SUPERSTABILITY OF SOME FUNCTIONAL EQUATION

BARBARA PRZEBIERACZ

Abstract. In this paper we establish superstability of the functional equation 
\[ \sup_{l \in L} f(x + l(y)) = f(x)f(y), \quad x, y \in G, \]
where an abelian group \( G \) is mapped into \( \mathbb{R} \), and by \( L \) we denote an arbitrary subset of \( G \) which 
includes the mappings \( x \mapsto x \) and \( x \mapsto -x \). We solve this equation in the 
particular case, when \( G \) is a complex vector space and \( L = \{x \mapsto \lambda x : \lambda \in \mathbb{C}, |\lambda| = 1\} \). Another special case of this equation, namely 
\[ \max\{f(x+y), f(x-y)\} = f(x)f(y), \quad x, y \in G, \]
was examined by A. Simon and P. Volkmann.

1. Introduction

Throughout this paper \( \mathbb{R} \) means the space of real and \( \mathbb{C} \) that of complex numbers. Let \( G \) be an abelian group. Suppose \( L \subset G^2 \) is such that \( \text{id}, -\text{id} \in L \). We are 
going to prove superstability of the functional equation

\[ \sup_{l \in L} f(x + l(y)) = f(x)f(y), \quad x, y \in G, \]

that is to show

Theorem 1. If a function \( f : G \to \mathbb{R} \) satisfies

\[ |\sup_{l \in L} f(x + l(y)) - f(x)f(y)| \leq \varepsilon, \quad x, y \in G, \]

then it is either bounded or is a solution of (1.1).

(Here we used the definition of superstability according to Z. Moszner’s Survey [3], 
Definition 4*). In paper [5] there was considered a particular case of (1.1), namely

\[ \max\{f(x+y), f(x-y)\} = f(x)f(y), \quad x, y \in G. \]

It has been proved that if \( G \) is divisible by 2 and 3, then the solutions \( f : G \to \mathbb{R} \) 
of (1.3) are \( f(x) = 0 \) and \( f(x) = \exp(|a(x)|) \) for some additive \( a : G \to \mathbb{R} \). In the 
present paper we will establish the solutions of (1.1) in another special case:

\[ \sup_{\lambda \in T} f(x + \lambda y) = f(x)f(y), \quad x, y \in V, \]

where \( V \) is a complex vector space and \( T = \{z \in \mathbb{C} : |z| = 1\} \). To do this, we need 
a result from [2]:

\[ f : V \to \mathbb{R} \] is a solution of

\[ \sup_{\lambda \in T} f(x + \lambda y) = f(x) + f(y), \quad x, y \in V \]

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if and only if there is a linear functional $\phi : V \to \mathbb{C}$, such that $f(x) = |\phi(x)|$.

Let us mention that the equations

$$\max\{f(x + y), f(x - y)\} = f(x) + f(y), \quad x, y \in G$$

and (1.5) are stable (both in the sense of Definition 1* and 2* from [3]), which follows from P. Volkmann [6] and from [4], respectively.

2. Stability of equation (1.1)

Before we turn to the proof of Theorem 1, we fix some notation. As in [4], we will write $A \simeq B$ instead of $|A - B| \leq \varepsilon$, whenever we find it convenient. Notice that inequality (1.2) can now be rewritten in the form

$$\sup_{l \in L} f(x + l(y)) \simeq f(x)f(y), \quad x, y \in G.$$  

**Proof of Theorem 1.** It is enough to show that if $f$ is unbounded then it satisfies

$$\sup_{l \in L} f(x + l(y)) = f(x)f(y), \quad x, y \in G.$$  

Since $f$ is unbounded there is a sequence $(w_n)_{n \in \mathbb{N}}$ such that $|f(w_n)| \nrightarrow \infty$, but taking into consideration that

$$\sup_{l \in L} f(w_n + l(w_n)) \simeq (f(w_n))^2 \nrightarrow \infty,$$

we infer that there is a sequence $(z_n)_{n \in \mathbb{N}}$ with

$$f(z_n) \nrightarrow \infty.$$  

Of course, we can assume that $f(z_n) > 0$ for every $n \in \mathbb{N}$. Suppose that $f(x) < 0$ for some $x \in G$. We have

$$\sup_{l \in L} f(x + l(z_n)) \simeq f(x)f(z_n) \nrightarrow -\infty,$$

whence, in particular,

$$f(x + z_n) \nrightarrow -\infty.$$  

Therefore,

$$f(x) \leq \sup_{l \in L} f(x + z_n + l(z_n)) \simeq f(x + z_n)f(z_n) \nrightarrow -\infty,$$

which is impossible. Assuming that $f(x) = 0$ for some $x \in G$ also leads to a contradiction. Namely, we would have

$$0 = f(z_n - x)f(x) \simeq \sup_{l \in L} f(z_n - x + l(x)) \geq f(z_n) \nrightarrow \infty.$$  

Hence we proved that $f > 0$.

In the rest of the proof we use ideas from [1]. Notice that

$$(2.1) \quad \sup_{l_1 \in L} f(x + l_1(y))f(z) \simeq \sup_{l_1 \in L} f(x + l_1(y) + l_2(z)) =$$

$$= \sup_{l_2 \in L} f(x + l_1(y) + l_2(z)) \simeq \sup_{l_2 \in L} f(x + l_2(z))f(y) =$$

$$= (\sup_{l_2 \in L} f(x + l_2(z))f(y) \leq f(x)f(z)f(y).$$
Thereby
\[ \sup_{l_1 \in L} f(x + l_1(y)) f(z)^{2x + \varepsilon f(y)} f(x) f(y), \quad x, y, z \in G. \]

Putting \( z = z_n \) and dividing by \( f(z_n) \) we get
\[ \sup_{l_1 \in L} f(x + l_1(y)) f(z_n)^{2x + \varepsilon f(y)} f(x) f(y), \quad x, y \in G, \quad n \in \mathbb{N}. \]

This implies that \( \sup_{l_1 \in L} f(x + l(y)) = f(x) f(y), \quad x, y \in G. \)

\[ \square \]

**Remark.** If \( f: G \to \mathbb{R} \) satisfying (1.2) is bounded, then
\[ |f(x)| \leq 1 + \sqrt{1 + 4 \varepsilon^2}, \quad x \in G. \]

**Proof.** Put \( M_0 = 1 + \sqrt{1 + 4 \varepsilon^2} \) and \( B = \sup_{x \in G} |f(x)|. \) Choose a sequence \( x_n \in G, \quad n \in \mathbb{N} \) with
\[ |f(x_n)| =: A_n \to B. \]

By (1.2) we have
\[ A_n^2 = f(x_n)^2 \leq \sup_{l_1 \in L} f(x_n + l(x_n)) \leq B, \quad n \in \mathbb{N}, \]
which results in \( A_n^2 \leq B + \varepsilon, \quad n \in \mathbb{N}. \) Therefore \( B^2 \leq B + \varepsilon, \) whence \( B \leq M_0. \)

\[ \square \]

### 3. Solution of equation (1.4)

**Lemma.** Let \( f: \mathbb{C} \to \mathbb{R} \) satisfy (1.4) (with \( V = \mathbb{C} \)) and suppose that \( f(0) = 1. \) Then \( f(z) > 0 \) for every \( z \in \mathbb{C}. \)

**Proof.** Notice that
\[ f(0) f(z) = \sup_{\lambda \in T} f(\lambda z) = f(0) f(|z|), \quad z \in \mathbb{C}, \]
whence
\[ (3.1) \quad f(z) = f(|z|), \quad z \in \mathbb{C}. \]

Using this we get
\[ f(a) f(b) = \sup_{\lambda \in T} f(b + \lambda \cdot a) = \sup_{\lambda \in T} f([b + \lambda \cdot a]), \]
\[ = \sup_{\lambda \in T} f([b - a, b + a]) \geq f(b), \quad 0 \leq a \leq b. \]

Furthermore, \( 0 \notin f(\mathbb{C}), \) since \( (f(z))^2 = \sup_{\lambda \in T} f(z + \lambda z) \geq f(0) = 1. \) Suppose that \( f(y) < 0 \) for some \( y \in \mathbb{C}. \) Owing to (3.1), we can assume that \( y \in (0, \infty). \) As follows from (3.2), \( f(x) < 0 \) for \( x > y. \) In particular, \( f(2y) < 0 \) and moreover
\[ 0 < f(2y) f(y) = \sup_{y, 3y} f([y, 3y]) \leq 0, \]
which is impossible.

\[ \square \]
**Theorem 2.** Let $f : V \to \mathbb{R}$ satisfy (1.4). Then either $f \equiv 0$ or $f(x) = \exp |\phi(x)|$, where $\phi : V \to \mathbb{C}$ is a linear functional.

**Proof.** Putting $x = y = 0$ in (1.4) we get $(f(0))^2 = f(0)$ whence either $f(0) = 0$ or $f(0) = 1$. In the first case we obtain

$$0 = f(x)f(0) = \sup_{\lambda \in T} f(x + \lambda \cdot 0) = f(x), \quad x \in V,$$

which means $f \equiv 0$. Now consider the other case, i.e., $f(0) = 1$. We will show that $f > 0$. Fix an arbitrary $x_0 \in V$. Define $f_{x_0} : \mathbb{C} \to \mathbb{R}$ by $f_{x_0}(\alpha) = f(\alpha x_0)$. It is easy to verify that $f_{x_0}$ is the solution of (1.4) with $V = \mathbb{C}$ and $f_{x_0}(0) = 1$. According to our Lemma we infer that $f_{x_0} > 0$, and, particularly, $f(x_0) = f_{x_0}(1) > 0$.

Put $g(x) := \log f(x)$ and notice that $g : V \to \mathbb{R}$ satisfies the equation (1.5). Hence, by the result in [2], already mentioned in the Introduction, we get $g(x) = |\phi(x)|$, for some linear functional $\phi : V \to \mathbb{C}$ and, therefore, $f(x) = \exp |\phi(x)|$. □

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**References**


Instytut Matematyki, Uniwersytet Śląski, Bankowa 14, 40-007 Katowice, Poland

E-mail address: przebieraczb@ux2.math.us.edu.pl