Series Mathematicae Catoviciensis et Debreceniensis, No. 32, 11 pp., 23. 06. 2010
http://www.math.us.edu.pl/smdk
(Manuscript received: 19. 02. 2010)

On functional equations in connection with the absolute value of additive functions

Witold Jarczyk and Peter Volkmann

1. Introduction. Let G be an abelian group. A function $a : G \to \mathbb{R}$ is *additive*, if the Cauchy functional equation

$$a(x+y) = a(x) + a(y) \qquad (x, y \in G)$$

is satisfied. It is easily seen that functions $f : G \to \mathbb{R}$ given by

$$(1) \qquad f(x) = |a(x)| \quad (x \in G), \quad \text{where } a : G \to \mathbb{R} \text{ is additive,}$$

fulfill the two functional equations

$$(2) \qquad \max\{f(x+y), f(x-y)\} = f(x) + f(y) \qquad (x, y \in G),$$

$$(3) \qquad \min\{f(x+y), f(x-y)\} = |f(x) - f(y)| \qquad (x, y \in G).$$

In a joint paper with Simon [14] it has been proved that (2) characterizes the functions (1); in the meantime there are new proofs by Fechner [4] and by Kochanek [8], cf. the general remarks in the next section. In a joint paper with Redheffer [13] a Pexider form of (2) has been solved, viz.

$$f(x) + g(y) = \max\{h(x+y), h(x-y)\} \qquad (x, y \in G)$$

for unknown functions $f, g, h : G \to \mathbb{R}$.

In the third section the stability of (2) will be shown: The proof was already presented during the 45th International Symposium on Functional Equations at Bielsko-Biała 2007; cf. [17]. Further stability results in connection with (2) are in Fechner's paper [4]. The stability of a generalization of (2) has been given in a joint paper with Gilányi and Nagatou [6], viz. the stability of

$$\max\{f((x \circ y) \circ y), f(x)\} = f(x \circ y) + f(y)$$

for real valued functions defined on a square-symmetric groupoid with a left unit element.

Some other recent results on stability should be mentioned, too. To explain them, let us first observe that in [14] also the equation

$$(4) \qquad \max\{f(x+y), f(x-y)\} = f(x)f(y) \qquad (x, y \in G)$$

has been considered. The functional equations (2), (3), (4) have complex analogues: Let V be a complex vector space, let $T = \{\chi \mid \chi \in \mathbb{C}, |\chi| = 1\}$ denote the unit circle in $\mathbb{C}$, and look at the equations

$$\sup_{\chi\in T} f(x+\chi y) = f(x) + f(y) \qquad (x, y \in V), \tag{5}$$

$$\inf_{\chi\in T} f(x+\chi y) = |f(x) - f(y)| \qquad (x, y \in V), \tag{6}$$

$$\sup_{\chi\in T} f(x+\chi y) = f(x)f(y) \qquad (x, y \in V) \tag{7}$$

for functions $f : V \to \mathbb{R}$. In a joint paper with Baron [3] it has been shown that the solutions of each of the equations (5), (6) are given by $f(x) = |\varphi(x)|$ $(x \in V)$, $\varphi : V \to \mathbb{C}$ being a linear functional, and from Przebieracz [11] it is known that the non-identically vanishing solutions of (7) are $f(x) = e^{|\varphi(x)|}$ $(x \in V)$, again $\varphi : V \to \mathbb{C}$ being linear.

According to Przebieracz [12] the functional equations (5), (6) are stable, and in [11] she gives a very general theorem, which implies the superstability of each of the equations (4), (7), the superstability being understood as in Moszner's survey [10].

The last section is devoted to (3) in the case $G = \mathbb{R}$, hence to

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)| \qquad (x, y \in \mathbb{R}) \tag{8}$$

for functions $f : \mathbb{R} \to \mathbb{R}$. In [14] a solution different from the functions (1) had been mentioned, viz. the continuous function f occuring below in (9) for $p = c = 1$. Here we determine all continuous solutions of (8): They are given by $f(x) = c|x|$ $(x \in \mathbb{R})$, where $c \geq 0$, and by

$$f(x+p) = f(x) \ (x \in \mathbb{R}), \qquad f(x) = c|x| \qquad (|x| \leq p/2), \tag{9}$$

where p, c are positive numbers.

This result was presented by the second author during the Conference on Inequalities and Applications '07 at Noszvaj 2007; unfortunately, neither title nor abstract of the talk are given in [1], nevertheless the abstract can be found in the internet, cf. [18].

Finally we get the continuous solutions of (8) under weaker regularity conditions; e.g., continuity at one point is sufficient.

According to Baron [2] a solution $f : \mathbb{R} \to \mathbb{R}$ of (8) is continuous, if it is Baire measurable in an open neighborhood of zero, and according to Kochanek and Lewicki [9] it is continuous, if it is Lebesgue measurable in an open neighborhood of zero. In fact, both papers [2] and [9] consider generalizations of (3) on some topological groups G.

Let us now show the existence of discontinuous solutions of (8) which are not absolute values of additive functions. For this, the following simple remark of Kochanek [7] is helpful: If $f : G \to \mathbb{R}$ solves (3) and $a : G \to G$ is additive, then $g(x) = f(a(x))$ $(x \in G)$ also solves (3). So, let us take a discontinuous, additive $a : \mathbb{R} \to \mathbb{R}$ and one of the functions $f : \mathbb{R} \to \mathbb{R}$ from (9) (p, c being positive). Then $g = f \circ a$ is a bounded, discontinuous solution of (8).

2. General remarks. The proof of the following simple theorem will be given without using results from the literature.

Theorem 1. *Let G be an abelian group. Then $f : G \to \mathbb{R}$ satisfies*

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \qquad (x, y \in G) \tag{2}$$

if and only if simultaneously

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)| \qquad (x, y \in G) \tag{3}$$

and

$$f(2x) = 2f(x) \qquad (x \in G) \tag{10}$$

hold true.

Proof. 1. Let f satisfy (2). Setting $y = x = 0$ gives $f(0) = 0$, and then $y = x$ leads to $f(x) \geq 0$ $(x \in G)$ and to (10). From the identity $\max\{\alpha, \beta\} + \min\{\alpha, \beta\} = \alpha + \beta$ $(\alpha, \beta \in \mathbb{R})$ we have

$$\max\{f(x+y), f(x-y)\} + \min\{f(x+y), f(x-y)\} = f(x+y) + f(x-y).$$

Applying (2) to both sides gives

$$f(x) + f(y) + \min\{f(x+y), f(x-y)\} = \max\{f(2x), f(2y)\},$$

and finally we use (10) to get

$$\min\{f(x+y), f(x-y)\} = 2\max\{f(x), f(y)\} - f(x) - f(y) = |f(x) - f(y)|.$$

Thus, (3) is shown.

2. Now let (3) and (10) hold. Here we use the identity $\max\{\alpha, \beta\} - \min\{\alpha, \beta\} = |\alpha - \beta|$ to get

$$\max\{f(x+y), f(x-y)\} - \min\{f(x+y), f(x-y)\} = |f(x+y) - f(x-y)|.$$

Applying (3) to both sides gives

$$\max\{f(x+y), f(x-y)\} - |f(x) - f(y)| = \min\{f(2x), f(2y)\},$$

and then we use (10) to get

$$\max\{f(x+y), f(x-y)\} = 2\min\{f(x), f(y)\} + |f(x) - f(y)| = f(x) + f(y).$$

This proves (2).

Kochanek [8] shows that a function $f : G \to \mathbb{R}$ solves (3), (10) if and only if (1) holds. Because of Theorem 1, this is a new proof of the corresponding result from [14] concerning the functional equation (2). Another new proof is due to Fechner [4]: Using a result of Ger [5] he shows that the functions (1) can be characterized as solutions of

$$(11) \qquad |f(x) - f(y)| = f(x+y) + f(x-y) - f(x) - f(y) \quad (x, y \in G)$$

having the property

$$(12) \qquad f(0) = 0.$$

Similarly to Theorem 1 we have

$$(11),\ (12) \iff (2).$$

Indeed, first of all (11) can be rewritten as

$$(13) \qquad 2\max\{f(x), f(y)\} = f(x+y) + f(x-y).$$

From this and (12), $y = x$ leads to (10), and then (13) gives

$$\max\{f(x+y), f(x-y)\} = \frac{1}{2}(f(2x) + f(2y)) = f(x) + f(y),$$

i.e., (2) holds. Conversely, (2) implies (12) as well as

$$f(x+y) + f(x-y) = \max\{f(2x), f(2y)\} = 2\max\{f(x), f(y)\},$$

i.e., (13) holds, hence also (11).

3. Stability of (2).

Theorem 2. *Let G be an abelian group, and let $g : G \to \mathbb{R}$ satisfy*

$$(14) \qquad |\max\{g(2x), g(0)\} - 2g(x)| \le \varepsilon \qquad (x \in G).$$

Then there exists a solution $f : G \to \mathbb{R}$ of

$$(15) \qquad \max\{f(2x), f(0)\} = 2f(x) \qquad (x \in G)$$

such that

$$(16) \qquad -3\varepsilon \le f(x) - g(x) \le \varepsilon \qquad (x \in G).$$

Moreover, f is given by

$$(17) \qquad f(x) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n x) \qquad (x \in G),$$

and this function is uniquely determined by (15) *and the requirement of* $f-g$ *to be bounded* (*hence by* (15) *and* (16)).

Proof. 1. With $x=0$ in (14) we have $|g(0)| \le \varepsilon$. Then (14) implies $-\varepsilon \le g(0) \le \varepsilon + 2g(x)$, which leads to

$$-\varepsilon \le g(x) \qquad (x \in G). \tag{18}$$

Using this with x replaced by $2x$, we have $g(0) \le \varepsilon = 2\varepsilon - \varepsilon \le 2\varepsilon + g(2x)$, and then (14) gives

$$2g(x) \le \varepsilon + \max\{g(2x), g(0)\} \le 3\varepsilon + g(2x).$$

Again using (14), we finally get

$$-3\varepsilon \le g(2x) - 2g(x) \le \varepsilon \qquad (x \in G). \tag{19}$$

2. Starting with (19), it is standard that $f : G \to \mathbb{R}$ given by (17) exists, this function satisfies

$$f(2x) = 2f(x) \qquad (x \in G) \tag{20}$$

as well as (16) (cf., e.g., [15]). Now (17), (18) imply $f(x) \ge 0$ for $x \in G$, therefore we get (15) from (20). On the other hand, (15) means just $f(2x) = 2f(x) \ge 0$ $(x \in G)$, and with this remark the uniqueness assertion in Theorem 2 easily follows.

Theorem 3. *Let* G *be an abelian group, and let* $g : G \to \mathbb{R}$ *satisfy*

$$|\max\{g(x+y), g(x-y)\} - g(x) - g(y)| \le \varepsilon \qquad (x, y \in G). \tag{21}$$

Then there exists a unique solution $f : G \to \mathbb{R}$ *of*

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \qquad (x, y \in G) \tag{2}$$

such that

$$-3\varepsilon \le f(x) - g(x) \le \varepsilon \qquad (x \in G). \tag{16}$$

Moreover, f *is given by* (17).

Proof. With $y = x$ in (21) we get (14), hence we have the function $f : G \to \mathbb{R}$ from Theorem 2. It remains to show that f solves (2). To do this, we write (21) with x, y replaced by $2^n x, 2^n y$, respectively, we divide by 2^n, and we let n tend to infinity; using (17) leads to (2).

Remark. Theorems 2, 3 show that the functional equations (15) and (2) are stable in the sense of Pólya-Szegő-Hyers-Ulam. Our main goal was the stability of (2), but we derived it from the stability of the single-variable equation obtained by taking $y = x$ in (2). The general method behind this has been presented during the 42nd International Symposium on Functional Equations at Opava 2004; cf. [16].

4. The continuous solutions of (8).

Theorem 4. a) *The continuous solutions* $f : \mathbb{R} \to \mathbb{R}$ *of*

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)| \qquad (x, y \in \mathbb{R}) \tag{8}$$

are given by $f(x) = c|x|$ $(x \in \mathbb{R})$, *where* $c \geq 0$, *and by*

$$f(x+p) = f(x)\ (x \in \mathbb{R}), \quad f(x) = c|x| \quad (|x| \leq p/2), \tag{9}$$

where p, c *are positive numbers.*

b) *If a solution* $f : \mathbb{R} \to \mathbb{R}$ *of* (8) *is continuous at zero, then it is continuous at every point* x *from* $\mathbb{R}$.

Proof. 1. Let f be an arbitrary solution of (8). It is easily seen that

$$f(-x) = f(x) \geq 0 = f(0) \qquad (x \in \mathbb{R}).$$

Furthermore, every zero of f is a period of this function, i.e.,

$$f(p) = 0 \Rightarrow f(x+p) = f(x) \qquad (x \in \mathbb{R}).$$

Indeed, from $f(p) = 0$ we get for $x \in \mathbb{R}$ that

$$f(x) = f(x) - f(p) = \min\{f(x+p), f(x-p)\} \leq f(x+p),$$

and because of $f(-p) = 0$ we have analogously

$$f(x+p) \leq f(x+p+(-p)) = f(x).$$

Let us observe that (8) implies

$$|f(x) - f(y)| \leq f(x-y) \qquad (x, y \in \mathbb{R}), \tag{22}$$

which also can be written as

$$|f(y+z) - f(y)| \leq f(z) \qquad (y, z \in \mathbb{R}). \tag{23}$$

Part b) now follows from (22) by taking the limit $y \to x$; we have $f(x-y) \to f(0) = 0$.

2. It remains the proof of a). Let us first mention that all the functions given in a) are indeed continuous solutions of (8). From now on let $f : \mathbb{R} \to \mathbb{R}$ be a continuous solution of (8).

Case I: $f(x) \neq 0$ for $x \neq 0$.

Then f is strictly increasing on $[0, \infty[$, for otherwise $f(x) = f(y)$ would be possible, where $0 < y < x$, and (8) would imply $f(x+y) = 0$ or $f(x-y) = 0$, which gives a contradiction. For $0 < y < x$ we now have $0 < x - y < x + y$, hence $f(x-y) < f(x+y)$, and (8) leads to

$$f(x-y) = f(x) - f(y) \qquad (0 \leq y \leq x).$$

f being continuous, we thus have $f(x) = cx$ $(x \geq 0)$, and finally we get $f(x) = c|x|$ $(x \in \mathbb{R})$, where $c > 0$.

Case II: $f(x) = 0$ for some $x \neq 0$ happens.

Without loss of generality we suppose $f(x) \not\equiv 0$. Now f has positive zeros. They are periods of the continuous function f, hence there is a smallest zero $p > 0$. This implies

$$f(0) = f(p) = 0, \quad f(x) > 0 \qquad (0 < x < p). \tag{24}$$

We can show:

$$f \text{ is strictly increasing on } [0, p/2]. \tag{25}$$

Indeed, otherwise $f(x) = f(y)$ would be possible, where $0 < y < x \leq p/2$. Then (8) implies $f(x+y) = 0$ or $f(x-y) = 0$, and because of $0 < x+y < p$ and $0 < x - y < p$ we arrive at a contradiction to (24).
From (8), (24) we get $\left|f\left(\frac{p}{2} + x\right) - f\left(\frac{p}{2} - x\right)\right| = \min\{f(p), f(2x)\} = 0$, which gives

$$f\left(\frac{p}{2} + x\right) = f\left(\frac{p}{2} - x\right) \qquad (x \in \mathbb{R}). \tag{26}$$

Consider $0 \leq y \leq x \leq p/4$. Then $0 \leq x - y \leq x + y \leq p/2$, and (25) implies $f(x-y) \leq f(x+y)$. We also have $f(y) \leq f(x)$, therefore (8) leads to $f(x) - f(y) = \min\{f(x+y), f(x-y)\} = f(x-y)$, hence we have shown

$$f(x) - f(y) = f(x-y) \qquad (0 \leq y \leq x \leq p/4).$$

f being continuous, we thus get

$$f(x) = cx \qquad (0 \leq x \leq p/4), \tag{27}$$

c being a positive number.
For $0 \leq y \leq p/4$ we obtain from (26), (8), (25), (27)

$$f\left(\frac{p}{2} - y\right) = \min\left\{f\left(\frac{p}{2} + y\right), f\left(\frac{p}{2} - y\right)\right\} = f\left(\frac{p}{2}\right) - f(y) = f\left(\frac{p}{2}\right) - cy.$$

When substituting $x = (p/2) - y$ $(p/4 \leq x \leq p/2)$, we see that on the interval $[p/4, p/2]$ the function f represents a straight line with slope c. Because of (27) we thus have $f(x) = cx$ $(0 \leq x \leq p/2)$; from this and from $f(x+p) = f(x) = f(-x)$ $(x \in \mathbb{R})$ we finally get (9).

In the proofs of the next two theorems we shall use the set $\mathbb{N} = \{1, 2, 3, \dots\}$ and the sequence space

$$c_0 = \{(y_n)_{n\in\mathbb{N}} \mid y_n \in \mathbb{R}\ (n \in \mathbb{N}),\ y_n \to 0\ (n \to \infty)\}.$$

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ solve*

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)| \qquad (x, y \in \mathbb{R}), \tag{8}$$

and suppose f to be continuous at a point x_0 from $\mathbb{R}$. Then f is continuous.

Proof. 1. Because of Theorem 4b) it is sufficient to check the continuity of f at zero. We have

$$|f(x_0) - f(y)| = \min\{f(x_0+y), f(x_0-y)\} \to f(x_0) \qquad (y \to 0), \tag{28}$$

and if $f(x_0) = 0$, then we get $f(y) \to 0$.

2. Now we suppose

$$f(x_0) \neq 0. \tag{29}$$

For $(y_n)_{n\in\mathbb{N}} \in c_0$ we get from (28) that $(f(y_n))_{n\in\mathbb{N}}$ is a bounded sequence with at most two accumulation points, namely zero and $2f(x_0)$. We shall exclude the second case, then we are done. Let us first observe that for $(y_n), (z_n) \in c_0$ we have

$$f(y_n) \to 0,\ f(z_n) \to 0 \Rightarrow f(y_n + z_n) \to 0;$$

this follows from (23). As a simple consequence we get:

$$(z_n) \in c_0,\ f(z_n) \to 0, k \in \mathbb{N} \Rightarrow f(kz_n) \to 0 \quad (n \to \infty). \tag{30}$$

3. To finish the proof, we like to rule out the existence of $(y_n) \in c_0$ such that

$$f(y_n) \to 2f(x_0). \tag{31}$$

(If $2f(x_0)$ is an accumulation point of $(f(y_n))_{n\in\mathbb{N}}$, we always arrive at (31) by taking an appropriate subsequence, if necessary.) Consider $k \in \mathbb{N}$. Our assumption (31) implies

$$f\left(\frac{1}{k}y_n\right) \to 2f(x_0) \qquad (n \to \infty), \tag{32}$$

for otherwise zero would be an accumulation point of $\left(f\left(\frac{1}{k}y_n\right)\right)_{n\in\mathbb{N}}$, and we could apply (30) to show that zero also is an accumulation point of the sequence $(f(y_n))$, which contradicts (31).

Now (32) implies

$$f\left(\frac{1}{2}y_n\right) \to 2f(x_0),\ f\left(\frac{1}{3}y_n\right) \to 2f(x_0),\ f\left(\frac{1}{6}y_n\right) \to 2f(x_0).$$

From (8) we get

$$\left|f\left(\frac{1}{3}y_n\right) - f\left(\frac{1}{6}y_n\right)\right| = \min\left\{f\left(\frac{1}{2}y_n\right), f\left(\frac{1}{6}y_n\right)\right\} \qquad (n \in \mathbb{N}),$$

and $n \to \infty$ leads to $0 = 2f(x_0)$, which contradicts (29).

The next theorem is a special case of those results from Baron [2] and from Kochanek and Lewicki [9], which have been mentioned in the Introduction; its proof is simple.

Theorem 6. *Let $f : \mathbb{R} \to \mathbb{R}$ solve*

$$\text{(8)} \qquad \min\{f(x+y), f(x-y)\} = |f(x) - f(y)| \qquad (x, y \in \mathbb{R}),$$

and suppose there is an open neighborhood U of zero, where f is bounded and lower semicontinuous. Then f is continuous.

Proof. Because of Theorem 4b) it is sufficient to check the continuity of f at zero. The boundedness of f in U implies $\gamma := \overline{\lim}_{x\to 0} f(x)$ to be finite. We choose $(x_n)_{n\in\mathbb{N}} \in c_0$ such that $f(x_n) \to \gamma$. Without loss of generality we assume $x_n \in U$ $(n \in \mathbb{N})$. For fixed $n \in \mathbb{N}$ we have

$$\underline{\lim}_{k\to\infty} f(x_n + x_k) \geq \underline{\lim}_{x\to x_n} f(x) \geq f(x_n)$$

and

$$\underline{\lim}_{k\to\infty} f(x_n - x_k) \geq \underline{\lim}_{x\to x_n} f(x) \geq f(x_n),$$

hence, by (8),

$$|f(x_n) - \gamma| \geq f(x_n).$$

Now $n \to \infty$ yields $\gamma = 0$.

Acknowledgement. This paper is based on joint research mainly during the second author's stay at the University of Zielona Góra in winter 2007/2008, and it had been written down during his stay at the Silesian University at Katowice in the academic year 2008/2009. He gratefully acknowledges the hospitality of both Universities.

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Typescript: Marion Ewald.

Authors' addresses:

W. Jarczyk, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, Szafrana 4a, 65-516 Zielona Góra, Poland.
(e-mail: w.jarczyk@wmie.uz.zgora.pl).

P. Volkmann, Institut für Analysis, Universität, 76128 Karlsruhe, Germany; and: Instytut Matematyki, Uniwersytet Śląski, Bankowa 14, 40-007 Katowice, Poland.