; last but not least it leads to some characterizations of strictly convex normed linear spaces as well as to some of their generalizations (see R. Ger [3]).

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Since the role of the exponential Cauchy equation

$$\text{(E)} \qquad f(x+y) = f(x)f(y)$$

is equally significant (for instance, in the theory of semigroups of operators) we think that its "normed" version

$$\text{(C)} \qquad \|f(x+y)\| = \|f(x)f(y)\|$$

seems to be noteworthy as well. The nature of (C) suggests Banach algebras as potential target spaces for the unknown function. It should be no surprise that, in contrast to the "additive case", even the scalar-valued solutions of (C) (with the norm reduced to the absolute value) can hardly be expected to satisfy equation (E). Therefore, the description of solutions of (C) presents itself as a nontrivial question. In the present paper we answer that question in the scalar case and, under some additional assumptions, in the case where the unknown function takes its values in some special Banach algebras.

2. Scalar solutions. The case where the target algebra reduces to the field of scalars poses no difficulties. The corresponding solutions of equation (C) may be expressed in terms of additive functionals on the group in question as follows.

THEOREM 1. *Let $(X,+)$ be a group. Then a function $f : X \longrightarrow \mathbb{C}$ yields a solution of equation*

$$\text{(S)} \qquad |f(x+y)| = |f(x)f(y)|, \quad x, y \in X,$$

if and only if $f = 0$ or there exists an additive functional $a : X \longrightarrow \mathbb{R}$ and a function $b : X \longrightarrow \mathbb{R}$ such that

$$f(x) = e^{a(x)+ib(x)} \qquad \textit{for all} \quad x \in X\,.$$

A real-valued function f on X is a solution of (S) *if and only if $f = 0$ or there exists an additive functional $a : X \longrightarrow \mathbb{R}$ and a function $\varepsilon : X \longrightarrow \{-1, 1\}$ such that*

$$f(x) = \varepsilon(x)e^{a(x)} \qquad \textit{for all} \quad x \in X\,.$$

P r o o f. Let $f : X \longrightarrow \mathbb{C}$ be a nonzero solution of equation (S). Clearly, $\varphi := |f|$ is then an *exponential* function, i.e. it satisfies the equation

$$\varphi(x+y) = \varphi(x)\varphi(y)$$

for all $x, y \in X$. Since each real valued nonzero exponential function on a group has to be strictly positive we may put

$$a := \log \varphi \qquad \text{and} \qquad b := \operatorname{Arg} f$$

to get the desired representation $f = \exp(a + ib)$; obviously, the fact that φ is exponential forces a to be additive.

Moreover, $f = \exp(a + ib)$ is real if and only if there exists a function k on X with integer values such that $b(x) = k(x) \cdot \pi$ for every x from X. If that is the case, it remains to put $\varepsilon := \exp(k\pi i)$ to obtain the form of real solutions of equation (S) that was claimed.

A straightforward verification shows that for any additive function $a : X \longrightarrow \mathbb{R}$, an arbitrary function $b : X \longrightarrow \mathbb{R}$ and every function $\varepsilon : X \longrightarrow \{-1, 1\}$ the function $f := \exp(a + ib)$ (resp. $f := \varepsilon \cdot \exp a$) yields a complex (resp. real) solution to (S). This ends the proof.

R e m a r k 1. Roughly speaking, each solution of (S) has to be exponential up to a factor whose absolute value is identically equal to 1.

Assuming that the domain group is a topological one, one may ask what are the continuous solutions of (S).

THEOREM 2. *Let $(X, +)$ be a topological group. Then a function $f : X \longrightarrow \mathbb{C}$ yields a continuous solution of equation* (S) *if and only if $f = 0$ or there exists a continuous exponential function $\varphi : X \longrightarrow (0, \infty)$ and a continuous function χ mapping X into the unit circle such that $f = \varphi \cdot \chi$.*

A continuous real-valued function f on a connected topological group $(X, +)$ is a solution of (S) *if and only if f or $-f$ yields a continuous exponential function.*

P r o o f. Results easily from Theorem 1.

3. Solutions with values in strictly convex algebras need not be exponential. Unlike "the additive case" (see e.g. P. Fischer & Gy. Muszély [2], J. Aczél & J. Dhombres [1, §10.3], R. Ger [3], R. Ger [4], R. Ger & B. Koclęga [5], Gy. Maksa & P. Volkmann [6]) whatever would be the target (complex) algebra $(A, \|\cdot\|)$ (say with a unit; in particular, strictly convex or not) equation (C) and that of exponential mappings

$$f(x + y) = f(x)f(y) \tag{E}$$

are not equivalent in the class of functions f mapping a given group $(X, +)$ into A. In fact, for every nonzero exponential mapping $f : X \longrightarrow A$ the function

$$F(x) := e^{ib(x)} f(x)\,, \quad x \in X,$$

where b is any real function on X, yields a solution to equation (C) and fails to be exponential unless

$$b(x+y)-b(x)-b(y)\in 2\pi\mathbb{Z} \qquad \text{for all} \quad x,y\in X\,.$$

One might then conjecture that the presence of the scalar factor in the formula defining F is entirely responsible for that lack of equivalence whenever we deal with strictly convex algebras; but that is not the case. Actually, let M_2^d be the algebra of all (real or complex) diagonal 2×2-matrices with

$$\left\|\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\right\| := \sqrt{|\alpha|^2+|\beta|^2}\,.$$

Clearly, the space $(M_2^d,\|\cdot\|)$ is strictly convex but it can be easily verified that for an arbitrary group $(X,+)$ and every additive functional a on X the map

$$f(x) := \begin{bmatrix}\sqrt{\frac{3}{5}}e^{a(x)} & 0\\ 0 & \sqrt{\frac{6}{5}}e^{a(x)}\end{bmatrix}, \quad x\in X,$$

is *not* exponential although it satisfies equation (C).

The following remark gives a better insight into qualitative differences between the behaviour of solutions of (C) and its “additive” counterpart.

R e m a r k 2. Let $(X,+)$ be an Abelian group and let

$$f(x) = \begin{bmatrix}\alpha(x) & 0\\ 0 & \beta(x)\end{bmatrix}, \quad x\in X,$$

be a solution of (C), mapping X into the algebra M_2^d equiped with the Euclidean norm. Then besides the exponential solution (or, what amounts the same, besides α,β being exponential) we have also solutions of the form

$$f(x) = \begin{bmatrix}\varepsilon m_1(x) & 0\\ 0 & \delta m_2(x)\end{bmatrix}, \quad x\in X,$$

where m_1, m_2 stand for exponential functionals on X and $\varepsilon,\delta\in\{-1,1\}$, or

$$f(x) = \begin{bmatrix}\varepsilon m(x) & 0\\ 0 & \delta m(x)\end{bmatrix}, \quad x\in X,$$

where m is an exponential functional on X and ε,δ are arbitrary solutions of the numerical equation

$$|\varepsilon|^2+|\delta|^2 = |\varepsilon|^4+|\delta|^4\,.$$

These solutions may be found by discussing some Levi-Cività type functional equations (see e.g. L. Székelyhidi [7]) involving $|\alpha|^2$, $|\beta|^2$ and $|\alpha|^2+|\beta|^2$ as the unknown functions.

On the other hand, the only functions $f : X \longrightarrow M_2^d$ of the form in question satisfying the equation

$$\|f(x+y)\| = \|f(x)+f(y)\|$$

for all $x, y \in X$ are just the additive ones (or, what amounts the same, α, β being additive). This is a consequence of the main result from R. Ger's paper [3].

4. A lemma. The following technical result sets the path for establishing the analytic form of solutions of (C) with values in the algebra of continuous functions on a compact domain. But the lemma itself is valid for every normed algebra.

LEMMA. *Let* $(X, +)$ *be an Abelian group and let* $(A, \|\cdot\|)$ *be a (real or complex) normed algebra. If* $f : X \longrightarrow A$ *is a nonzero solution of equation* (C) *such that*

$$(\mathrm{i}_-) \qquad \|f(x)\| \cdot \|f(-x)\| \le 1\,, \quad x \in X,$$

and for every positive integer $n \ge 2$ *we have*

$$(\mathrm{i}_n) \qquad \|f(x_1+\cdots+x_n)\| \le \|f(x_1)\cdots f(x_n)\| \quad \textit{for all} \quad x_1, \dots, x_n \in X,$$

then

$$(\mathrm{i}_-^0) \qquad \|f(x)\| \cdot \|f(-x)\| = 1\,, \quad x \in X,$$

and for every $n \in \mathbb{N}$ *the equalities*

$$(\mathrm{i}_n^0) \quad \|f(x_1+\cdots+x_n)\| = \|f(x_1)\cdots f(x_n)\| = \|f(x_1)\|\cdots\|f(x_n)\|\,, \quad x_1, \dots, x_n \in X,$$

hold true.

P r o o f. Since $f \ne 0$, equation (C) implies that $f(0) \ne 0$, whence

$$0 < \|f(0)\| = \|f(0)^2\| \le \|f(0)\|^2$$

and, consequently, $\|f(0)\| \ge 1$. On the other hand, (i_1) applied for $x = 0$ says that $\|f(0)\|^2 \le 1$. Thus

$$1 = \|f(0)\| \qquad \text{and} \qquad \|f(x)\| \cdot \|f(-x)\| = 1 \quad \text{for all} \quad x \in X.$$

Fix arbitrarily an $n \in \mathbb{N} \setminus \{1\}$ and points $x_1, \dots x_n \in X$. Then

$$1 = \|f(0)\| \;=\; \|f((x_1+\cdots+x_n)+(-x_1\cdots-x_n))\|$$

$$
\begin{aligned}
&= \|f(x_1+\cdots+x_n)\cdot f(-x_1-\cdots-x_n)\| \\
&\le \|f(x_1+\cdots+x_n)\|\cdot\|f(-x_1)\cdots f(-x_n)\| \\
&\le \|f(x_1+\cdots+x_n)\|\cdot\|f(-x_1)\|\cdots\|f(-x_n)\| \\
&\le \|f(x_1)\cdots f(x_n)\|\cdot\|f(-x_1)\|\cdots\|f(-x_n)\| \\
&\le \|f(x_1)\|\cdots\|f(x_n)\|\cdot\|f(-x_1)\|\cdots\|f(-x_n)\| = 1\,.
\end{aligned}
$$

Therefore, all the inequalities above are equalities. In particular,

$$\|f(x_1+\cdots+x_n)\|\frac{1}{\|f(x_1)\|}\cdots\frac{1}{\|f(x_n)\|}=1\,,$$

i.e.

$$(*) \qquad \|f(x_1+\cdots+x_n)\| = \|f(x_1)\|\cdots\|f(x_n)\|$$

as well as

$$\|f(x_1)\cdots f(x_n)\|\frac{1}{\|f(x_1)\|}\cdots\frac{1}{\|f(x_n)\|}=1\,,$$

i.e.

$$(**) \qquad \|f(x_1)\cdots f(x_n)\| = \|f(x_1)\|\cdots\|f(x_n)\|\,.$$

Relations $(*)$ and $(**)$ imply the assertion (i_n^0), which completes the proof.

5. Main results. We proceed with the examination of solutions to equation (C) with values in some special Banach algebras.

THEOREM 3. *Let $(X,+)$ be an Abelian group and let $C(K)$ denote the algebra of all continuous functions from a compact Hausdorff topological space K into the field $\mathbf{K}\in\{\mathbb{R},\mathbb{C}\}$, equipped with the maximum norm. If $f:X\longrightarrow C(K)$ is a solution of equation* (C) *such that conditions* (i_-) *and* (i_n), *$n\ge 2$, are satisfied, then there exist: an exponential function $\varphi:X\longrightarrow[0,\infty)$ and a function $F:X\times K\longrightarrow\mathbf{K}$ continuous with respect to the second variable with $|F|\le 1$ and $|F(\cdot,t_0)|=1$ for some $t_0\in K$ such that*

$$(1) \qquad f(x)(t)=\varphi(x)F(x,t)\,,\quad (x,t)\in X\times K\,.$$

Conversely, each function $f:X\longrightarrow C(K)$ of that form yields a solution to (C) *and satisfies conditions* (i_-) *and* (i_n), *$n\ge 2$.*

P r o o f. We may assume that $f\ne 0$ (otherwise it suffices to take $\varphi=0$ and $F=1$). Thus (cf. the Lemma) we have (i_-^0) and (i_n^0) for all $n\in\mathbb{N}$. Put $\varphi(x):=\|f(x)\|$, $x\in X$. Clearly, (C) jointly with (i_2^0) says that

$$\varphi(x+y)=\|f(x+y)\|=\|f(x)f(y)\|=\|f(x)\|\cdot\|f(y)\|=\varphi(x)\varphi(y)$$

for all $x,y\in X$, i.e. φ is exponential.

Now, observe that given an $n \in \mathbb{N}$ and points $x_1, ..., x_n \in X$, we may find a point $t \in K$ such that

$$\|f(x_k)\| = |f(x_k)(t)| \quad \text{for all} \quad k \in \{1, ..., n\}\,. \tag{2}$$

In fact, otherwise $n \geq 2$ and for every point $t \in K$ we would find a $k \in \{1, ..., n\}$ such that

$$\|f(x_k)\| > |f(x_k)(t)|\,,$$

and, consequently, since

$$\|f(x_1) \cdots f(x_n)\| = |f(x_1) \cdots f(x_n)(\bar{t})| = |f(x_1)(\bar{t})| \cdots |f(x_n)(\bar{t})|$$

for some $\bar{t} \in K$, we would get

$$\|f(x_1) \cdots f(x_n)\| = |f(x_1)(\bar{t})| \cdots |f(x_n)(\bar{t})| < \|f(x_1)\| \cdots \|f(x_n)\|\,,$$

contradicting (i_n^0) (note that, in view of (i_-^0), one has $\|f(x)\| > 0$ for all $x \in X$).

Now, we infer that

(3) there exists a $t_0 \in K$ such that $\|f(x)\| = |f(x)(t_0)|$ for all $x \in X$.

Indeed, for an arbitrarily fixed $x \in K$, the set

$$A_x := \{t \in K : \|f(x)\| = |f(x)(t)|\}$$

yields a nonempty closed subset of K. Moreover, on account of (2), the family

$$\mathcal{F} := \{A_x : x \in X\}$$

has the finite intersection property. Therefore, due to the compactness of K, we have

$$\bigcap \mathcal{F} \neq \emptyset\,,$$

i.e. there exists a $t_0 \in K$ such that for every $x \in X$ one has $t_0 \in A_x$. In other words,

$$\|f(x)\| = |f(x)(t_0)|$$

for all $x \in X$; thus (3) has been proved.

Now, we have

$$f(x) = \|f(x)\| \frac{f(x)}{\|f(x)\|} = \varphi(x) \frac{f(x)}{|f(x)(t_0)|}$$

for all $x \in X$, i.e.

$$f(x)(t) = \varphi(x) \frac{f(x)(t)}{|f(x)(t_0)|}\,, \quad x \in X,\, t \in K\,,$$

and it suffices to put

$$F(x,t) := \frac{1}{|f(x)(t_0)|} f(x)(t)\,, \quad (x,t) \in X \times K\,.$$

Then, obviously, $|F| \le 1$ as well as $|F(x,t_0)| = 1$ for all $x \in X$. Finally, for every $x \in X$ the function $F(x,\cdot) = \mathrm{const} \cdot f(x)$ belongs to $C(K)$ and hence is continuous, as claimed.

Conversely, let $f : X \longrightarrow C(K)$ be given by formula (1) where $\varphi : X \longrightarrow [0,\infty)$ is exponential and $F : X \times K \longrightarrow \mathbf{K}$ be such that the sections $F(x,\cdot)$ are continuous for all $x \in X$ and $|F| \le 1$ with $|F(x,t_0)| = 1, x \in X$, for some $t_0 \in X$. Then, for every $x, y \in X$ one has

$$\begin{aligned} \|f(x+y)\| &= \varphi(x+y) \max_{t\in K} |F(x+y,t)| \\ &= \varphi(x)\varphi(y)|F(x+y,t_0)| = \varphi(x)\varphi(y)\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|f(x)f(y)\| &= \max_{t\in K} |\varphi(x)F(x,t) \cdot \varphi(y)F(y,t)| \\ &= \varphi(x)\varphi(y) \max_{t\in K} |F(x,t)F(y,t)| = \varphi(x)\varphi(y)\,, \end{aligned}$$

whence

$$\|f(x+y)\| = \|f(x)f(y)\| \quad \text{for all} \quad x,y \in X\,.$$

Similarly one can check the validity of conditions (i_-) and (i_n), $n \ge 2$. Thus the proof has been completed.

R e m a r k 3. Representation (1) is unique unless the function f in question vanishes identically. Actually, if we had

$$\varphi(x)F(x,t) = \psi(x)G(x,t)\,, \quad (x,t) \in X \times K\,,$$

with $\varphi \ne 0 \ne \psi$, $|F| \le 1$, $|G| \le 1$ and $|F(\cdot,t_0)| = |G(\cdot,t_1)| = 1$ for some $t_0, t_1 \in X$, then $\varphi(x) > 0$ and $\psi(x) > 0$ for all $x \in X$ and putting $m := \varphi//\psi$ we would get

$$m(x) = m(x)|F(x,t_0)| = |G(x,t_0)| \le 1, \quad x \in X,$$

i.e. $m = 1$ because m itself is exponential and positive. Consequently, $\varphi = \psi$ and, therefore, $F = G$.

In the exceptional case where $f = 0$ the exponential factor φ in (1) must necessarily be zero but F may be taken arbitrarily.

In what follows we shall extend this result for Banach algebras with isometric Gelfand transforms.

THEOREM 4. *Let* $(A, \|\cdot\|)$ *be a complex commutative Banach algebra with a unit* j *and such that*

$$\|a^2\| = \|a\|^2 \qquad \textit{for all} \quad a \in A\,. \tag{4}$$

If $(X,+)$ *is an Abelian group then a map* $f : X \longrightarrow A$ *enjoying the properties* (i_-) *and* (i_n), $n \in \mathbb{N}$, *satisfies equation* (C) *if and only if the functional*

$$\varphi(x) := \|f(x)\|\,, \quad x \in X\,,$$

is exponential and there exist a maximal ideal $M_0 \subset A$ *and a function* $\beta : X \longrightarrow \mathbb{R}$ *such that* f *yields a selection of the multifunction*

$$X \ni x \longmapsto \varphi(x)e^{i\beta(x)}j + M_0\,. \tag{5}$$

P r o o f. Let $\hat{f} : X \longrightarrow \hat{A}$ be defined by the formula

$$\hat{f}(x) := (f(x))\hat{}\,, \quad x \in X\,,$$

where $\hat{}$ stands for the Gelfand transform of the algebra A. Since condition (4) forces $\hat{}$ to be an isometry, conditions (i_-) and (i_n), $n \in \mathbb{N}$, remain valid when the symbols f and $\|\cdot\|$ are replaced by $\hat{f}$ and $\|\cdot\|_\infty$, respectively.

S u f f i c i e n c y. Without loss of generality we may assume that $f \neq 0$; consequently $\varphi \neq 0$ and, *a fortiori*, $\varphi(x) > 0$ for all $x \in X$ because φ is supposed to be exponential. The Gelfand transform, being an isometry, is injective; therefore the formula

$$g_0^*(\hat{a}) := a_0^*(a)\,, \quad a \in A\,,$$

where a_0^* is the (unique) linear multiplicative functional on A with $M_0 = \ker a_0^*$, correctly defines a linear multiplicative functional on $\hat{A} \subset C(\mathfrak{M})$ (here $\mathfrak{M}$ stands for the space of all maximal ideals in A, equipped with the Gelfand topology). By means of (5) we have

$$g_0^*(\hat{f}(x)) = a_0^*(f(x)) = \varphi(x)e^{i\beta(x)} \quad \text{for all} \quad x \in X\,.$$

Since, obviously, $\hat{A}$ is closed in $C(\mathfrak{M})$, the functional g_0^* admits a linear multiplicative extension g^* on $C(\mathfrak{M})$. Therefore, there exists an $M_1 \in \mathfrak{M}$ such that

$$g^*(u) = u(M_1) \quad \text{for every} \quad u \in C(\mathfrak{M}).$$

In particular,

$$\varphi(x)e^{i\beta(x)} = g_0^*(\hat{f}(x)) = g^*(\hat{f}(x)) = \hat{f}(x)(M_1) \quad \text{for all} \quad x \in X.$$

Now, setting

$$F(x,M) := \frac{1}{\varphi(x)}\hat{f}(x)(M)\,, \quad (x,M) \in X \times \mathfrak{M}\,,$$

we define a map $F : X \times \mathfrak{M} \longrightarrow \mathbb{C}$ such that

$$|F(x,M_1)| = \frac{1}{\varphi(x)}|\hat{f}(x)(M_1)| = \frac{1}{\varphi(x)}|\varphi(x)e^{i\beta(x)}| = 1$$

for all $x \in X$ and

$$|F(x,M)| = \frac{1}{\varphi(x)}|\hat{f}(x)(M)| \le \frac{1}{\varphi(x)}\|\hat{f}(x)\|_\infty = \frac{1}{\varphi(x)}\|f(x)\| = 1\,, \quad (x,M) \in X \times \mathfrak{M}\,.$$

Clearly, $F(x,\cdot)$ is continuous for each $x \in X$ whence, by Theorem 1 the map $\hat{f} : X \longrightarrow \hat{A}$ satisfies the equation

$$\|\hat{f}(x+y)\|_\infty = \|\hat{f}(x)\hat{f}(y)\|_\infty, \quad x,y \in X\,.$$

Thus

$$\|f(x+y)\| = \|\hat{f}(x+y)\|_\infty = \|(f(x)f(y))\hat{}\ \|_\infty = \|f(x)f(y)\|$$

for all $x, y \in X$, which was to be proved.

N e c e s s i t y. That $\varphi(x) := \|f(x)\|$, $x \in X$, is exponential results directly from the Lemma. We have also

$$\|\hat{f}(x+y)\|_\infty = \|\hat{f}(x)\hat{f}(y)\|_\infty\,, \quad x,y \in X\,,$$

whence, by virtue of Theorem 3,

$$\hat{f}(x)(M) = \|\hat{f}(x)\|_\infty F(x,M) = \|f(x)\|F(x,M) = \varphi(x)F(x,M)\,, \quad (x,M) \in X \times \mathfrak{M}\,,$$

with $|F| \le 1$, $|F(x,M_1)| \equiv 1$ for some $M_1 \in \mathfrak{M}$ and with $F(x,\cdot)$ being continuous for all $x \in X$. Obviously, there exists a function $\beta : X \longrightarrow \mathbb{R}$ such that $F(x,M_1) = e^{i\beta(x)}, x \in X,$ so that $\hat{f}(x)(M_1) = \varphi(x)e^{i\beta(x)}, x \in X.$

Now, consider the functional $a_0^* : A \longrightarrow \mathbb{C}$ given by the formula

$$a_0^*(a) := \hat{a}(M_1),\ a \in A, \qquad \text{and put} \qquad M_0 := \ker a_0^*\,.$$

Plainly, a_0^* is linear multiplicative and for every $x \in X$ one has

$$a_0^*(f(x) - \varphi(x)e^{i\beta(x)}j) = a_0^*(f(x)) - \varphi(x)e^{i\beta(x)} = \hat{f}(x)(M_1) - \varphi(x)e^{i\beta(x)} = 0\,,$$

which shows that f yields a selection of the multifunction (5), as claimed. This completes the proof.

R e m a r k 4. In the nontrivial case ($f \neq 0$) the multifunction (5) may equivalently be written as

$$X \ni x \longmapsto e^{\alpha(x)+i\beta(x)} j + M_0 \,,$$

where $\alpha : X \longrightarrow \mathbb{R}$ is additive. Since

$$X \ni x \longmapsto e^{\alpha(x)+i\beta(x)} \in \mathbb{C}$$

yields then the general solution of equation (S) (see Theorem 1) we can say that, under the assumptions of Theorem 4, vector-valued solutions of equation (C) are, in a sense, determined by their scalar counterparts, i.e. solutions of equation (S).

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References

[1] János Aczél and Jean Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.

[2] Pál Fischer and György Muszély, *On some new generalizations of the functional equation of Cauchy*, Canad. Math. Bull. 10 (1967), 197-205.

[3] Roman Ger, *On a characterization of strictly convex spaces*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 127 (1993), 131-138.

[4] Roman Ger, *Fischer-Muszély additivity on Abelian groups*, Comment. Math. Prace Mat., Tomus Specialis in honorem Iuliani Musielak, (2004), 83-96.

[5] Roman Ger and Barbara Koclęga, *Isometries and a generalized Cauchy equation*, Aequationes Math. 60 (2000), 72-79.

[6] Gyula Maksa and Peter Volkmann, *Characterization of group homomorphisms having values in an inner product space*, Publ. Math. Debrecen 56 (2000), 197-200.

[7] László Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups*, World Scientific, Singapore, 1991.

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