UNIFORM DENSITY THEOREM

by

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Abstract. We strenght the lemma 2.2 from V. Bergelson, N. Hindman and B. Weiss [1] replacing the word *many*, by words *for almost all* in the conclusion. A new commentaries are added to the problem 146 in [3].

1. Introduction. Let $\mu(X)$ denotes the Lebesgue measure of X. Let $\varrho(t, X) = \inf\{|t - y| : y \in X\}$ and $B(X, \varepsilon) = \{y : \varrho(y, X) < \varepsilon\}$. The set X is *measurably large* if X is Lebesgue measurable, and for every $\varepsilon > 0$ there holds $\mu(X \cap (0, \varepsilon)) > 0$. We assume that the reader is familiar with density points, so we list properties only, which we shall use. Let X^* denotes the set of all density points of X. For any Lebesgue measurable set X we have

(A) $X^* \subseteq X$ and $\mu(X) = \mu(X^*)$;

(B) If the set X is Lebesgue measurable, and $p + t$ is a density point of X, then t is a density point of the set $X - p$;

(C) If sets X and Y are Lebesgue measurable, and t is a density point of X, and t a density point of Y, then t is a density point of the intersection $X \cap Y$.

2. The Bergelson-Hindman-Weiss lemma and an uniform density theorem. In [1], Lemma 2.2, there is stated the following lemma.

Lemma. *Let* $A \subseteq (0, 1]$ *be measurably large. There exist* (*many*) $t \in A$ *such that* $A \cap (A - t)$ *is measurably large.*

We give the following modification of this lemma.

Theorem. *If X is measurably large, then for almost all $t \in X$ the intersection $X \cap (X - t)$ is measurably large.*

Proof. Fix a measurably large set $D \subseteq X^*$ such that $D_1 = \{0\} \cup D$ is a compact set. Since (A) we have $D \subseteq X$. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \mu(D)$: the number $\sum_{n=1}^{\infty} \alpha_n$ could be arbitrary small. Choose $\varepsilon_1 > 0$ such that $\mu(B(D_1, \varepsilon_1)) - \mu(D_1) < \alpha_1$. Then choose $t_1 < \varepsilon_1$ such that $t_1 \in D$. Whereafter put $D_2 = D_1 \cap (D_1 - t_1)$. The set D_2 is compact and $\mu(D_1) < \mu(D_2) + \alpha_1$.

Suppose there have been defined compact sets (with positive Lebesgue measure) $D_1, D_2, \ldots, D_n$ and points $\{t_1, t_2, \ldots, t_{n-1}\} \subseteq D$ such that

$$D_{k+1} = D_k \cap (D_k - t_k)$$

and $\mu(D_k) < \mu(D_{k+1}) + \alpha_k$, for $0 < k < n$. Choose $\varepsilon_n \in (0, \frac{1}{n})$ such that $\mu(B(D_n, \varepsilon_n)) - \mu(D_n) < \alpha_n$. Then choose $t_n < \varepsilon_n$ such that $t_n \in D$. Whereafter put $D_{n+1} = D_n \cap (D_n - t_n)$. The set D_{n+1} is compact and $\mu(D_n) < \mu(D_{n+1}) + \alpha_n$.

So, there have been defined compact sets $D_1, D_2, \ldots$ such that

$$\beta + \mu(D_1 \cap D_2 \cap \ldots) > \mu(D) + \sum_{n=1}^{\infty} \alpha_n,$$

where $\beta > 0$ can be arbitrary small. The number $\sum_{n=1}^{\infty} \alpha_n - \beta$ could be arbitrary small, so we infer that there exists a point $p \in D_1 \cap D_2 \cap \ldots$, where $p \neq 0$. We have

$$p \in \cap\{D_n : n = 1, 2, \ldots\} = \cap\{D_n \cap (D_n - t_n) : n = 1, 2, \ldots\}.$$

Hence, for each n there holds $p \in D_n - t_n$, so $p + t_n \in D_n \subseteq D \subseteq X^*$, i.e. each t_n is a density point of the set $X - p$ because of (B). Since this and since each t_n is a density point of X following (C) we obtain that each t_n is a density point of $X \cap (X - p)$; but $\lim_{t \to \infty} t_n = 0$, so the intersection $X \cap (X - p)$ is measurably large.

The finish conclusion one could deduce from this that the above argumentations work for every number $p \in D_1 \cap D_2 \cap \ldots$; and since a careful choise of sets D_n (because of $\mu(X) = \mu(X^*)$) is possible such that $\mu(X \setminus (D_1 \cap D_2 \cap \ldots))$ is arbitrary small, whenever $\mu(X) < \infty$. $\square$

3. Commentary. In [3] on page 228 S. Ulam asked: *It is known that in sets of positive measure there exist points of density 1* [*that is to say, points with the property that the ratio of the length of intervals to the measure of the part of the set contained in these intervals tends to 1* (*if the length of the interval converges to* 0)]. *Can one determine the speed of convergence of this ratio for almost all points of the set?*

From a geometric point of view "the speed of convergence" could be understood as in our theorem. A reasumption of our reasonning is the following.

Corollary. *If X is measurably large and Y is Lebesgue measurable, then for almost all $t \in Y$ the intersection $X \cap (Y - t)$ is measurably large.*

Proof. Take $X \cap B(\{0\}, \varepsilon) \cup (Y \setminus B(\{0\}, \varepsilon))$ us X in Theorem. $\square$

Corollary could be treated as a geometric commentary to Ulam's problem. It complements the commentary of J. Mycielski in [2].

References

[1] V. Bergelson, N. Hindman and B. Weiss. *All-sum sets in* $(0, 1]$– *category and measure*. Mathematika, 44 (1997), p. 61 - 87.

[2] J. Mycielski. *Commentary.* The Scottish Book, eddited by R. D. Mauldin, Brikhäuser Boston (1981), p. 228.

[3] S. Ulam. *The problem 146.* The Scottish Book, eddited by R. D. Mauldin, Brikhäuser Boston (1981), p. 228.

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28A75: *MEASURE AND INTEGRATION; classical measure theory; other geometric measure theory.*