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PROPERTY (P) AND GAME IDEALS

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ABSTRACT. We prove that, for each family $\mathcal{K} \subseteq [\omega]^\omega$ of size less than dominating number $\mathfrak{f}$ and for each perfect set $P \subseteq 2^\omega$, there is a perfect subset $Q \subseteq P$ fulfilling $Q|X \neq 2^X$ for each $X \in \mathcal{K}$. We apply this to show that some game ideals on 2^ω are not perfect isomorphic.

1. Introduction

In [B1] and [B2] the following property (P) for ideals of subsets of an uncountable Polish space X was investigated: *For each perfect set $P \subseteq X$ there is a perfect subset $Q \subseteq P$ belonging to a given ideal.* Several examples of ideals with the property (P) were given and an application of this property to a generalized version of the Erdös–Sierpiński theorem (cf. [O, Th.19.5]) was considered. Here we continue those studies. We show a combinatorial result which decides the property (P) and we use it to exclude a natural kind of isomorphism between some game ideals on the Cantor space.

Let us note that the property (P) can be considered not only for ideals. It can be also interesting for hereditary families. Recall that a family $\mathcal{K}$ is *hereditary* whenever $A \subseteq B \in \mathcal{K}$ implies $A \in \mathcal{K}$. A simple fact holds: *if $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{P}(X)$ and $\mathcal{I}$ has the property* (P), *then $\mathcal{J}$ has this property too.*

Suppose that X and Y are uncountable Polish spaces. A bijection $f\colon X \to Y$ is called a *perfect isomorphism* if for each perfect set $P \subseteq X$ the image $f[P]$ contains a perfect set and for each perfect set $Q \subseteq Y$ the preimage $f^{-1}[Q]$ contains a perfect set. From the classical theorems of Alexandrov, Hausdorff and Souslin [§ 37, I; § 39, IV–V] it follows that each Borel isomorphism between X and Y is a perfect isomorphism. We say that families $\mathcal{I} \subseteq \mathcal{P}(X)$ and $\mathcal{J} \subseteq \mathcal{P}(Y)$ are *perfect isomorphic* provided that there exists a perfect isomorphism $f\colon X \to Y$ such that for each $E \subseteq X$ we have $E \in \mathcal{I}$ iff $f[E] \in \mathcal{J}$.

Proposition. *Assume that hereditary families $\mathcal{I} \subseteq \mathcal{P}(X)$ and $\mathcal{J} \subseteq \mathcal{P}(Y)$ are perfect isomorphic. If $\mathcal{I}$ has the property (P), then $\mathcal{J}$ has the property (P).*

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Proof. Let $f\colon X \to Y$ be a perfect isomorphism witnessing that $\mathcal{I}$ and $\mathcal{J}$ are perfect isomorphic. Consider a perfect set $P \subseteq Y$. Then $f^{-1}[P]$ contains a perfect set Q since f is a perfect isomorphism. By the property (P) for $\mathcal{I}$, we can assume that $Q \in \mathcal{I}$. Since families $\mathcal{I}$ and $\mathcal{J}$ are perfect isomorphic, the image $f[Q]$ belongs to $\mathcal{J}$. Since f is a perfect isomorphism, the image $f[Q]$ contains a perfect set R. We have $R \in \mathcal{J}$ because the family $\mathcal{J}$ is hereditary and $R \subseteq f[Q] \in \mathcal{J}$. Obviously, $R \subseteq P$. $\square$

2. Combinatorial result

In the sequel ω denotes the set of all natural numbers, $[Y]^\omega$ denotes the family of all infinite (countable) subsets of Y and 2^ω denotes the Cantor space, i.e., the set of all infinite sequences of zeros and ones endowed with the product topology. For a subset $A \subseteq 2^\omega$ and a set $X \in [\omega]^\omega$ we put

$$A|_X = \{f|_X \colon f \in A\},$$

where $f|_X$ denotes the restriction of the function f to the set X.

Recall (cf. [V]) that a set $X \subseteq \omega^\omega$ (here ω^ω denotes the set of all infinite sequences with terms in ω) is $\leq^*$ *dominating* provided that for each $f \in \omega^\omega$ there exists $g \in X$ such that $f \leq^* g$. The order $\leq^*$ is defined by: $f \leq^* g$ iff there exists $k \in \omega$ such that $f(n) \leq g(n)$ for all $n \geq k$. The *dominating number* $\mathfrak{d}$ is given by the formula

$$\mathfrak{d} = \min\{|D| \colon D \text{ is } \leq^* \text{dominating}\},$$

where $|D|$ denotes the size of the set D.

It is known that $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$, where $\mathfrak{c}$ stands for the size of the continuum, and that $\omega_1 < \mathfrak{d}$ or $\mathfrak{d} < \mathfrak{c}$ in some models of ZFC (cf. [V]). Observe that in the definition of $\mathfrak{d}$ one can restrict considerations to sets consisting of increasing sequences. Moreover, since $\mathfrak{d} \cdot \omega = \mathfrak{d}$, the order $\leq^*$ can be replaced by the order $\leq$, where $f \leq g$ iff $f(n) \leq g(n)$ for all $n \in \omega$. Further, one can identify an increasing sequence $f \in \omega^\omega$ with its range and, conversely, one can treat a set $K \in [\omega]^\omega$ as the increasing sequence $\langle k_0, k_1, \ldots\rangle$, where $k_n < k_{n+1}$ for each $n \in \omega$. We will use that identification below. So, one can consider the order $\leq$ for $[\omega]^\omega$, where $\langle k_0, k_1, \ldots\rangle \leq \langle l_0, l_1, \ldots\rangle$ means that $k_n \leq l_n$ for each $n \in \omega$. Similarly as $\leq^*$ dominating sets one can define $\leq$ *dominating* sets for families contained in $[\omega]^\omega$ and use the formula

$$\mathfrak{d} = \min\{|D| \colon D \text{ is } \leq \text{dominating}\},$$

which gives an equivalent definition of the dominating number $\mathfrak{d}$.

For $x \in 2^\omega$ and $n \in \omega$ we denote $x = \langle x_0, x_1, \ldots \rangle$ and $x|_n = \langle x_0, x_1, \ldots, x_{n-1} \rangle$. For $E \subseteq 2^\omega$ and $n \in \omega$ we say that n is a *branching number for* E, if there are $x, y \in E$ such that $x|_n = y|_n$ and $x_n \neq y_n$. The set of all branching numbers for E is denoted by $\mathrm{br}(E)$.

THEOREM. *For each family $\mathcal{K} \subseteq [\omega]^\omega$ of size less than the dominating number $\mathfrak{d}$ and for each perfect set $P \subseteq 2^\omega$ there is a perfect subset $Q \subseteq P$ fulfilling $Q|_X \neq 2^X$ for each $X \in \mathcal{K}$.*

We need the two following lemmas.

LEMMA 1. *If $P \subseteq 2^\omega$ is a perfect set and $K \in [\omega]^\omega$, then there exists a perfect subset $Q \subseteq P$ such that $K \leq \mathrm{br}(Q)$.*

Proof. Let $P = [T]$, where $[T]$ denotes a set of all infinite branches of a perfect tree $T \subseteq 2^{<\omega}$. For the representation of a perfect set by perfect trees, see, e.g., [J]. We will use this representation of perfect sets below. Thus we shall find a perfect subtree $S \subseteq T$ such that $K \leq \mathrm{br}([S])$.

We shall define auxiliary increasing sequences $\langle r_0, r_1, \ldots \rangle$ and $\langle m_0, m_1, \ldots \rangle$ with terms in ω. Choose an arbitrary sequence $x_0^0 \in T$ with length $r_0 = k_0$. Let $m_0 = 0$ and $X_0 = \{x_0^0\}$. Next, find $m_1 \in \omega$ such that $m_1 > \max\{k_{2^2}, r_0\}$ and choose distinct extensions x_1^1, x_2^1 of x_0^0 both of length r_1 such that $x_1^1|_{m_1} = x_2^1|_{m_1}$. Put $X_1 = \{x_1^1, x_2^1\}$.

Assume that we have defined $m_{n-1} \in \omega$ and distinct sequences $x_i^{n-1} \in T$, where $i = 1, \ldots, 2^{n-1}$, all of length r_{n-1}. Moreover, $X_{n-1} = \{x_i^{n-1} \colon i = 1, \ldots, 2^{n-1}\}$.

Choose $m_n \in \omega$ such that $m_n > \max\{k_{2^{n+1}}, m_{n-1}, r_{n-1}\}$ and choose distinct extensions

$$(x_i^{n-1})^1, (x_i^{n-1})^2 \in T$$

of x_i^{n-1}, where $i = 1, \ldots, 2^{n-1}$, all of length r_n such that

$$(x_i^{n-1})^1\big|_{m_n} = (x_i^{n-1})^2\big|_{m_n}.$$

Obviously, $r_n > r_{n-1}$. Put

$$X_n = \big\{(x_i^{n-1})^j \colon\ 1 \leq i \leq 2^{n-1} \text{ and } j = 1, 2\big\}$$

to finish the induction, and define

$$S = \bigcup_{n \in \omega} \{t|_j \colon\ t \in X_n \text{ and } j \leq r_n\}.$$

Clearly, S is a perfect subtree of T. It remains to show that $K \leq \mathrm{br}([S])$. Let $L = \mathrm{br}([S])$. By construction $l_0 \geq m_1$. Consider an arbitrary l_i. There is a unique $n \in \omega$ such that $m_n \leq l_i < m_{n+1}$. By construction, there is a unique branching number for $[S]$ in the interval $[m_1, m_2)$ and the number of branching numbers in each consecutive interval $[m_j, m_{j+1})$ at most doubles. Hence

$$l_i \leq \sum_{j=1}^{n} 2^{j-1} = 2^{n+1} - 1 < 2^{n+1}.$$

Consequently, $k_i < k_{2^{n+1}} \leq m_n \leq l_i$. □

Lemma 2. *If $Q \subseteq 2^\omega$ and $M \in [\omega]^\omega$ is such that $2^M = Q|_M$, then $\mathrm{br}(Q) \leq M$.*

Proof. Let $\mathrm{br}(Q) = L$. For each $n \in \omega$ we have

$$\left|Q|_{\{0,\dots,l_n\}}\right| \leq 2^{n+1} \quad \text{and} \quad \left|Q|_{\{m_0,\dots,m_n\}}\right| = 2^{n+1}.$$

Suppose that $l_n > m_n$ for some $n \in \omega$. Then $\{m_0, \dots, m_n\} \subseteq \{0, \dots, l_{n-1}\}$. Hence

$$2^{n+1} = \left|Q|_{\{m_0,\dots,m_n\}}\right| \leq \left|Q|_{\{0,\dots,l_{n-1}\}}\right| \leq 2^n,$$

a contradiction. □

Proof of Theorem. Assume that a family $\mathcal{K} \subseteq [\omega]^\omega$ is of size less than $\mathfrak{d}$. Suppose that the assertion is false. Thus there is a perfect subset $P \subseteq 2^\omega$ such that for each perfect set $Q \subseteq P$ there exists $X \in \mathcal{K}$ with $Q|_X = 2^X$. Consider an arbitrary $K \in [\omega]^\omega$. By Lemma 1 we can find a perfect set $Q \subseteq P$ such that $K \leq \mathrm{br}(Q)$. Now, let $M \in \mathcal{K}$ be such that $Q|_M = 2^M$ — it exists, by our supposition. By Lemma 2 we have $\mathrm{br}(Q) \leq M$. Hence $K \leq M$, but this means that $\mathcal{K}$ is $\leq$ dominating. This yields a contradiction since $|\mathcal{K}| < \mathfrak{d}$. □

We do not know whether, without extra axioms added to ZFC, one can replace $\mathfrak{d}$ by $\mathfrak{c}$ in the above theorem.

3. Applications

Now we recall basic information about game ideals on 2^ω considered by Mycielski [M] and Rosłanowski [R1, R2]. For $A \subseteq 2^\omega$ and $X \in [\omega]^\omega$ we denote by $\Gamma(A, X)$ the infinite game with perfect information defined as follows (cf. [M]). Two players choose consecutive terms of $x \in 2^\omega$. Player I chooses x_i whenever $i \in \omega \setminus X$, and player II chooses x_i whenever $i \in X$. Player I wins when $x \in A$, and player II — when $x \notin A$. The family of all sets $A \subseteq 2^\omega$ for which player II has a winning strategy in $\Gamma(A, X)$ is denoted by $V(X)$. For a precise definition of a strategy, see, e.g., [R2]. Now, consider a special case when player II has a winning strategy in $\Gamma(A, X)$ which does not depend on the moves of player I. That situation is equivalent to $A|_X \neq 2^X$. Put

$$V^*(X) = \left\{A \subseteq 2^\omega \colon A|_X \neq 2^X\right\}.$$

Observe that $V^*(X) \subseteq V(X)$ and the inclusion can be proper. For $\mathcal{K} \subseteq [\omega]^\omega$ let

$$M(\mathcal{K}) = \bigcap_{X\in\mathcal{K}} V(X) \qquad \text{and} \qquad M^*(\mathcal{K}) = \bigcap_{X\in\mathcal{K}} V^*(X).$$

Obviously, $M^*(\mathcal{K}) \subseteq M(\mathcal{K})$ and both families are hereditary.

We say that $\mathcal{K} \subseteq [\omega]^\omega$ is a *normal system* whenever for each $X \in \mathcal{K}$ there are $Y, Z \in \mathcal{K}$ such that $Y \cup Z \subseteq X$ and $Y \cap Z = \emptyset$.

If $\mathcal{K}$ is a normal system, $M(\mathcal{K})$ and $M^*(\mathcal{K})$ form σ-ideals in 2^ω. In fact, the normality of $\mathcal{K}$ is needed to show that those families are countable additive. Ideals $M(\mathcal{K})$ for countable normal systems were first considered by Mycielski [M]. In a general case, $M(\mathcal{K})$ and $M^*(\mathcal{K})$ were studied by Rosłanowski [R1, R2]. In [B2] it was shown that the ideal $M([\omega]^\omega)$ has the property (P). However, the smaller ideal $M^*([\omega]^\omega)$ does not have the property (P) — this follows from [DKP], where the problem stated in [B2] was solved. On the other hand, it is not hard to check that $M^*(\mathcal{K})$ has the property (P) for each countable $\mathcal{K}$. So, it would be interesting to know what is the least size of a $\mathcal{K}$ for which $M^*(\mathcal{K})$ does not have the property (P). The following result being a straightforward reformulation of our Theorem enables us to estimate that cardinal from below.

THEOREM*. *For each family $\mathcal{K} \subseteq [\omega]^\omega$ of size less than the dominating number $\mathfrak{d}$ the ideal $M^*(\mathcal{K})$ has the property (P).*

Observe that among families $\mathcal{K} \subseteq [\omega]^\omega$ of size $\mathfrak{c}$ there are both kinds: those which produce the property (P) for $M^*(\mathcal{K})$ and those which exclude this property.

EXAMPLE 1. For a fixed $n \in \omega$ let

$$\mathcal{K}_n = \{X \in [\omega]^\omega : n \in X\}.$$

We will show that $M^*(\mathcal{K}_n)$ has the property (P). Let $S \subseteq 2^\omega$ be an arbitrary perfect set. Pick any $x \in S$ and let Q consist of all extensions of $\langle x_0, \ldots, x_n\rangle$ belonging to S. Then Q is a perfect set and $Q|_X \neq 2^X$ for each $X \in \mathcal{K}_n$ since for each $y \in Q$ we have $y_n = x_n$.

EXAMPLE 2. For a fixed $Y \in [\omega]^\omega$ let

$$\mathcal{K}_Y = \{X \in [\omega]^\omega : X \subseteq Y\}.$$

We will show that $M^*(\mathcal{K}_Y)$ does not have the property (P). Since 2^Y is homeomorphic to 2^ω, therefore $M^*([Y]^\omega)$ in the space 2^Y does not have the property (P) by [DKP]. Hence there exists a perfect set $P \subseteq 2^Y$ such that for each perfect subset $Q \subseteq P$ in 2^Y we can find $X \in [Y]^\omega$ such that $Q|_X = 2^Y$. We extend $x \in P$ to $x^* \in 2^\omega$ letting $x^*_n = 0$ for $n \in \omega \setminus Y$. Then the perfect set $P^* = \{x^* : x \in P\}$ witnesses that $M^*(\mathcal{K}_Y)$ does not have the property (P).

Now, using Proposition we can derive:

COROLLARY 1. *Ideals $M([\omega]^\omega)$ and $M^*([\omega]^\omega)$ are not perfect isomorphic.*

Proof. The ideal $M([\omega]^\omega)$ has the property (P) by [B2] but the ideal $M^*([\omega]^\omega)$ does not have (P) by [DKP]. □

COROLLARY 2. *Let $\mathcal{I}, \mathcal{J} \subseteq [\omega]^\omega$ be such that $|\mathcal{I}| < \mathfrak{d}$ and $\mathcal{J} = \mathcal{K}_Y$ for some $Y \in [\omega]^\omega$. Then $M^*(\mathcal{I})$ and $M^*(\mathcal{J})$ are not perfect isomorphic.*

Proof. The ideal $M^*(\mathcal{I})$ has the property (P) by Theorem* but the ideal $M^*(\mathcal{J})$ does not have the property (P) by Example 2. □

REFERENCES

[B1] BALCERZAK, M.: *On σ-ideals having perfect members in all perfect sets*, Demonstratio Math. **22** (1989), 1159–1168.

[B2] BALCERZAK, M.: *Some properties of ideals of sets in Polish spaces (thesis)*, Acta Univ. Łódź, Łódź, 1991.

[DKP] DĘBSKI, W.—KLESZCZ, J.—PLEWIK, SZ.: *Perfect sets of independent functions*, Acta Univ. Carolin.-Math. Phys. **33** (1992), 31–33.

[J] JECH, T.: *Set Theory*, Academic Press, New York, 1978.

[K] KURATOWSKI, K.: *Topology*, Vol. I, PWN–Academic Press, Warszawa–New York, 1966.

[M] MYCIELSKI, J.: *Some new ideals of sets on the real line*, Colloq. Math. **20** (1969), 71–76.

[O] OXTOBY, J. C.: *Measure and Category*, Springer-Verlag, Berlin, 1971.

[R1] ROSLANOWSKI, A.: *On game ideals*, Colloq. Math. **54** (1990), 159–168.

[R2] ROSLANOWSKI, A.: *Mycielski ideals generated by uncountable systems*, Colloq. Math. **66** (1994), 187–200.

[V] VAUGHAN, J. E.: *Small uncountable cardinals and topology*, in: Open Problems in Topology (J. van Mill and G.M. Reed, eds.), North Holland, Amsterdam, 1990, pp. 197–218.

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