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NO b-CONCENTRATED MEASURES WHENEVER $b^{nk}(b^n - 1) = 1$

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Introduction

In this note μ is a non-trivial, finite and continuous Borel measure on the reals, and

$$f(x,h) = \mu\big((x-h, x+h)\big)$$

for every real number x and every positive real number h.

Directly from definitions we infer that $\operatorname{supp}(\mu)$, i.e. the reals minus the union of all open sets with μ-measure null, is a perfect set and for any real numbers a, b, c the set

$$\left\{x \in \operatorname{supp}(\mu) : \frac{f(x,bh)}{f(x,h)} \leqq a \text{ for each } 0 < h < c\right\}$$

is closed in view of continuity of μ.

A measure μ is *b-concentrated* if for any real number x, which belongs to $\operatorname{supp}(\mu)$, there holds

$$\limsup_{h \to 0+} \frac{f(x,bh)}{f(x,h)} < b.$$

The concept of b-concentrated measures was introduced and examined in [1] and [2]. In these two papers Z. Buczolich and M. Laczkovich verified that for $b < 1.01$ and $b = \sqrt[n]{2}$, where $n = 1, 2, \ldots$, there are no b-concentrated measures, while for $b > 2$ there are such ones. We prove that if a positive real number b satisfies the equality $1 = b^{nk}(b^n - 1)$, where k and n are natural numbers, then there does not exist any b-concentrated measure. Thus, we answer a problem stated in [1] on page 349 — *whether b-concentrated measures exist for $b \in (1,2)$* — in some cases. For $k = 0$ we reprove the result of Z. Buczolich and M. Laczkovich that there are no b-concentrated measures whenever $b = 2$.

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Results

LEMMA 1. *If μ is a b-concentrated measure, then there is a set H, which is homeomorphic to the Cantor set and is open relative to* $\operatorname{supp}(\mu)$, *such that*

$$H \subseteqq \left\{x \in \operatorname{supp}(\mu) : f(x, bh) < bf(x, h) \text{ for each } 0 < h < \frac{1}{n}\right\}$$

for some natural number n.

PROOF. For every natural number n the set

$$H_n = \left\{x \in \operatorname{supp}(\mu) : f(x, bh) \leqq \left(b - \frac{1}{n}\right) f(x, h) \text{ for each } 0 < h < \frac{1}{n}\right\}$$

is closed and

$$H_0 \cup H_1 \cup \ldots = \operatorname{supp}(\mu).$$

Therefore and by the Baire theorem, some H_n has non-empty interior relative to $\operatorname{supp}(\mu)$. Since $\operatorname{supp}(\mu)$ does not contain non-degenerate intervals by Corollary 2.5 of [1], there are real numbers a and c, which do not belong to $\operatorname{supp}(\mu)$, such that the set

$$H = [a, c] \cap \operatorname{supp}(\mu) = [a, c] \cap H_n$$

is not empty. Clearly, the set H is perfect, compact, open relative to $\operatorname{supp}(\mu)$ and does not contain non-degenerate intervals, i.e. is homeomorphic to the Cantor set and is open relative to $\operatorname{supp}(\mu)$. □

LEMMA 2. *If b is a positive real number such that there does not exist any b-concentrated measure, then there does not exist any $\sqrt[n]{b}$-concentrated measure whenever $n > 0$ is a natural number.*

PROOF. In the proof of 4.2 Theorem of [1] it was shown that if a measure is b-concentrated, then it is b^n-concentrated for every natural number $n > 0$. The contraposition of this implies the lemma. □

Hereinafter we assume that $1 < b \leqq 2$ and μ is a b-concentrated measure such that

$$\operatorname{supp}(\mu) = \left\{x : f(x, bh) < bf(x, h) \text{ for each } 0 < h < \frac{1}{n}\right\},$$

for some natural number n. These assumptions do not loose generality, since by Lemma 1 we can consider μ restricted to H and such a restriction is b-concentrated.

Let (x, y) be an interval contiguous to $\operatorname{supp}(\mu)$ such that

$$h = y - x < \frac{1}{n}.$$

We put

$$D = \mu\big((x - h, y - bh)\big), \qquad E = \mu\big((y - bh, x)\big),$$

$$F = \mu\big((y, x + bh)\big), \qquad G = \mu\big((x + bh, y + h)\big).$$

LEMMA 3.

$$D + G > (E + F)\frac{2 - b}{b - 1}.$$

PROOF. Directly from the definitions we have

$$bf(y, h) > f(y, bh) \quad \text{and} \quad bf(x, h) > f(x, bh),$$

which implies

$$b(F + G) > F + G + E \quad \text{and} \quad b(E + D) > E + D + F.$$

Therefore

$$F > \frac{E}{b - 1} - G \quad \text{and} \quad E > \frac{F}{b - 1} - D.$$

By adding these inequalities, we obtain the statement of the lemma. □

Let us note that if $b = 2$, then $D = 0 = G$ and Lemma 3 says that $0 > 0$. This is a contradiction, which implies 4.2 Theorem in [1], i.e. that there are no 2-concentrated measures.

THEOREM. *Let k and n be natural numbers. If a positive real number b satisfies*

$$b^{nk}(b^n - 1) = 1,$$

then there does not exist any b-concentrated measure.

PROOF. If $k = 0$, then $b^n = 2$. In these cases we are done by 4.2 Theorem in [1]. Because of Lemma 2 we can assume $n = 1$.

If $k > 0$, then for every natural number $m < k$ we put

$$X_m = \mu\big(\,(y + b^m(b - 1)h, y + b^{m+1}(b - 1)h)\,\big).$$

Since μ is a b-concentrated measure there holds

$$F + X_0 < bF,$$
$$F + X_0 + X_1 < b^2 F,$$
$$\dots$$
$$F + X_0 + \ldots + X_{k-1} < b^k F.$$

Because of $G = X_0 + \ldots + X_{k-1}$ and $b^k(b-1) = 1$ we obtain

$$G < (b^k - 1)F.$$

If we add to this inequality the similarly obtained inequality

$$D < (b^k - 1)E,$$

then we get

$$D + G < (b^k - 1)(E + F).$$

Combining this inequality with the inequality from Lemma 3 we get

$$b^k - 1 > \frac{2-b}{b-1}.$$

This inequality is equivalent to $b^k(b-1) > 1$, a contradiction, which implies that there are no b-concentrated measures with $b^k(b-1) = 1$. □

References

[1] Z. Buczolich and M. Laczkovich, Concentrated Borel measures, *Acta Math. Hungar.*, **57** (1991), 349–362.

[2] Z. Buczolich, No b-concentrated measures with $b < 1.01$, *Real Analysis Exchange*, **19** (1993–94), 612–615.

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