

## ON A THEOREM OF E. HELLY

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ABSTRACT. E. Helly's theorem asserts that *any bounded sequence of monotone real functions contains a pointwise convergent subsequence*. We reprove this theorem in a generalized version in terms of monotone functions on linearly ordered sets. We show that the cardinal number responsible for this generalization is exactly the splitting number. We also show that a positive answer to a problem of S. Saks is obtained under the assumption of the splitting number being strictly greater than the first uncountable cardinal.

### 0. INTRODUCTION

E. Helly's theorem ([3]) asserts that *any bounded sequence of monotone real functions contains a pointwise convergent subsequence*. In the present paper, we prove the following generalization of the theorem: *for linearly ordered sets  $X$  and  $Y$ , if  $Y$  is sequentially compact with density less than the splitting number  $\mathfrak{s}$ , then any sequence of monotone functions from  $X$  to  $Y$  contains a pointwise convergent subsequence* (Theorem 7). We also show that this theorem characterizes the splitting number (Theorem 9).

We begin with reviewing some definitions and elementary facts needed for our results.

### 1. PRELIMINARIES: LINEARLY ORDERED SETS

A linearly ordered set  $X$  is said to be *dense linear order*, if, for any  $x, y \in X$ ,  $x < y$  implies that there exists  $z \in X$  such that  $x < z < y$ . A subset  $D$  of a linearly ordered set  $X$  is said to be *dense in  $X$* , if, for any  $x, y \in X$ ,  $x < y$  implies that there exists  $z \in D$  such that  $x \leq z \leq y$ . The *density* of  $X$  is defined by

$$d(X) = \min\{|D| : D \subseteq X \text{ and } D \text{ is dense in } X\},$$

where  $|D|$  denotes the cardinality of the set  $D$ .

Let  $X$  and  $Y$  be linearly ordered sets. A function  $f : X \rightarrow Y$  is said to be *increasing*, if, for any  $x, y \in X$ ,  $x < y$  implies  $f(x) \leq f(y)$ ; *decreasing*, if  $x < y$  implies  $f(y) \leq f(x)$ . A function is monotone if it is either increasing or decreasing. A sequence  $(x_n)_{n \in \mathbb{N}}$  of elements of  $X$  is called increasing, decreasing or monotone respectively, if it is increasing, decreasing or monotone respectively as a function from the set of all natural numbers  $\mathbb{N}$  into  $X$ .

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For a linearly ordered set  $X$ , the notion of convergence can be introduced in a canonical way: an increasing sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  converges to a point  $x$ , if  $x$  is the supremum of the set of all elements of this sequence; a decreasing sequence converges to  $x$ , if  $x$  is the infimum of the set of all elements of this sequence; in general, a sequence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$ , if every monotone subsequence of this sequence converges to  $x$ . We say also that a sequence is *convergent* if it converges to some  $x$ . A linearly ordered set is said to be *sequentially compact* if each monotone sequence of its elements converges to some point in it. If a sequence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$ , we denote this as usual by  $\lim_{n \rightarrow \infty} x_n = x$ . For a sequence  $(x_n)_{n \in I}$  indexed by an infinite subset  $I$  of  $\mathbb{N}$ , its convergence to a point  $x$  is defined similarly and denoted by  $\lim_{n \in I} x_n = x$ . For an infinite  $I \subseteq \mathbb{N}$  and a sequence  $(f_n)_{n \in I}$  of functions from a set  $X$  to a linearly ordered set  $Y$ , we say that  $(f_n)_{n \in I}$  converges *pointwise* to  $f : X \rightarrow Y$ , if  $\lim_{n \in I} f_n(x) = f(x)$  holds for every  $x \in X$ . We shall also say that a sequence  $(f_n)_{n \in I}$  is pointwise convergent if there is some function  $f$  to which the sequence converges pointwise.

Since any sequence in a linearly ordered set has a monotone subsequence, every sequence in a sequentially compact linearly ordered set has a convergent subsequence. Using this fact, we can see easily the following.

**Lemma 1.** *If  $(x_n)_{n \in \mathbb{N}}$  is a non-convergent sequence of elements of a sequentially compact linearly ordered set  $X$ , then there exist infinite subsets of natural numbers  $I$  and  $J$  such that subsequences  $(x_n)_{n \in I}$  and  $(x_n)_{n \in J}$  converge to different points of  $X$ .  $\square$*

**Lemma 2.** *Any infinite linearly ordered set  $X$  can be embedded into a dense linear order  $\bar{X}$  such that  $d(\bar{X}) = d(X)$ . If  $X$  is sequentially compact, then  $\bar{X}$  can be also chosen to be so. Also, convergent sequences in  $X$  remain convergent in  $\bar{X}$  with the same limit.*

*Proof.* Let  $D$  be a dense subset of  $X$  of cardinality  $d(X)$ . For points  $x, y \in X$ , let us call  $(x, y)$  a jump in  $X$  if  $x < y$  and there is no  $z \in X$  such that  $x < z < y$ . By definition of dense subsets, for each jump  $(x, y)$  one of the points  $x$  or  $y$  must be in  $D$ . Hence there are at most  $d(X)$  jumps. Let  $\bar{X}$  be the linearly ordered set constructed from  $X$  by inserting a copy of the reals into each of the jumps in  $X$ . Noting that the density of the reals  $\mathbb{R}$ , with respect to the canonical ordering, is countable and  $\mathbb{R} \cup \{-\infty, +\infty\}$  is sequentially compact, it is easy to see that  $\bar{X}$  is as desired.  $\square$

**Lemma 3.** *Suppose that  $X$  and  $Y$  are linearly ordered sets and  $(f_n)_{n \in I}$  is a sequence of increasing functions from  $X$  to  $Y$ . If  $(f_n)_{n \in I}$  converges pointwise to a function  $f : X \rightarrow Y$ , then  $f$  is also increasing.*

*Proof.* By Lemma 2, we may assume that  $Y$  is a dense linear order. The rest of the proof can be done just like the usual proof of the corresponding assertion on increasing real functions.  $\square$

Any linearly ordered set  $X$  can be densely embedded into a sequentially compact linearly ordered set  $\bar{X}$ . E.g. we can take the Dedekind completion of  $X$  as  $\bar{X}$ . Note that we have  $d(X) = d(\bar{X})$  since  $X$  is dense in  $\bar{X}$ . In general, the Dedekind completion of  $X$  is not the minimal sequentially compact linearly ordered set containing a dense copy of  $X$  since there can be an unfilled Dedekind cut  $(D, E)$  of  $X$  such that  $D$  has uncountable cofinality and  $E$  uncountable coinitality. Let us call a sequence

$(x_n)_{n \in \mathbb{N}}$  in a linearly ordered set  $X$  *potentially convergent* if it converges to some point in some  $\overline{X}$  as above. By virtue of Lemma 1, this is equivalent to saying that there are no  $x, y \in X$  and no infinite  $I, J \subseteq \mathbb{N}$  such that  $x_n \leq x < y \leq x_m$  for every  $n \in I$  and  $m \in J$ .

2. THE SPLITTING NUMBER

A family  $\mathcal{S}$  of infinite subsets of  $\mathbb{N}$  is said to be *splitting* if, for every infinite subset  $I \subseteq \mathbb{N}$ , there exists a set  $J \in \mathcal{S}$  such that  $I \cap J$  and  $I \setminus J$  are both infinite. The splitting number  $\mathfrak{s}$  is defined by

$$\mathfrak{s} = \min\{ |\mathcal{S}| : \mathcal{S} \text{ is a splitting family} \}.$$

In particular, if  $\mathcal{S}$  is a family of infinite subsets of  $\mathbb{N}$  of cardinality less than  $\mathfrak{s}$ , then there exists an infinite subset  $I \subseteq \mathbb{N}$  such that  $I$  is almost included either in  $J$  or in  $\mathbb{N} \setminus J$  for every  $J \in \mathcal{S}$ . It is readily seen that  $\mathfrak{s}$  is uncountable and less than or equal to the cardinality of the reals. On the other hand, it is known that the value of  $\mathfrak{s}$  cannot be decided from the axioms of set theory alone. A splitting family was considered first by Sierpiński in [5]. He showed that under the Continuum Hypothesis there is a splitting family  $\mathcal{S}$  with the property that every uncountable subfamily of  $\mathcal{S}$  is still splitting. For more about the cardinal  $\mathfrak{s}$  and its relation to other cardinal invariants of reals the reader may consult [1], [2] or [7]. The role of splitting number in connection with convergence was also studied in [8].

The following lemma is the set-theoretic core of the generalization of Helly’s theorem.

**Lemma 4.** *If  $X$  is a set of cardinality less than  $\mathfrak{s}$  and  $Y$  is a sequentially compact linearly ordered set of density less than  $\mathfrak{s}$ , then for any sequence  $(f_n)_{n \in \mathbb{N}}$  of functions from  $X$  to  $Y$  there exists an infinite subset  $I \subseteq \mathbb{N}$  such that the sequence of functions  $(f_n)_{n \in I}$  converges pointwise.*

*Proof.* By Lemma 2 we may assume that  $Y$  is a dense linear order. Let  $D$  be a dense subset of  $Y$  of cardinality less than  $\mathfrak{s}$ . For  $x \in X$  and  $y \in D$ , let

$$C_x^y = \{ n \in \mathbb{N} : f_n(x) < y \}.$$

Since  $|X \times D| < \mathfrak{s}$ , there exists an infinite  $I \subseteq \mathbb{N}$  such that  $I$  is almost included either in  $C_x^y$  or in  $\mathbb{N} \setminus C_x^y$  for any  $x \in X$  and  $y \in D$ .

We shall show that the set  $I$  is as desired. Otherwise there would be some point  $a \in X$  such that the sequence  $(f_n(a))_{n \in I}$  of points in  $Y$  is not convergent. Then, by Lemma 1, there are infinite subsets  $J$  and  $K$  of  $I$  and a point  $d \in D$  such that sequences  $(f_n(a))_{n \in J}$  and  $(f_n(a))_{n \in K}$  of points in  $Y$  are convergent and we have

$$\lim_{n \in J} f_n(a) < d < \lim_{n \in K} f_n(a).$$

Hence we have  $f_n(a) < d$  for all but finitely many  $n \in J$  and  $d < f_n(a)$  for all but finitely many  $n \in K$ . It follows that the sets  $I \cap C_a^d$  and  $I \setminus C_a^d$  are both infinite; but this contradicts the choice of  $I$ . □

Lemma 4 gives the consistency of a positive answer to the following question of S. Saks studied in [5]:

*For arbitrary sequence  $(f_n)_{n \in \mathbb{N}}$  of real functions, do there exist an infinite  $I \subseteq \mathbb{N}$  and an uncountable  $X \subset \mathbb{R}$  such that, for each  $x \in X$ , the sequence of real numbers  $(f_n(x))_{n \in I}$  has a finite or infinite limit?*

Under the Continuum Hypothesis, Sierpiński gave a negative answer to the question in [5]. By applying Lemma 4 for the sequentially compact linearly ordered set  $\mathbb{R} \cup \{-\infty, +\infty\}$ , we see that, under  $\mathfrak{s} > \aleph_1$ , a positive answer to the question is obtained.

Since every linearly ordered set can be embedded densely into a sequentially compact linearly ordered set, the next lemma follows immediately from Lemma 4.

**Lemma 5.** *If  $X$  is a set of cardinality less than  $\mathfrak{s}$  and  $Y$  is a linearly ordered set of density less than  $\mathfrak{s}$ , then for any sequence  $(f_n)_{n \in \mathbb{N}}$  of functions from  $X$  to  $Y$  there exists an infinite subset  $I \subseteq \mathbb{N}$  such that the sequence  $(f_n(x))_{n \in I}$  is potentially convergent for every  $x \in X$ .  $\square$*

Lemma 5 can be yet slightly improved. For any infinite  $I \subseteq \mathbb{N}$ , let us call a sequence  $(x_n)_{n \in I}$  in a linearly ordered set  $X$  *semi-monotone* if there is a bijection  $\varphi : \mathbb{N} \rightarrow I$  such that  $(x_{\varphi(n)})_{n \in \mathbb{N}}$  is eventually monotone, i.e. monotone from some  $m \in \mathbb{N}$  on. It is clear that a semi-monotone sequence is potentially convergent. If  $x = \lim_{n \in I} x_n$  exists, then  $(x_n)_{n \in I}$  is semi-monotone if and only if  $(x_n)_{n \in I}$  approaches to  $x$  eventually from one side — i.e. for some  $m \in \mathbb{N}$  either  $x_n \leq x$  for every  $n \geq m$  or  $x \leq x_n$  for every  $n \geq m$ .

**Lemma 6.** *If  $X$  is a set of cardinality less than  $\mathfrak{s}$  and  $Y$  is a linearly ordered set of density less than  $\mathfrak{s}$ , then for any sequence  $(f_n)_{n \in \mathbb{N}}$  of functions from  $X$  to  $Y$  there exists an infinite subset  $I \subseteq \mathbb{N}$  such that the sequence  $(f_n(x))_{n \in I}$  is semi-monotone for every  $x \in X$ .*

*Proof.* Without loss of generality, we may assume that  $Y$  is sequentially compact. By Lemma 4, there is an infinite  $I \subseteq \mathbb{N}$  such that  $(f_n)_{n \in I}$  is pointwise convergent. For each  $x \in X$  let  $y_x = \lim_{n \in I} f_n(x)$ . Let  $\tilde{Y}$  be the linearly ordered set obtained from  $Y$  by inserting a new point  $y'_x$  between  $y_x$  and  $\{y \in Y : y_x < y\}$  for each  $x \in X$ .  $\tilde{Y}$  is still sequentially compact and  $d(\tilde{Y}) < \mathfrak{s}$  since only fewer than  $\mathfrak{s}$  new points are added. Hence we can apply Lemma 4 again to  $(f_n)_{n \in I}$  as a sequence of functions from  $X$  to  $\tilde{Y}$  to obtain an infinite  $J \subseteq I$  such that  $(f_n)_{n \in J}$  is pointwise convergent as a sequence of functions from  $X$  to  $\tilde{Y}$ . For each  $x \in X$ , as  $(f_n(x))_{n \in J}$  should converge to  $y_x$  or  $y'_x$ , it follows that, for each  $x \in X$ ,  $(f_n(x))_{n \in J}$  as a sequence of points in  $Y$  approaches  $y_x$  eventually from one side. Hence by the remark before this lemma,  $(f_n(x))_{n \in J}$  is a quasi-monotone sequence in  $Y$ .  $\square$

### 3. GENERALIZED HELLY'S THEOREM

Since the density of the reals is countable, Helly's theorem ([3]) as cited in the Introduction is just a special case of the following theorem.

**Theorem 7** (Generalized Helly's Theorem). *Let  $X$  and  $Y$  be linearly ordered sets. If  $Y$  is sequentially compact with density less than  $\mathfrak{s}$ , then any sequence of monotone functions from  $X$  to  $Y$  contains a pointwise convergent subsequence.*

*Proof.* Without loss of generality, we may assume that the sequence  $(f_n)_{n \in \mathbb{N}}$  consists of increasing functions. By Lemma 2 we may also assume that  $Y$  is a dense linear order. Let  $D \subseteq Y$  be a dense subset of cardinality less than  $\mathfrak{s}$ . For  $b, d \in D$  and  $n \in \mathbb{N}$ , let  $x_{b,d}^n$  be an element of  $X$  such that  $b \leq f_n(x_{b,d}^n) \leq d$  if such an element exists; otherwise let  $x_{b,d}^n$  be an arbitrary element of  $X$ . Let

$$Z = \{x_{b,d}^n : b, d \in D, n \in \mathbb{N}\}.$$

Then we have  $|Z| < \mathfrak{s}$ . Hence, by Lemma 4, there exists an infinite subset  $L \subseteq \mathbb{N}$  such that the sequence of functions  $(f_n \upharpoonright Z)_{n \in L}$  converges pointwise where  $f_n \upharpoonright Z$  denotes the restriction of the function  $f_n$  to the set  $Z$ . Let

$$T = \{x \in X : (f_n(x))_{n \in L} \text{ is a convergent sequence of points in } Y\}.$$

We have  $Z \subseteq T$ . Let  $h : T \rightarrow Y$  be the function such that the sequence of functions  $(f_n \upharpoonright T)_{n \in L}$  converges pointwise to  $h$ . By Lemma 3,  $h$  is an increasing function. Now let

$$\mathcal{U} = \{U : U \text{ is a maximal interval in } X \text{ such that } U \subseteq X \setminus T\}.$$

For each interval  $U \in \mathcal{U}$  we choose  $x_U \in U$ . By definition of  $Z$ ,  $f_n$  is constant on each  $U \in \mathcal{U}$  for every  $n \in \mathbb{N}$ . Hence, for any subset  $M \subseteq L$ , we have:

$$(*) \quad (f_n(x_U))_{n \in M} \text{ converges if and only if } (f_n \upharpoonright U)_{n \in M} \text{ converges pointwise.}$$

Letting  $W = \{x_U : U \in \mathcal{U}\}$ , we claim that  $|W| < \mathfrak{s}$ . To see this, let  $x \in W$ . The sequence  $(f_n(x))_{n \in L}$  is not convergent. Since  $Y$  is a dense linear order and sequentially compact, by Lemma 1, there are infinite subsets  $J, K \subseteq L$  and points  $b_x, c_x, d_x \in D$  such that sequences of points  $(f_n(x))_{n \in J}$  and  $(f_n(x))_{n \in K}$  are convergent and

$$\lim_{n \in J} f_n(x) < b_x < d_x < c_x < \lim_{n \in K} f_n(x).$$

If  $y \in T$  and  $y < x$ , then  $h(y) \leq b_x$ , since  $\lim_{n \in L} f_n(y) = h(y)$  and  $f_n(y) < b_x$  for infinitely many  $n \in L$ . Likewise, for any  $z \in T$  with  $x < z$  we have  $h(z) \geq c_x$ . For  $x_1, x_2 \in W$  with  $x_1 < x_2$ , there is  $y \in T$  such that  $x_1 < y < x_2$ . Hence the mapping from  $W$  to  $D$  defined by  $x \mapsto d_x$  is injective. As  $|D| < \mathfrak{s}$ , it follows that  $|W| < \mathfrak{s}$ .

Again by Lemma 4 we can find an infinite  $I \subseteq L$  such that  $(f_n \upharpoonright W)_{n \in I}$  is pointwise convergent. By definitions and  $(*)$  above, we have that the sequence of functions  $(f_n)_{n \in I}$  converges pointwise.  $\square$

#### 4. THE SPLITTING NUMBER IS OPTIMAL

For an infinite subset  $V \subseteq \mathbb{N}$ , let

$$\mathcal{D}(V) = \sum_{n \in V} \frac{1}{2^{n+1}}.$$

Note that  $\mathcal{D}$  is a bijective mapping from infinite subsets of  $\mathbb{N}$  to the real numbers in the half-open interval  $(0, 1]$ . For a family  $\mathcal{S}$  of subsets of  $\mathbb{N}$ , let us denote by  $\mathcal{D}(\mathcal{S})$  the set  $\{\mathcal{D}(V) : V \in \mathcal{S}\}$ . Thus  $\mathcal{D}(\mathcal{S})$  is a subset of the unit interval of cardinality  $|\mathcal{S}|$ .

Assume now that  $\mathcal{S}$  is a splitting family of cardinality  $\mathfrak{s}$ . Let

$$H = (\mathcal{D}(\mathcal{S}) \times [0, 1]) \cup ([0, 1] \times \{0\})$$

be the linearly ordered set equipped with the lexicographical ordering, i.e. we let  $(x, y) < (p, q)$ , whenever  $x < p$ , or  $x = p$  and  $y < q$ . Here,  $x < p$  and  $y < q$  denote the canonical ordering on the reals.

**Lemma 8.** *The linearly ordered set  $H$  is sequentially compact and its density is equal to  $\mathfrak{s}$ .*

*Proof.* Suppose that  $S = ((x_n, y_n))_{n \in \mathbb{N}}$  is a monotone sequence of points in  $H$ . Then  $(x_n)_{n \in \mathbb{N}}$  is monotone as well. If  $(x_n)_{n \in \mathbb{N}}$  is eventually constant, say  $x_n = x$  for all  $n > m$ , then  $(y_n)_{n > m}$  is a monotone sequence. Hence  $\lim_{n \rightarrow \infty} y_n$  exists and  $S$  converges to  $(x, \lim_{n \rightarrow \infty} y_n)$ . Otherwise there are infinitely many distinct  $x_n$ 's. If  $S$  is increasing, then  $S$  converges to  $(\lim_{n \rightarrow \infty} x_n, 0)$ . If  $S$  is decreasing, then  $S$  converges to  $(\lim_{n \rightarrow \infty} x_n, 1)$  provided that  $\lim_{n \rightarrow \infty} x_n \in \mathcal{D}(S)$ ; otherwise it converges to  $(\lim_{n \rightarrow \infty} x_n, 0)$ .

Let  $Q$  be the set of rational numbers in the unit interval  $[0, 1]$ . Then

$$H_0 = (\mathcal{D}(S) \times Q) \cup (Q \times \{0\})$$

is dense in  $H$  and of cardinality  $\mathfrak{s}$ . This shows that  $d(H) \leq \mathfrak{s}$ . If  $H' \subseteq H$  is of cardinality less than  $\mathfrak{s}$ , then there is some  $s \in \mathcal{D}(S)$  such that  $\{s\} \times [0, 1]$  is disjoint from  $H'$ . Hence  $H'$  is not dense in  $H$ . Thus we also have  $d(H) \geq \mathfrak{s}$ .  $\square$

The following theorem is a variation of an example in [6].

**Theorem 9.** *There exists a sequence  $(f_n)_{n \in \mathbb{N}}$  of increasing functions from the subset  $\mathcal{D}(S)$  of the unit interval to the linearly ordered set  $H$  such that  $(f_n)_{n \in \mathbb{N}}$  does not have any pointwise convergent subsequence.*

*Proof.* For each  $n \in \mathbb{N}$  and  $V \in \mathcal{S}$ , let

$$f_n(\mathcal{D}(V)) = \begin{cases} (\mathcal{D}(V), 1), & \text{if } n \in V; \\ (\mathcal{D}(V), 0), & \text{otherwise.} \end{cases}$$

Each function  $f_n$  is obviously increasing. For any infinite subsequence  $(f_n)_{n \in I}$ , let  $V \in \mathcal{S}$  be such that the sets  $I \cap V$  and  $I \setminus V$  are both infinite. Then we have  $f_n(\mathcal{D}(V)) = (\mathcal{D}(V), 1)$  for every  $n \in I \cap V$ , and  $f_n(\mathcal{D}(V)) = (\mathcal{D}(V), 0)$  for every  $n \in I \setminus V$ . In particular, the sequence of points  $(f_n(\mathcal{D}(V)))_{n \in I}$  is not convergent.  $\square$

The theorem above shows that the condition  $d(Y) < \mathfrak{s}$  in Theorem 7 is optimal. Using this fact, we obtain the following characterization of the splitting number.

Let  $\tau_1$  be the supremum of the cardinals  $\kappa$  with the property that for every set  $X$  of cardinality less than  $\kappa$  and for every sequentially compact linearly ordered set  $Y$  of density less than  $\kappa$ , any sequence of functions from  $X$  to  $Y$  has a pointwise convergent subsequence. Likewise, let  $\tau_2$  be the least cardinal  $\kappa$  such that, for some set  $X$  of cardinality  $\kappa$ , it is not the case that any sequence of functions from  $X$  to  $\{0, 1\}$  has a pointwise convergent subsequence, where we consider  $\{0, 1\}$  as a linearly ordered set with  $0 < 1$ . Finally, let  $\mu$  be the supremum of the cardinals  $\kappa$  with the property that, for any linearly ordered set  $X$  and any sequentially compact linearly ordered set  $Y$  with  $d(Y) < \kappa$ , any sequence of monotone functions from  $X$  to  $Y$  has a pointwise convergent subsequence.

**Theorem 10.**  $\mathfrak{s} = \tau_1 = \tau_2 = \mu$ .

*Proof.* By definition we have  $\tau_1 \leq \tau_2$ . Lemma 4 implies  $\mathfrak{s} \leq \tau_1$ ;  $\mathfrak{s} \leq \mu$  follows from Theorem 7. Theorem 9 implies  $\mathfrak{s} \geq \mu$ .

To see  $\tau_2 \leq \mathfrak{s}$ , we use a variant of Rademacher's functions (see [4]): for each  $n \in \mathbb{N}$ , the function  $\varphi_n$  on the family of all infinite subsets of  $\mathbb{N}$  to  $\{0, 1\}$  is defined by

$$\varphi_n(V) = \begin{cases} 1, & \text{if } n \in V; \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\mathcal{S}$  be a splitting family of cardinality  $\mathfrak{s}$ . For any infinite subset  $I \subseteq \mathbb{N}$ , let  $V \in \mathcal{S}$  be such that the sets  $I \cap V$  and  $I \setminus V$  are both infinite. Then the 0-1 sequence  $(\varphi_n(V))_{n \in I}$  is not convergent as 0 and 1 both appear infinitely many times in this sequence. This shows that no subsequence of the sequence  $(\varphi_n \upharpoonright \mathcal{S})_{n \in \mathbb{N}}$  of functions from  $\mathcal{S}$  to  $\{0, 1\}$  can be pointwise convergent.  $\square$

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