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ON SUBSPACES OF Exp(N)

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Let $\exp(X)$ denote the exponential space of a topological space X introduced by Vietoris [14]. In this paper, we study the subspaces of the space $\exp(N)$, where $N = \{1, 2, 3, \ldots\}$ is the discrete space of natural numbers. We also show that if the metrizability number of $\exp(X)$ is countable, then X (and $\exp(X)$) must be compact and metrizable.

1. Preliminaries.

For every topological space X, let $\exp(X)$ denote the exponential space of X with the Vietoris topology (see the definition below or [3]; 2.7.20). Also, recall that the *metrizability number* of a space X, denoted by $m(X)$, is the smallest cardinal κ such that X can be represented as a union of κ many metrizable subspaces. It was shown (cf. [5]; Corollary 5.3) that if X is a compact Hausdorff space and $m(\exp(X)) \leq \omega$, then X and $\exp(X)$ are metrizable. We will show below (cf. Corollary 27) that the assumption in this theorem that X be compact is redundant. This is shown by first determining the metrizability number of the space $\exp(N)$, where $N\{1, 2, 3, \ldots\}$ is the discrete space of natural numbers. This motivated us to study the space $\exp(N)$ and some of its subspaces in more detail.

Topological properties of the space $\exp(N)$ have already been investigated by many authors, for instance Michael [10], Keesling [7], Ellentuck [2], Plewik [11] or [12]. It is known, for example, that the space $\exp(N)$

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is first countable, zero-dimensional, completely regular, but not normal. In this note we prove more topological properties of the space $\exp(N)$.

2. Notation.

Throughout this paper, let $N = \{1, 2, 3, \ldots\}$ denote the discrete space of natural numbers.

Let X be an arbitrary set. Following the standard notation we set:

$$2^X = \{Y : Y \subseteq X\}.$$

$$[X]^\omega = \{Y : |Y| \geq \omega\},$$

$$[X]^{<\omega} = \{Y : |Y| < \omega\}.$$

A *filter* on X is a non-empty family $\mathcal{F}$ of subsets of X satisfying the following two conditions:

(1) if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$;

(2) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

If $\emptyset \notin \mathcal{F}$, then the filter $\mathcal{F}$ is called a *Proper filter*; if $\emptyset \in \mathcal{F}$, then $\mathcal{F} = 2^X$.

A pair A, B of subsets of X is said to be *almost disjoint* if $A \cap B$ is a finite set.

A set A is said to be *almost contained* in a set B if $A \backslash B$ is a finite set.

For any pair F, A of subsets of X we set:

$$\langle F; A\rangle = \{B \subseteq X : F \subseteq B \subseteq A\}.$$

For any ordered triple A, x, y of subsets of X we set:

$$A(x, y) = x \cap (A \backslash y).$$

For any set X and a filter $\mathcal{F}$ on X we set $T_{\mathcal{F}}$ to be the topology on 2^X generated by the base consisting of sets of the form

$$\langle x, A(x, y)\rangle,$$

where $x, y \in [X]^{<\omega}$ and $A \in \mathscr{F}$. In the case $X = N$, the topologies $T_{\mathscr{F}}$ were studied by A. Louveau (cf. [9]). In the whole spectrum of the topologies $T_{\mathscr{F}}$, there are two topologies that could be thought of as standing at the oposite ends: one, when $\mathscr{F} = \{X\}$, and the other one when $\mathscr{F} = 2^X$. The former is the product (= Cantor) topology and the latter is the Vietoris topology (see the definition, below).

For every topological space X, let $\exp(X)$ denote the set of all non-empty closed subsets of the space X endowed with the Vietoris topology, i.e., topology generated by the base consisting of sets of the form

$$\langle U_1, U_2, \ldots, U_n \rangle = V_1 \cap V_2,$$

where $U_1, U_2, \ldots, U_n$ are open subsets of X and

$$V_1 = \{E \in \exp(X) : E \subseteq U_1 \cup U_2 \cup \ldots \cup U_n\}$$

and

$$V_2 = \{E \in \exp(X) : E \cap U_i \neq \emptyset \ \text{ for each } \ i = 1, 2, \ldots, n\}.$$

Thus, in the case $X = N$, the Vietoris topology on $\exp(N)$ is the topology generated by the base consisting of sets of the form

$$\langle F; A \rangle$$

where $A \in \exp(N)$ and F is a finite subset of the set A.

Equivalently, the Vietoris topology on $\exp(N)$ is the topology generated by the base consisting of sets of the form

$$W_n(A) = \langle \{a_1\}, \{a_2\}, \ldots, \{a_n\}, A \rangle = \{B \subseteq N : \{a_1, a_2, \ldots, a_n\} \subseteq B \subseteq A\},$$

where $a_1, a_2, \ldots, a_n$ are the first n elements of the set A with respect to the natural ordering of the set N. Obviously, the sets $W_1(A)$, $W_2(A), \ldots$ form a base at the point $A \in \exp(N)$.

Let $[N]^\omega$ denote the subspace of $\exp(N)$ consisting of infinite subsets of N. For each $A \in [N]^\omega$, let

$$V_n(A) = W_n(A) \cap [N]^\omega.$$

We refer to these sets as the basic open subsets of the space $[N]^\omega$.

3. Introductory Results.

LEMMA 1. *If* $B \in V_n(A)$*, then the first* n *elements of the set* B *are exactly the same as the first* n *elements of the set* A.

Proof. Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$, where $a_1 < a_2 < \ldots < a_n < \ldots$, and $B = \{b_1, b_2, \ldots, b_m, \ldots\}$, where $b_1 < b_2 < \ldots < b_m < \ldots$. Since $\{a_1, a_2, \ldots, a_n\} \subseteq B \subseteq A$, $\{1, 2, \ldots, a_n\} \cap B = \{1, 2, \ldots, a_n\}$ and therefore $a_1 = b_1, \ldots, a_n = b_n$. □

LEMMA 2. *Let* $A = \{a_1, a_2, \ldots, a_n, \ldots\}$*, where* $a_1 < a_2 < \ldots < a_n < \ldots$*, and* $C = \{c_1, c_2, \ldots, c_m, \ldots\}$*, where* $c_1 < c_2 < \ldots < c_m < \ldots$ *If* $n \leq m$ *and* $V_n(A) \cap V_m(C) \neq \emptyset$*, then* $a_i = c_i$ *for each* $i = 1, 2, \ldots, n$*, and* $c_i \in A$ *for each* $i = n+1, n+2, \ldots, m$.

Proof. Let $B \in V_n(A) \cap V_m(C)$. If $B = \{b_1, b_2, \ldots, b_m, \ldots\}$, where $b_1 < b_2 < \ldots < b_m < \ldots$, then, by Lemma 1, $a_1 = b_1 = c_1, \ldots, a_n = b_n = c_n$ and $c_{n+1} = b_{n+1}, \ldots, c_m = b_m$. Since $B \subseteq A$, $c_i \in A$ for each $i = n+1, n+2, \ldots, m$. □

LEMMA 3. *If* $B \in \cap\{V_k(A_k) : k = 1, 2, \ldots\}$*, then* $B = \cap\{A_k : k \in M\}$ *for each* $M \in [N]^\omega$.

Proof. Since $B \in V_k(A_k)$ for each $k = 1, 2, \ldots, B \subseteq \cap\{A_k : k \in M\}$. To prove the converse inclusion, let $m \in \cap\{A_k : k \in M\}$. Choose $k \in M$ such that $k \geq m$. Let $A_k = \{a_1^k, a_2^k, \ldots\}$, where $a_1^k < a_2^k < \ldots$ Since the mth element of an arbitrary subset of N is always $\geq m$, $m \leq a_m^k \leq a_k^k$. Since $m \in A_k$, $m \in \{a_1^k, a_2^k, \ldots, a_m^k, \ldots, a_k^k\}$. Since $B \in V_k(A_k)$, $\{a_1^k, a_2^k, \ldots, a_k^k\} \subseteq B$, and hence $m \in B$. □

LEMMA 4. *Let* $n_1 < n_2 < \ldots$ *be an increasing sequence of natural numbers. Let* U_k *denote the basic open set* $V_{n_k}(A_k)$ *for some infinite set* $A_k \subseteq N$*. If any two elements of the family* $\{U_k : k = 1, 2, \ldots\}$ *have non-empty intersection, then* $\cap\{U_k : k = 1, 2, \ldots\}$ *is a one-point set.*

Proof. For each k, let $A_k = \{a_1^k, a_2^k, \ldots\}$, where $a_1^k < a_2^k < \ldots$ Let

$$B = \cup\{\{a_1^k, a_2^k, \ldots, a_{n_k}^k\} : k = 1, 2, \ldots\}.$$

We shall show that

$$\cap\{U_k : k = 1, 2, \ldots\} = \{B\}.$$

To prove that $B \in \cap\{U_k : k = 1, 2, \ldots\}$, let $k \in N$ and let us show that $B \in U_k$, which is to show that $\{a^k_1, a^k_2, \ldots, a^k_{n_k}\} \subseteq B \subseteq A_k$. The first inclusion follows immediately from the definition of the set B. To prove the second inclusion let $b \in B$. There exists m such that $b = a^m_j$ for some $j \leq n_m$. Let $l \geq \max\{k, m\}$. Since $n_m \leq n_l$ and $U_m \cap U_l \neq \emptyset$, $b = a^m_j = a^l_j$, by Lemma 2. Since $n_k \leq n_l$ and $U_l \cap U_k \neq \emptyset$, $a^l_j = a^k_j$ or $a^l_j \in A_k$ depending on whether $j \leq n_k$ or not. Since $b = a^l_j$, $b \in A_k$.

The fact that the intersection is a one-point set follows immediately from Lemma 3. □

PROPOSITION 5. *Let $\mathscr{S}$ be a family of basic open sets. Then $\cap\mathscr{S}$ is the empty set or $\cap\mathscr{S}$ is a one-point set or $\cap\mathscr{S}$ is a basic open set.*

Proof. Suppose that $\cap\mathscr{S}$ is non-empty. If for each $n \in N$ there exist $m \geq n$ and $A \subseteq N$ such that $V_m(A) \in \mathscr{S}$, then $\cap\mathscr{S}$ is a one-point set according to the above lemma. Otherwise, there exist $m \in N$ and $B \subseteq N$ such that $V_m(B) \in \mathscr{S}$ and, if $V_n(A) \in \mathscr{S}$, then $n \leq m$. Let

$$C = \cap\{A : V_n(A) \in \mathscr{S}, \quad \text{for some} \quad n \in N\}.$$

Claim. The sets B and C have the same first m elements.

Proof of the claim. Let $V_n(A) \in \mathscr{S}$. Since $V_n(A) \cap V_m(B) \neq \emptyset$, by Lemma 2, the first m elements of B are contained in A. Hence the first m elements of B are contained in C. Since $C \subseteq B$, B and C have the same first m elements.

We shall show that

$$\cap\mathscr{S} = V_m(C).$$

Let $D \in V_m(C)$, and let $U \in \mathscr{S}$. Then $U = V_n(A)$ and so $n \leq m$. Clearly, $D \subseteq A$. Since $V_n(A) \cap V_m(B) \neq \emptyset$, by Lemma 2, the first n elements of A are the same as the first n elements of B. By the above Claim, the first n elements of A are the same as the first n elements of C. Hence $D \in U$, which shows that $\cap\mathscr{S} \supseteq V_m(C)$.

To prove the converse inclusion, let $D \in \cap\mathscr{S}$. Then $D \subseteq \cap\{A : V_n(A) \in \mathscr{S}\} = C$. Since $V_m(B) \in \mathscr{S}$, $D \in V_m(B)$. Thus D and B

have the same first m elements. Hence by the above claim, D and C must have the same first m elements. Therefore $D \in V_m(C)$, which shows that $\cap \mathscr{S} \subseteq V_m(C)$. □

THEOREM 6. *The space $[N]^\omega$ is a Baire space.*

Proof. Let U be a non-empty open subset of the space $[N]^\omega$, and let $Z_1, Z_2, \ldots$ be nowhere dense subsets of $[N]^\omega$. By induction, one can choose basic open sets $V_{n_k}(A_k)$, $k = 1, 2, \ldots,$ such that $U \supseteq V_{n_1}(A_1) \supseteq V_{n_2}(A_2) \supseteq \ldots$, $n_1 < n_2 < \ldots$, and $V_{n_k}(A_k) \cap Z_k = \emptyset$. By Lemma 4, $U \setminus \cup \{Z_k : k = 1, 2, \ldots\} \neq \emptyset$. □

PROPOSITION 7. *If A_n, $A \in \exp(N)$, then $A = \lim_{n\to\infty} A_n$ if and only if for each $n \in N$ all but finitely many elements of the sequence $\{A_k\}$ are subsets of A and they have the same first n elements as those of the set A.*

Proof. Suppose that $A = \lim_{n\to\infty} A_n$ and let $n \in N$. There exists a $k \geq n$ such that $A_m \in V_n(A)$ for each $m \geq k$. Then for each $m \geq k$, $A_m \subseteq A$ and the first n elements of A_m are the same as the first n elements of the set A.

To prove the converse statement, let U be an arbitrary open neighborhood of A. There exists n such that $V_n(A) \subseteq U$. Since all but finitely many elements of the sequenc $\{A_k\}$ are subsets of A and they have the same first n elements as those of the set A, all but finitely many elements of the sequence $\{A_k\}$ are elements of $V_n(A) \subseteq U$. Thus $A = \lim_{n\to\infty} A_n$. □

PROPOSITION 8. *If the set differences $N\setminus B$, $N\setminus A_1$, $N\setminus A_2, \ldots$ are infinite and $B \in V_n(A_n)$ for each $n \in N$, then $N\setminus B = \lim_{n\to\infty}(N\setminus A_n)$.*

Proof. Since $B \in \cap\{V_k(A_k) : k = 1, 2, \ldots\}$, by Lemma 3, $B = \cap\{A_k : k = 1, 2, \ldots\}$. Thus $N\setminus B = \cup\{N\setminus A_k : k = 1, 2, \ldots\}$. Let $n \in N$ and let m be the nth element of $N\setminus B$. Then the first $m - n$ elements of the set B are contained in the set $\{1, 2, \ldots, m\}$. Since the set B and every set A_k have the same first $m - n$ elements for each $k \geq m - n$, the set $N\setminus B$ and every set $N\setminus A_k$ have the same first n elements for each $k \geq m - n$. Hence, by Proposition 7, $N\setminus B = \lim_{n\to\infty}(N\setminus A_n)$. □

LEMMA 9. *If* $B \in \cap\{V_k(A_k) : k = 1, 2, \ldots\}$ *and the sequence* $\{A_k\}$ *is convergent in* $[N]^\omega$*, then* $B = A_n$ *for all but finitely many* $n \in N$.

Proof. Suppose that $A \in [N]^\omega$ and that $A = \lim_{k \to \infty} A_k$.

Claim. $B = A$.

Indeed, let m be an arbitarry member of B. Suppose it is the ith element of the set B. Let n_k be such that $n_k \geq i$ and $A_{n_k} \subseteq A$. Since $B \in V_{n_k}(A_{n_k})$, $m \in A_{n_k} \subseteq A$. Thus $B \subseteq A$. To prove the converse inclusion, take an arbitrary element m of A. By Proposition 6, there exists k such that $m \in A_{n_k}$ and $n_k \geq m$. Since $B \in V_{n_k}(A_{n_k})$, $m \in B$. Thus $A \subseteq B$ and the Claim is proved.

Let m be such that $A_n \subseteq A$ for each $n \geq m$. Since $B \in V_n(A_n)$, $B \subseteq A_n \subseteq A = B$. Hence $B = A_n$ for each $n \geq m$. □

THEOREM 10. (V. Popov) *Every countably compact subset of the space* $\exp(N)$ *is countable.*

Proof. Let Y be the set $\exp(N)$ endowed with the Cantor set topology. The idnetity map $i : \exp(N) \to Y$ is continuous. Hence the function i restricted to any countably compact subspace of $\exp(N)$ is a homeomorphism. Since any countably compact subspace of the Cantor set Y is compact, any countably compact subspace of $\exp(N)$ is also compact. Thus the theorem will be shown if we prove that any compact subspace of $\exp(N)$ is countable.

Let Z be a compact subspace of the space $\exp(N)$. For each $n \in N$, the family $\{W_n(A) : A \in Z\}$ is an open cover of Z. Hence for each $n \in N$, there is a finite subset F_n of Z such that $Z \subseteq \cup\{W_n(A) : A \in F_n\}$. We shall show that $Z \subseteq \cup\{F_n : n \in N\}$.

Let $B \in Z$. Then for each $n \in N$ there exists $A_n \in F_n$ such that $B \in W_n(A_n)$. Since Z (being compact and first countable) is sequentially compact, the sequence $\{A_n\}$ has a subsequence that converges to a point of Z. Thus B is one of the sets A_n by virtue of Lemma 9. □

A family D of subsets of N is almost disjoint if for each $A, B \in D$, $A \neq B$, the intersection $A \cap B$ is a finit set.

LEMMA 11. *The cellularity of* $[N]^\omega$, $c([N]^\omega)$, *equals* 2^ω.

Proof. Let $D \subset [N]^\omega$ be an almost disjoint family of cardinality 2^ω. Then $\{\langle \emptyset, A\rangle \cap [N]^\omega : A \in D\}$ is a disjoint family of open subsets of the space $[N]^\omega$ and the cardinality of this family equals 2^ω. □

LEMMA 12. *Let* $A, B \in [N]^\omega$ *and let* $F \in [N]^\omega$ *be such that* $F \subseteq A \cap B$. *If* $\langle F, A\rangle \cap \langle F, B\rangle \cap [N]^\omega = \emptyset$, *then* A *and* B *are almost disjoint.*

Proof. If $A \cap B$ were infinite, then $A \cap B$ would belong to $\langle F, A\rangle \cap \langle F, B\rangle \cap [N]^\omega$. □

THEOREM 13. *Every subspace of* $[N]^\omega$ *of cellularity* $< 2^\omega$ *is nowhere dense in* $[N]^\omega$.

Proof. Let Z be a subspace of $[N]^\omega$ and $c(Z) < 2^\omega$. Assume that Z is not nowhere dense in $[N]^\omega$. Then there exists a basic open set V such that $Z \cap V$ is dense in V. Then $c(Z \cap V) < 2^\omega$ and therefore $c(V) < 2^\omega$. Since V is homeomorphic to $[N]^\omega$, this contradicts Lemma 11. □

DEFINITION 14. *Let* h *be the smallest cardinal* κ *for which there exists a collection* $\mathscr{H} = \{D_\alpha : \alpha < \kappa\}$ *such that the following conditions are satisfied:*

1. *Each* D_α *is a maximal almost disjoint family contained in* $[N]^\omega$;
2. *For each* $A \in [N]^\omega$ *there exists* $B \in \cup\mathscr{H}$ *such that* B *is almost contained in* A.

The cardinal h was introduced by Balcar, Pelant and Simon [1]. Therein it was shown that there exists a family $\mathscr{H} = \{D_\alpha : \alpha < h\}$ that, in addition to (1) and (2), also satisfies

(3). If $\alpha < \beta < h$, then each element of the family D_β is almost contained in some element of the family D_α.

It was also shown that $\omega_1 \leq h \leq 2^\omega$ and that it is consistent with ZFC that $h < 2^\omega$.

THEOREM 15. *The space* $[N]^\omega$ *contains a* π*-base which can be represented as a union of* h *many disjoint families.*

Proof. Let $\mathscr{H} = \{D_\alpha : \alpha < h\}$ be a collection satisfying conditions (1) and (2), above. For any pair x, y of finite subsets of N and for each $\alpha < h$, let $D_\alpha(x, y) = \{\langle x, A(x, y)\rangle \cap [N]^\omega : A \in D_\alpha\}$. Then each $D_\alpha(x, y)$ is a disjoint family of basic open subsets of $[N]^\omega$. Let us show that

$$\mathscr{D} = \cup\{D_\alpha(x, y) : \alpha < h \text{ and } x, y \in [N]^{<\omega}\}$$

is a π-*base* in $[N]^\omega$.

Let $B \in [N]^\omega$ and consider a basic neighborhood $V_n(B)$ of B. There exist $\alpha, \alpha < h$, and $A \in D_\alpha$ such that $A\backslash B$ is finite. Let x be the set of first n elements of B and let $y = A\backslash B$. Then $\langle x, A(x, y)\rangle \subseteq V_n(B)$. Thus $\mathscr{D}$ is a required π-*base* in the space $[N]^\omega$.

Also, $\mathscr{D}$ is a union of $h \cdot \omega = h$ many disjoint families. □

THEOREM 16. *Let κ be a cardinal and let Z be a subspace of $[N]^\omega$ such that Z has a π-base which can be represeted as a union of κ many disjoint subfamilies. If $\kappa < h$, then Z is a nowhere dense subset of $[N]^\omega$.*

Proof. Assume the contrary and suppose that there exists a basic open set V such that $Z \cap V$ is dense in V. Then $Z \cap V$ also has a π-*base* which can be represented as a union of κ many disjoint subfamilies. Since V is homeomorphic to $[N]^\omega$, we can assume, without loss of generality, that Z is dense in $[N]^\omega$. Thus there exists a π-*base* $\mathscr{P}$ of $[N]^\omega$ such that $\mathscr{P} = \cup\{P_\alpha : \alpha < \kappa\}$, where each P_α is a disjoint family. Moreover, we may assume that $\mathscr{P}$ consists of basic open sets.

For each $\alpha < \kappa$ and for each $F \in [N]^{<\omega}$, let

$$P(\alpha, F) = \{A \in [N]^\omega : \langle F, A\rangle \cap [N]^\omega \in P_\alpha\}.$$

By Lemma 12, each $P(\alpha, F)$ is an almost disjoint family. We may assume, without loss of generality, that each $P(\alpha, F)$ is even a maximal almost disjoint family. Since $\mathscr{P}$ is a π-*base* of $[N]^\omega$, given $B \in [N]^\omega$, there exist $a < \kappa$, $F \in [N]^{<\omega}$ and $A \in P(\alpha, F)$ such that $\langle F, A\rangle \cap [N]^\omega \subseteq \langle \emptyset, B\rangle$. Then A is almost contained in B. Hence $\mathscr{H} = \{P(\alpha, F) : \alpha < \kappa$ and $F \in [N]^{<\omega}\}$ satisfies conditions (1) and (2) of Definition 14. Since $\kappa \cdot \omega < h$, we have a contradiction. □

If one assumes that $h < 2^\omega$, then by Theorem 15, the space $[N]^\omega$ may contain a π-*base* which can be represented as a union of less that 2^ω

many disjoint families. However this is not true for any base of $[N]^\omega$ as the following proposition shows.

PROPOSITION 17. *No base of the space* $[N]^\omega$ *can be represented as a union of less than* 2^ω *many disjoint families.*

Proof. It is known that the space $[N]^\omega$ contains a subspace, say Z, that is separable and of weight 2^ω (cf. [11]). Since the space Z is separable, any disjoint family of open subsets of the space Z must be countable. Since $w(Z) = 2^\omega$, no base of the space Z can be represented as a union of less than 2^ω many disjoint families. Thus the same conclusion holds for $[N]^\omega$. □

Remark 1. In connection with the above proposition, let us remark that the space $[N]^\omega$ contains a dense subspace Z such that Z has a base that can be represented as a union of h many disjoint families. To show this, let $\mathscr{P}$ be a π*-base* of $[N]^\omega$ such that $\mathscr{P} = \cup\{P_\alpha : \alpha < h\}$, where each P_α is a disjoint family. For each $\alpha < h$ and for each $U \in P_\alpha$ fix $A(U) \in U$ and fix a countable base $\mathscr{B}(U)$ of $A(U)$ in U. Let $Z = \{A(U) : U \in P_\alpha, \alpha < h\}$, and let $\mathscr{B} = \{V \cap Z : V \in \mathscr{B}(U), U \in P_\alpha, \alpha < h\}$. Then Z is dense in $[N]^\omega$ and $\mathscr{B}$ is a base of Z that can be represented as a union of h many disjoint families. □

4. Metrizable subspaces of $[N]^\omega$.

THEOREM 18. *Every subspace of* $[N]^\omega$ *with a* σ*-disjoint* π*-base (in particular, every metrizable subspace of* $[N]^\omega$*) is nowhere dense in* $[N]^\omega$.

Proof. This fact follows immediately from Theorem 16. □

A topological space is called a σ*-space* if it has a σ*-discrete* network.

THEOREM 19. *Every subspace of* $\exp(N)$ *which is a* σ*-space is* σ*-discrete.*

Proof. Let Z be a subspace of $\exp(N)$ and let Z be a σ*-space*. Let $\mathscr{S} = \cup\{S_i : i \in N\}$ be a network of Z, where each S_i is a discrete family of subsets of Z. For each $i \in N$, let

$$Z_i = \{A \in Z : \text{ there exists } U \in S_i \text{ such that } A \in U \subseteq \langle \emptyset, A\rangle\}.$$

Since $\mathscr{S}$ is a network of Z, $Z = \cup\{Z_i : i \in N\}$. Let us show that each Z_i is discrete (and closed) in Z. For each $A \in Z_i$, fix $U_A \in S_i$ such that $A \in U_A \subseteq \langle \emptyset, A\rangle$. Then for $A, B \in Z_i$, $A \neq B$, $U_A \neq U_B$ (for otherwise, $A \in \langle \emptyset, B\rangle$ and $B \in \langle \emptyset, A\rangle$ which would imply that $A = B$). Since $\{U_A : A \in Z_i\}$ is a discrete family in Z, Z_i is a closed discrete subset of Z. □

A topological space Z is said to be *developable* if there exists a collection $\{\mathscr{B}_i : i = 1, 2, \ldots\}$, called a *development* for Z, possessing the following properties:

1. For each i, $\mathscr{B}_i$ is an open cover of Z;
2. For each $p \in Z$, if $U_i \in \mathscr{B}_i$ is such that $p \in U_i$ for each $i \in N$, then the family $\{U_i : i \in N\}$ is a base at p.

An extensive discussion of developable spaces can be found, for example, in [3] or [4].

Metrizable spaces are developable and every regular developable space is a σ*-space* [4]. We therefore have the following corollary.

COROLLARY 20. *Every developable subspace of* $\exp(N)$ *is* σ*-discrete.*

THEOREM 21. *Every subspace of* $\exp(N)$ *possessing a* σ*-point - finite base is* σ*-discrete.*

Proof. Let Z be a subspace of $\exp(N)$ and let $\mathscr{B} = \cup\{B_i : i \in N\}$ be a base of Z, where each B_i is point-finite. For each $i \in N$, let

$$Z_i = \{A \in Z : \text{ there exists } U \in B_i \text{ such that } A \in U \subseteq \langle \emptyset, A\rangle\}.$$

Since $\mathscr{B}$ is a base of Z, $Z = \cup\{Z_i : i \in N\}$. Let us show that each Z_i is σ*-discrete*.

For each $n \in N$, let

$$Z_i(n) = \{A \in Z_i : A \text{ belongs to exactly } n \text{ elements of } B_i\}.$$

Since B_i is point-finite, $Z_i = \cup\{Z_i(n) : n \in N\}$. It is enough to show that each $Z_i(n)$ is a discrete subset of Z.

For each $A \in Z_i(n)$, let $W_1, W_2, \ldots, W_n \in B_i$ be such that $A \in W_j$, for each $j = 1, 2, \ldots, n$. We set $W_A = W_1 \cap W_2 \cap \ldots \cap W_n$. Thus W_A is an open neighborhood of A. Notice also that $W_A \subseteq \langle \emptyset, A\rangle$. It follows that $W_A \cap Z_i(n) = \{A\}$. indeed, if $B \in W_A \cap Z_i(n)$, then $W_A = W_B$ for W_1, $W_2, \ldots, W_n$ are also the only elements of B_i such that $B \in W_j$, for each $j = 1, 2, \ldots, n$. Hence $A \in \langle \emptyset, B\rangle$ and $B \in \langle \emptyset, A\rangle$ which implies that $A = B$. □

THEOREM 22. *If Z is a developable subspace of $[N]^\omega$, then $|Z| = w(Z)$.*

Proof. The theorem is trivial in the case Z is finite. So suppose Z is infinite. Let $\kappa = w(Z)$ and let $\{\mathscr{B}_i : i = 1, 2, \ldots\}$ be a development for Z. One may assume that the development has the following additional properties:

(a) For each i, $|\mathscr{B}_i| \leq \kappa$;

(b) For each i, $\mathscr{B}_i$ consists of basic open sets of the form $V_n(A) \cap Z$, where $A \in Z$.

We shall show that

$$Z \subseteq \{A : V_n(A) \cap Z \in \mathscr{B}_i \text{ for some } i \text{ and for some } n\}.$$

To this end, let B be a point of Z. For each i, chose $V_{n_i}(A_i) \cap Z \in \mathscr{B}_i$ such that $B \in V_{n_i}(A_i) \cap Z$. Since the family $\{\mathscr{B}_i : i = 1, 2, \ldots\}$ is a development for Z, the family $\{V_{n_i}(A_i) \cap Z : i \in N\}$ is a base at the point B in the subspace Z. If B is an isolated point of Z, then $\{B\} = V_{n_i}(A_i) \cap Z$ for some i; Hence $B = A_i$. Otherwise, the sequence $\{A_i : i \in N\}$ contains a subsequence converging to B. Hence, by virtue of Lemma 9, $B = A_i$ for some i and the inclusion is shown.

Since $|\{A : V_n(A) \cap Z \in \mathscr{B}_i$ for some i and for some $n\}|$

$\leq \kappa \cdot \omega$, $|Z| \leq \kappa \cdot \omega = \kappa$. Since Z has countable base at every point, $\kappa = w(Z) \leq |Z|$. Thus $|Z| = \kappa$. □

COROLLARY 23. *The weight of any uncountable subspace of the space $[N]^\omega$ is uncountable.*

The following fact was discovered by V. Popov (cf. [13], Example 5) in 1978. We provide a slightly different proof for the sake of completeness.

THEOREM 24. *The space* $[N]^{\omega}$ *contains a subspace homeomorphic to the Sorgenfrey line.*

Proof. Let Q denote the set of rational numbers with the discrete topology. Then $\exp(Q)$ is homeomorphic to $\exp(N)$. Let $X = \{C \in \exp(Q) : C \textit{ is a cut}\}$. Recall that a proper subset C of Q is a cut if C has no largest element and for each $p \in C$, $(-\infty, p] \cap Q \subseteq C$. Also, if C and D are cuts and C is a proper subset of D, then we write $C < D$.

Let us show that the subspace X of $[N]^{\omega}$ is homeomorphic to the Sorgenfrey line whose basic neighborhoods point to the left.

Let C and D be cuts such that $C < D$, and let $E \in (C, D]$. Then for any $q \in E \backslash C$, $E \in \langle\{q\}, E\rangle \cap X \subseteq (C, D]$. This shows that $(C, D]$ is open in X. Conversely, let $W = \langle F, A\rangle \cap X$, where F is a finite subset of Q, and let $D \in W$. Let p be the largest element of F and let $C = \{r \in Q : r < p\}$. Then $C < D$ and $(C, D] \subseteq W$. This shows that W is open in the topology generated by sets of the form $(C, D]$. □

COROLLARY 25. $m(\exp(N)) = 2^{\omega}$.

Proof. Let X be a subspace of $[N]^{\omega}$ which is homeomorphic to the Sorgenfrey line. Then X is hereditarily separable and the netweight of X is 2^{ω}. Hence $m(X) = 2^{\omega}$ and in consequence, $m(\exp(N)) = 2^{\omega}$. □

COROLLARY 26. *Let* X *be a* T_1 *space such that* $m(\exp(X)) < 2^{\omega}$. *Then* X *is countably compact.*

Proof. Assume X is not countably compact. Then X contains a closed subspace homeomorphic to N. Therefore $\exp(N)$ is embedded into $\exp(X)$. Hence $m(\exp(X)) \geq m(\exp(N)) = 2^{\omega}$. This is a contradiction. □

COROLLARY 27. *Let* X *be a* T_1 *space such that* $m(\exp(X)) \leq \omega$. *Then* X *(hence* $\exp(X)$*) is compact and* $m(X) = m(\exp(X)) = 1$.

Proof. By the previous corollary, X is countably compact. Also, since $m(X) \leq \omega$, X is *ω-refinable* in the sense of [6]. Therefore, by [6]; Theorem

1, X is compact. Hence, by [5]; Corollary 5.3, $m(X) = m(\exp(X)) = 1.\square$

Given a partition $\mathscr{P} = \{N_j : j \in J\}$ of N into pairwise disjoint infinite sets, let us define a mapping

$$\eta_{\mathscr{P}} : \prod\{[N_j]^\omega : j \in J\} \to [N]^\omega$$

by

$$\eta_{\mathscr{P}}((A_j : j \in J)) = \cup\{A_j : j \in J\}.$$

Let

$$Y_{\mathscr{P}} = \eta_{\mathscr{P}}\left(\prod\{[N_j]^\omega : j \in J\}\right).$$

LEMMA 28. *The mapping $\eta_{\mathscr{P}}$ is one-to-one.*

Proof. Let $(A_j : j \in J)$, $(B_j : j \in J) \in \prod\{[N_j]^\omega : j \in J\}$ be such that $(A_j : j \in J) \neq (B_j : j \in J)$. Then $A_j \neq B_j$, for some $j \in J$. Since $\mathscr{P}$ is a partition of N, $\cup\{A_j : j \in J\} \neq \cup\{B_j : j \in J\}$. $\square$

LEMMA 29. *Let X_j be a closed subspace of $[N_j]^\omega$ for each $j \in J$. Then $\eta_{\mathscr{P}}\left(\prod\{X_j : j \in J\}\right)$ is a closed subspace of $[N]^\omega$.*

Proof. Let $A \in [N]^\omega \setminus \eta_{\mathscr{P}}\left(\prod\{X_j : j \in J\}\right)$. Then $A \cap N_j \notin X_j$ for some $j \in J$. There exists a finite set $F \subseteq A \cap N_j$ such that $\langle F, A \cap N_j \rangle \cap X_j = \emptyset$. Hence $\langle F, A \rangle \cap \eta_{\mathscr{P}}\left(\prod\{X_j : j \in J\}\right) = \emptyset$. $\square$

LEMMA 30. *The mapping $\eta_{\mathscr{P}}$ is continuous when $\prod\{[N_j]^\omega : j \in J\}$ is equipped with the box product topology.*

Proof. Let $(A_j : j \in J) \in \prod\{[N_j]^\omega : j \in J\}$, let $A = \eta_{\mathscr{P}}((A_j : j \in J))$, and let $\langle F, A \rangle$ be a basic neighborhood of A. For each $j \in J$, let $F_j = F \cap N_j$ and let $U = \prod\{\langle F_j, A_j \rangle : j \in J\}$. Then U is a neighborhood of $(A_j : j \in J)$ and $\eta_{\mathscr{P}}(U) \subseteq \langle F, A \rangle$. $\square$

LEMMA 31. *If $\mathscr{P} = \{N_j : j \in J\}$ is a finite partition of N, then the mapping $\eta_{\mathscr{P}}$ is a homeomorphism onto $Y_{\mathscr{P}}$.*

Proof. By virtue of the preceding lemmas, it is enough to show that the mapping $\eta_{\mathscr{P}}^{-1}$ is continuous.

It is easy to verify that for any basic open set $U = \prod\{\langle F_j, A_j\rangle : j \in J\}$ of the space $\prod\{[N_j]^\omega : j \in J\}$, $\eta_{\mathscr{P}}(U) = \langle F, A\} \cap Y_{\mathscr{P}}$, where $A = \cup\{A_j : j \in J\}$ and $F = \cup\{F_j : j \in J\}$. Thus $\eta_{\mathscr{P}}$ is an open mapping and therefore $\eta_{\mathscr{P}}^{-1}$ is continuous. □

THEOREM 32. *(1) Any finite power* $([N]^\omega)^n$ *of the space* $[N]^\omega$ *is embedded into* $[N]^\omega$ *as a closed subspace. (2) Any finite power of the Sorgenfrey line is embedded into* $[N]^\omega$.

Proof. If follows imemdiately from the preceding lemmas and Theorem 24. □

The space $[N]^\omega$ contains an uncountable subspace which is hereditarily Lindelöf and hereditarily separable: the Sorgenfrey line is an instance of such a subspace. The following theorems give characterizations of hereditarily Lindelöf and hereditarily separable subspaces of the space $\exp(N)$.

THEOREM 33. *A subspace* X *of* $\exp(N)$ *is hereditarily Lindelöf if and only if for each* $Y \subseteq X$ *there exists a countable subset* Z *of* Y *such that for each* $A \in Y$ *there exists* $B \in Z$ *such that* $A \subseteq B$.

Proof. Suppose that $X \subseteq \exp(N)$ is hereditarily Lindelöf and let $Y \subseteq X$. Since the family $\{\langle \emptyset, A\rangle \cap Y : A \in Y\}$ is an open cover of Y, there exists a countable subset Z of Y such that $\{\langle \emptyset, A\rangle \cap Y : A \in Z\}$ covers Y Thus for each $A \in Y$ there exists $B \in Z$ such that $A \subseteq B$.

Conversely, suppose that X satisfies the above condition. Let Y be a subspace of X and let $\mathscr{G}$ be a cover of Y by sets of the form $\langle F, A\rangle \cap Y$, where $A \in Y$ and F is a finite subset of A. For each $F \in [N]^{<\omega}$, let $Y(F) = \{A \in Y : \langle F, A\rangle \cap Y \in G\}$. Let $Z(F)$ be a counatble subset of $Y(F)$ such that for each $A \in Y(F)$ there exists $B \in Z(F)$ such that $A \subseteq B$. Let $\mathscr{G}(F) = \{\langle F, B\rangle \cap Y : B \in Z(F)\}$ and, finally, let $\mathscr{H} = \cup\{\mathscr{G}(F) : F \in [N]^{<\omega}\}$. Then $\mathscr{H}$ is a countable subfamily of $\mathscr{G}$. Since for each $F \in [N]^{<\omega}$, $\cup\{\langle F, A\rangle \cap Y : A \in Y(F)\} \subseteq \cup\mathscr{G}(F)$, the family $\mathscr{H}$ is a cover of Y. □

THEOREM 34. *A subspace* X *of* $\exp(N)$ *is hereditarily separable if and only if for each* $Y \subseteq X$ *there exists a countable subset* Z *of* Y *such that for each* $A \in Y$ *there exists* $B \in Z$ *such that* $B \subseteq A$.

Proof. Suppose that $X \subseteq \exp(N)$ is hereditarily separable and let $Y \subseteq X$. Let Z be a countable and dense subset of Y. Then for each $A \in Y$, $\langle \emptyset, A\rangle \cap Z \neq \emptyset$. Thus for each $A \in Y$ these exists $B \in Z$ such that $B \subseteq A$.

Conversely, suppose that X satisfies the baove condition. Let Y be a subspace of X. For each $F \in [N]^{<\omega}$, let $Y(F) = \{A \in Y : F \subseteq A\}$. Let $Z(F)$ be a countable subset of $Y(F)$ such that for each $A \in Y(F)$ there exists $B \in Z(F)$ such that $B \subseteq A$. Then $Z = \cup\{Z(F) : F \in [N]^{<\omega}\}$ is a countable dense subset of Y. □

COROLLARY 35. *A subspace X of* $\exp(N)$ *is hereditarily Lindelöf if and only if the subspace $X^c = \{N \setminus A : A \in X\} \setminus \{\emptyset\}$ is hereditarily separable.*

Any non-empty subset of $\exp(N)$ that is linearly ordered by $\subseteq$ is called a chain in $\exp(N)$. The Sorgenfrey line constructed in Theorem 24 is a chain in $\exp(N)$ of cardinality 2^ω.

THEOREM 36. *Any chain in* $\exp(N)$ *is both hereditarily Lindelöf and hereditarily separable.*

Proof. Since the set of all complements of a chain is a chain again, in view of the preceding corollary, it is enough to show that every subspace of $\exp(N)$ that is a chain is hereditarily Lindelöf.

Since every chain of subsets of a countable set contains a countable cofinal subset, every chain of subsets of N satisfies the condition of Theorem 33 and thus it is hereditarily Lindelöf. □

For every subspace X of $\exp(N)$, let $X^c = \{N \setminus A : A \in X\} \setminus \{\emptyset\}$. As the above corollary shows, a subspace X of $\exp(N)$ is hereditarily Lindelöf if and only if X^c is hereditarily separable. This duality between X and X^c holds only in one direction if the word "hereditarily" is omitted from the above statement.

PROPOSITION 37. *If a subspace X of* $\exp(N)$ *is Lindelöf, then X^c is separable.*

Proof. We can assume, without loss of generality, that $N \notin X$. For each $n \in N$, there exists a countable subset Z_n of X such that $X \subseteq \cup\{W_n(B) : B \in Z_n\}$. Let $Z = \cup\{Z_n : n \in N\}$. Let us show that the set

$$D = Z^c \cup (X^c \cap [N]^{<\omega})$$

is dense in X^c.

Let $N\backslash A \in X^c$. if $N\backslash A$ is finite, then $N\backslash A \in D$. Assume that $N\backslash A$ is infinite and let us consider an arbitrary basic neighborhood $W_m(N\backslash A)$. Suppose that k is the mth element of $N\backslash A$. There exists $B \in Z_k$ such that $A \in W_k(B)$. Then $N\backslash B \in D \cap W_m(N\backslash A)$. □

EXAMPLE 38. There exists a separable subspace X of $[N]^\omega$ such that X^c is homeomorphic to the Sorgenfrey plane and hence X^c is not Lindelöf.

Let Q be the set of all rational numbers with discrete topology and let Q_1 and Q_2 be two disjoint dense (with respect to the usual topology) subsets of Q such that $\{Q_1, Q_2\}$ is a partition of Q. Let $Y = \{A \in \exp(Q) : A \cap Q_1$ is a cut in Q_1 and $A \cap Q_2$ is a cut in $Q_2\}$. By Theorem 24 and Lemma 31, it follows that the subspace Y of $\exp(Q)$ is homeomorphic to the Sorgenfrey plane, and thus, Y is not Lindelöf. Let $X = Y^c$. For each $(s,t) \in Q_1 \times Q_2$, let $A(s,t) = ([s,\infty) \cap Q_1) \cup ([t,\infty) \cap Q_2)$. Then $\{A(s,t) : (s,t) \in Q_1 \times Q_2\}$ is a countable dense subset of the space X. Therefore X is separable but $X^c = Y$ is not Lindelöf. □

EXAMPLE 39. There exists a separable subspace X of $\exp(N)$ such that X^c is not separable.

Let $\mathscr{D}$ be an almost disjoint family of infinite subsets of N of cardinality continuum and let Cof be the family of all cofinite subsets of N that are different from N. Then $\mathscr{D}^c$ is an open discrete subset of the space $Y = Cof \cup \mathscr{D}^c$ because for each $N\backslash A \in \mathscr{D}^c$, $\langle \emptyset, N\backslash A\rangle \cap Y = \{N\backslash A\}$. Therefore Y cannot be separable. However the dual of Y contains $[N]^{<\omega}$ and thus it is separable. Setting $X = Y^c$ we get an example of a separable subspace X of $\exp(N)$ such that X^c is not separable. □

EXAMPLE 40. If no subset of reals of cardinality continuum is concentrated about a countable set, then there exists a metrizable subspace X of $\exp(N)$ such that X^c is not metrizable.

Let $Y = Cof \cup \mathscr{D}^c$ be as in the above example. Then, when Y is viewed as a subset of the Cantor set, there exists an open neighborhood U of the countable set Cof such that $\mathscr{D}^c - U$ is uncountable. Let$X = Cof \cup (\mathscr{D}^c - U)$. Then X, being the disjoint sum of two metrizable subspaces of $\exp(N)$, is metrizable. Since X^c contains a countable dense set $[N]^{<\omega}$ and an uncountable discrete subspace $(\mathscr{D}^c - U)^c$, X^c is not metrizable. □

Remark 2. It was shown by Lavre [8] that it is consistent that no uncountable subset of real is concentrated about a countable set.

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