Theory of Means:
Comparison, Equality, Homogeneity, Characterization

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1. Introduction

The systematic treatment of the theory of means started with the book [14] by Hardy, Littlewood and Pólya on Inequalities. In this book, the notion of quasi-arithmetic means was introduced and its comparison, equality and homogeneity problem was completely solved. The solution to these problems require the characterizations of Jensen convexity and some basic facts from the theory of functional equations. The characterization theorems of quasi-arithmetic means that were found independently by Kolmogorov [15], Nagumo [29] and de Finetti [12] also enlightened the importance and the applicability of quasi-arithmetic means. This explains why these theories are interdisciplinary between the fields of inequalities, functional equations, and also probability theory. All these notions and results have had an enormous impact on the theory of means, and starting from the fifties of the last century, hundreds of papers and dozens of monographs (e.g., Kuczma [16], Robert–Varberg [36], Mitrinović [27], Bullen–Mitrinović–Vasić [6], Mitrinović–Pečarić–Fink [28]) were published on this subject. In these works, several generalizations of the notion of quasi-arithmetic means were invented and their comparison, equality and homogeneity theorems got far reaching extensions.

Given an interval $I \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, a function $M_n : I^n \to I$ is called an $n$-variable mean if, for all $(x_1, \ldots, x_n) \in I^n$, the following inequality holds

$$\min(x_1, \ldots, x_n) \leq M_n(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n).$$

If both inequalities are strict whenever $\min(x_1, \ldots, x_n) < \max(x_1, \ldots, x_n)$, then $M_n$ called a strict $n$-variable mean. If $M_n$ is symmetric and continuous on $I^n$, then it is called a symmetric and continuous $n$-variable mean, respectively.

A function $M : \bigcup_{n=1}^{\infty} I^n = I$ is called a (strict, symmetric, continuous) mean if, for all $n \in \mathbb{N}$, the restriction $M_n$ of $M$ to the Cartesian product $I^n$ is a (strict, symmetric, continuous) $n$-variable mean.

The basic problems of the theory of means can be described as follows. Given a class of $n$-variable means $\mathcal{M}_n(I)$ (of means $\mathcal{M}(I)$ defined on the interval $I$, the comparison problem in $\mathcal{M}_n(I)$ (in $\mathcal{M}(I)$) is to find necessary and sufficient conditions in order that, for two given $n$-variable means $M_n, N_n \in \mathcal{M}_n(I)$ (for two given means $M, N \in \mathcal{M}(I)$), for all $x_1, \ldots, x_n \in I$ (and for all $n \in \mathbb{N}$), the comparison inequality

$$M_n(x_1, \ldots, x_n) \leq N_n(x_1, \ldots, x_n)$$

be valid. The equality problem in $\mathcal{M}_n(I)$ (in $\mathcal{M}(I)$) can be described analogously.

An $n$-variable mean $M_n$ is called homogeneous on $I$ if, for all $x_1, \ldots, x_n \in I$ and $t \in \mathbb{R}$ with $tx_1, \ldots, tx_n \in I$ the equality

$$M_n(tx_1, \ldots, tx_n) \leq tM_n(x_1, \ldots, x_n)$$

be valid. The equality problem in $\mathcal{M}_n(I)$ (in $\mathcal{M}(I)$) can be described analogously.
holds. A mean $M$ is called homogeneous on $I$ if, for all $n \in \mathbb{N}$, the restriction $M_n$ of $M$ to the Cartesian product $I^n$ is a homogeneous $n$-variable mean. The problem of homogeneity in a given class of $n$-variable means $M_n(I)$ (in a given class of means $M(I)$) is to determine all $n$-variable homogeneous means of $M_n(I)$ (all homogeneous means of $M(I)$).

Finally, we mention the characterization problem of a class of $n$-variable means $M_n(I)$ (of a class of means $M(I)$), which is to find a system of functional equations or a system of functional inequalities and additional regularity, monotonicity, convexity properties which is satisfied exactly by the elements of $M_n(I)$ (of $M(I)$).

The aim of the series of 5 lectures is to give an account and overview of results and open problems in several classes of means. In the first lecture, we deal with the classes of Hölder (power) means and quasi-arithmetic means. The problems of comparison, equality, homogeneity and characterization are completely solved in these two classes. The presentation is based on the book [14] and the papers by Kolmogorov [15] and Aczél [12]. It turns out the homogeneous means among quasi-arithmetic means on the interval of positive real numbers are exactly the Hölder means. In the second lecture, we consider a possible generalization of quasi-arithmetic means which was introduced by Matkowski in [26] in 2010, where he also described the solution of the equality problem. We present the solution to the comparison problem which requires new techniques, such as the use of the Brower Fixed Point Theorem. In the third lecture, we recall the notion of Bajraktarević means introduced in [4, 5] and we clarify its connection to Chebyshev systems. We also recall the notion of Daróczy means that are defined via deviations. Based on the results of the papers by Aczél and Daróczy [3], Daróczy [7, 8], Daróczy–Losonczi [9], Daróczy and Páles [10, 11], Páles [30, 31] the problems of comparison, equality, homogeneity and characterization, will also be completely described. It is important to mention that, in view of the result of Aczél and Daróczy, the homogeneous means of the class of Bajraktarević means turn out to be the Gini means introduced by Gini in [13]. In the fourth lecture we consider two-variable generalizations of quasi-arithmetic means and Bajraktarević means that are defined in terms of a Borel probability measure. We discuss their equality, comparison and homogeneity problems based on the results by Losonczi [17, 18, 19, 20, 21, 22, 23], Losonczi and Páles [24], Makó and Páles [25]. In the last lecture, we consider the comparison problem of two parametric classes of homogeneous means, these are the two-variable Gini means and Stolarsky means [37, 38]. The approach is based on the papers by Páles [32, 33, 34, 35]. We will emphasize the similarities and the differences and also the dependence of these comparison theorems on the underlying subinterval of positive real numbers. Several open problems will be formulated as well.

References


