ON SOME PROBLEM OF A. ROSLANOWSKI

BY

SYMON PLEWIK (KATOWICE)

We present a negative answer to problem 3.7(b) posed on page 193 of [2], where, in fact, A. Roslanowski asked: *Does every set of Lebesgue measure zero belong to some Mycielski ideal?*

We identify a set \( X \in \omega^\omega \) with its characteristic function, i.e. with the sequence \((X(0), X(1), \ldots) \in 2^\omega\) such that \( X(n) = 1 \) iff \( n \in X \). A set \( X \in \omega^\omega \) has asymptotic density \( d \) whenever

\[
\lim_{n \to \infty} \frac{|X \cap n|}{n} = d,
\]

where \(|X \cap n|\) denotes the number of natural numbers from \( X \) less than \( n \).

We consider the family of all sets of asymptotic density not equal to \( 1/2 \), i.e. the set

\[
A = 2^\omega \setminus \{ X \in \omega^\omega : X \text{ is of asymptotic density } 1/2 \}.
\]

An old result of E. Borel [1] says: *The set \( A \) has Lebesgue measure zero.* A direct consequence of this result is

**Theorem.** The set \( A \) does not belong to any Mycielski ideal.

**Proof.** Our notation follows [2]. If \( K \) is a normal system, i.e. for each \( X \in K \) there exist two disjoint subsets of \( X \) which belong to \( K \), then \( K \) contains three disjoint sets \( X, Y \) and \( Z \). Since

\[
|X \cap n| + |Y \cap n| + |Z \cap n| \leq n,
\]

one of the sets: \( X, Y \) or \( Z \) does not contain any subset of asymptotic density \( 1/2 \). Suppose \( X \) is such a set. If Player I always chooses zero, then he wins the game \( \Gamma(X, A) \), because any set (sequence) which can be the result of that game is not of asymptotic density \( 1/2 \) and thus belongs to \( A \). This means that the set \( A \) does not belong to the Mycielski ideal generated by \( K \).

If one considers Mycielski ideals on \( k^\omega \), where \( k > 2 \) is a natural number, then our theorem can be slightly modified. The Lebesgue measure and

1991 Mathematics Subject Classification: 03E05, 04A20, 28A05.
Mycielski ideals can also be considered on $k^\omega$ because of the definition of the Lebesgue measure given in [2], p. 188. Similarly to the asymptotic density, one can define the asymptotic frequency of functions from $k^\omega$. Again, it is a result of E. Borel [1] that: The set of all sequences from $k^\omega$ in which every natural number $n$ occurs asymptotically with frequency $1/k$ has full measure. Its complement $A^*$ has Lebesgue measure zero and does not belong to any Mycielski ideal, since Player I wins the game $\Gamma(\omega \setminus X, A^*)$ whenever he always chooses the same number and $X$ does not contain any subset with asymptotic frequency $(k - 1)/k$.

REFERENCES
