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SUPERSTABILITY OF SOME FUNCTIONAL EQUATION

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Abstract. In this paper we establish superstability of the functional equation $\sup_{l\in L} f(x+l(y)) = f(x)f(y)$, $x, y \in G$. The unknown function f maps an abelian group G into $\mathbb{R}$, and by L we denote an arbitrary subset of G^G which includes the mappings $x \overset{\mathrm{id}}{\mapsto} x$ and $x \overset{-\mathrm{id}}{\longmapsto} -x$. We solve this equation in the particular case, when G is a complex vector space and $L = \{x \mapsto \lambda x : \lambda \in \mathbb{C}, |\lambda| = 1\}$. Another special case of this equation, namely $\max\{f(x+y), f(x-y)\} = f(x)f(y)$, $x, y \in G$, was examined by A. Simon and P. Volkmann.

1. Introduction

Throughout this paper $\mathbb{R}$ means the space of real and $\mathbb{C}$ that of complex numbers. Let G be an abelian group. Suppose $L \subset G^G$ is such that $\mathrm{id}, -\mathrm{id} \in L$. We are going to prove superstability of the functional equation

$$\sup_{l\in L} f(x+l(y)) = f(x)f(y), \qquad x, y \in G, \tag{1.1}$$

that is to show

Theorem 1. *If a function $f\colon G \to \mathbb{R}$ satisfies*

$$|\sup_{l\in L} f(x+l(y)) - f(x)f(y)| \le \varepsilon, \qquad x, y \in G, \tag{1.2}$$

then it is either bounded or is a solution of (1.1).

(Here we used the definition of superstability according to Z. Moszner's Survey [3], Definition 4*). In paper [5] there was considered a particular case of (1.1), namely

$$\max\{f(x+y), f(x-y)\} = f(x)\cdot f(y), \qquad x, y \in G. \tag{1.3}$$

It has been proved that if G is divisible by 2 and 3, then the solutions $f\colon G \to \mathbb{R}$ of (1.3) are $f(x) = 0$ and $f(x) = \exp(|a(x)|)$ for some additive $a\colon G \to \mathbb{R}$. In the present paper we will establish the solutions of (1.1) in another special case:

$$\sup_{\lambda\in T} f(x+\lambda y) = f(x)\cdot f(y), \qquad x, y \in V, \tag{1.4}$$

where V is a complex vector space and $T = \{z \in \mathbb{C} : |z| = 1\}$. To do this, we need a result from [2]:
$f\colon V \to \mathbb{R}$ is a solution of

$$\sup_{\lambda\in T} f(x+\lambda y) = f(x) + f(y), \qquad x, y \in V \tag{1.5}$$

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if and only if there is a linear functional $\phi\colon V \to \mathbb{C}$, such that $f(x) = |\phi(x)|$.

Let us mention that the equations

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y), \qquad x, y \in G$$

and (1.5) are stable (both in the sense of Definition 1* and 2* from [3]), which follows from P. Volkmann [6] and from [4], respectively.

2. Stability of equation (1.1)

Before we turn to the proof of Theorem 1, we fix some notation. As in [4], we will write $A \overset{\varepsilon}{\sim} B$ instead of $|A - B| \leq \varepsilon$, whenever we find it convenient. Notice that inequality (1.2) can now be rewritten in the form

$$\sup_{l\in L} f(x + l(y)) \overset{\varepsilon}{\sim} f(x)f(y), \qquad x, y \in G.$$

Proof of Theorem 1. It is enough to show that if f is unbounded then it satisfies

$$\sup_{l\in L} f(x + l(y)) = f(x)f(y), \qquad x, y \in G.$$

Since f is unbounded there is a sequence $(w_n)_{n\in\mathbb{N}}$ such that $|f(w_n)| \overset{n\to\infty}{\to} \infty$, but taking into consideration that

$$\sup_{l\in L} f(w_n + l(w_n)) \overset{\varepsilon}{\sim} (f(w_n))^2 \overset{n\to\infty}{\longrightarrow} \infty,$$

we infer that there is a sequence $(z_n)_{n\in\mathbb{N}}$ with

$$f(z_n) \overset{n\to\infty}{\longrightarrow} \infty.$$

Of course, we can assume that $f(z_n) > 0$ for every $n \in \mathbb{N}$. Suppose that $f(x) < 0$ for some $x \in G$. We have

$$\sup_{l\in L} f(x + l(z_n)) \overset{\varepsilon}{\sim} f(x)f(z_n) \overset{n\to\infty}{\longrightarrow} -\infty,$$

whence, in particular,

$$f(x + z_n) \overset{n\to\infty}{\longrightarrow} -\infty.$$

Therefore,

$$f(x) \leq \sup_{l\in L} f(x + z_n + l(z_n)) \overset{\varepsilon}{\sim} f(x + z_n)f(z_n) \overset{n\to\infty}{\longrightarrow} -\infty,$$

which is impossible. Assuming that $f(x) = 0$ for some $x \in G$ also leads to a contradiction. Namely, we would have

$$0 = f(z_n - x)f(x) \overset{\varepsilon}{\sim} \sup_{l\in L} f(z_n - x + l(x)) \geq f(z_n) \overset{n\to\infty}{\longrightarrow} \infty.$$

Hence we proved that $f > 0$.

In the rest of the proof we use ideas from [1]. Notice that

$$\begin{aligned}
(2.1)\qquad \sup_{l_1\in L} f(x + l_1(y))f(z) &\overset{\varepsilon}{\sim} \sup_{l_1\in L}\sup_{l_2\in L} f(x + l_1(y) + l_2(z)) = \\
&= \sup_{l_2\in L}\sup_{l_1\in L} f(x + l_1(y) + l_2(z)) \overset{\varepsilon}{\sim} \sup_{l_2\in L}(f(x + l_2(z))f(y)) = \\
&= (\sup_{l_2\in L} f(x + l_2(z)))f(y) \overset{\varepsilon\cdot f(y)}{\sim} f(x)f(z)f(y).
\end{aligned}$$

Thereby

$$\sup_{l_1 \in L} f(x + l_1(y))f(z) \overset{2\varepsilon + \varepsilon \cdot f(y)}{\sim} f(x)f(z)f(y), \qquad x, y, z \in G.$$

Putting $z = z_n$ and dividing by $f(z_n)$ we get

$$\sup_{l_1 \in L} f(x + l_1(y)) \overset{\frac{2\varepsilon + \varepsilon \cdot f(y)}{f(z_n)}}{\sim} f(x)f(y), \qquad x, y \in G,\ n \in \mathbb{N}.$$

This implies that $\sup_{l \in L} f(x + l(y)) = f(x)f(y)$, $x, y \in G$. $\square$

Remark. If $f \colon G \to \mathbb{R}$ satisfying (1.2) is bounded, then

$$|f(x)| \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2}, \qquad x \in G.$$

Proof. Put $M_0 = \frac{1+\sqrt{1+4\varepsilon}}{2}$ and $B = \sup_{x \in G}|f(x)|$. Choose a sequence $x_n \in G$, $n \in \mathbb{N}$ with

$$|f(x_n)| =: A_n \overset{n\to\infty}{\longrightarrow} B.$$

By (1.2) we have

$$A_n^2 = f(x_n)^2 \overset{\varepsilon}{\sim} \sup_{l \in L} f(x_n + l(x_n)) \leq B, \qquad n \in \mathbb{N},$$

which results in $A_n^2 \leq B + \varepsilon$, $n \in \mathbb{N}$. Therefore $B^2 \leq B + \varepsilon$, whence $B \leq M_0$. $\square$

3. Solution of equation (1.4)

Lemma. *Let $f \colon \mathbb{C} \to \mathbb{R}$ satisfy (1.4) (with $V = \mathbb{C}$) and suppose that $f(0) = 1$. Then $f(z) > 0$ for every $z \in \mathbb{C}$.*

Proof. Notice that

$$f(0)f(z) = \sup_{\lambda \in T} f(\lambda z) = f(0)f(|z|), \qquad z \in \mathbb{C},$$

whence

$$f(z) = f(|z|), \qquad z \in \mathbb{C}. \tag{3.1}$$

Using this we get

$$\begin{aligned} f(a)f(b) &= \sup_{\lambda \in T} f(b + \lambda \cdot a) = \sup_{\lambda \in T} f(|b + \lambda \cdot a|) = \\ &= \sup f([b - a, b + a]) \geq f(b), \qquad 0 \leq a \leq b. \end{aligned} \tag{3.2}$$

Furthermore, $0 \notin f(\mathbb{C})$, since $(f(z))^2 = \sup_{\lambda \in T} f(z + \lambda z) \geq f(0) = 1$. Suppose that $f(y) < 0$ for some $y \in \mathbb{C}$. Owing to (3.1), we can assume that $y \in (0, \infty)$. As follows from (3.2), $f(x) < 0$ for $x > y$. In particular, $f(2y) < 0$ and moreover

$$0 < f(2y)f(y) = \sup f([y, 3y]) \leq 0,$$

which is impossible. $\square$

Theorem 2. *Let* $f\colon V \to \mathbb{R}$ *satisfy* (1.4)*. Then either* $f \equiv 0$ *or* $f(x) = \exp|\phi(x)|$*, where* $\phi\colon V \to \mathbb{C}$ *is a linear functional.*

Proof. Putting $x = y = 0$ in (1.4) we get $(f(0))^2 = f(0)$ whence either $f(0) = 0$ or $f(0) = 1$. In the first case we obtain

$$0 = f(x)f(0) = \sup_{\lambda \in T} f(x + \lambda \cdot 0) = f(x), \qquad x \in V,$$

which means $f \equiv 0$. Now consider the other case, i.e., $f(0) = 1$. We will show that $f > 0$.
Fix an arbitrary $x_0 \in V$. Define $f_{x_0}\colon \mathbb{C} \to \mathbb{R}$ by $f_{x_0}(\alpha) = f(\alpha x_0)$. It is easy to verify that f_{x_0} is the solution of (1.4) with $V = \mathbb{C}$ and $f_{x_0}(0) = 1$. According to our Lemma we infer that $f_{x_0} > 0$, and, particularly, $f(x_0) = f_{x_0}(1) > 0$.

Put $g(x) := \log f(x)$ and notice that $g\colon V \to \mathbb{R}$ satisfies the equation (1.5). Hence, by the result in [2], already mentioned in the Introduction, we get $g(x) = |\phi(x)|$, for some linear functional $\phi\colon V \to \mathbb{C}$ and, therefore, $f(x) = \exp|\phi(x)|$. □

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