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HAUSDORFF GAPS RECONSTRUCTED FROM LUZIN GAPS

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Abstract. We consider a question: Can a given AD-family be ADR for two orthogonal uncountable towers? If $b > \omega_1$, then we rebuilt any AD-family of the cardinality ω_1 onto a Hausdorff pre-gap. Moreover, if a such AD-family is a Luzin gap, then we obtain a Hausdorff gap. Under $b = \omega_1$, a similar rebuilding is impossible.

1. Introduction

A family $\mathcal{Q}$ is called *almost disjoint*, briefly AD-*family*, whenever any two members of $\mathcal{Q}$ are almost disjoint, i.e. their intersection is finite. A set C *separates* a family $\mathcal{Q}$ from a family $\mathcal{H}$, whenever each member of $\mathcal{Q}$ is almost contained in C, i.e. $B \setminus C$ is finite for any $B \in \mathcal{Q}$, and each member of $\mathcal{H}$ is almost disjoint with C. Whenever sets A and B are almost disjoint for any $A \in \mathcal{Q}$ and $B \in \mathcal{H}$, then families $\mathcal{Q}$ and $\mathcal{H}$ are called *orthogonal*. If no set C separates $\mathcal{Q}$ from $\mathcal{H}$, then families $\mathcal{Q}$ and $\mathcal{H}$ are called *non-separated*. Below, $A \subset^* B$ means that A is almost contained in B, but not conversely. A pair of indexed families $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ is called *Hausdorff pre-gap*, whenever $\alpha < \beta < \omega_1$ implies $A_\alpha \subset^* A_\beta \subset^* B_\beta \subset^* B_\alpha$. A Hausdorff pre-gap $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ is called *Hausdorff gap*, whenever orthogonal towers $\{A_\alpha : \alpha < \omega_1\}$ and $\{\omega \setminus B_\alpha : \alpha < \omega_1\}$ are non-separated. Establish that, a family $\{A_\alpha : \alpha < \lambda\}$ is a *tower*, whenever $\alpha < \beta$ implies $A_\alpha \subset^* A_\beta$. An AD-family $\mathcal{Q}$ of the cardinality ω_1 is called *Luzin gap*, whenever no two disjoint uncountable subfamilies of $\mathcal{Q}$ are separated. An AD-family$\mathcal{Q}$ is *almost disjoint refinement* of a family $\mathcal{P}$ (briefly $\mathcal{Q}$ is ADR of $\mathcal{P}$), whenever there exists a bijection $f : \mathcal{Q} \to \mathcal{P}$ such that X is almost contained in $f(X)$ for every $X \in \mathcal{Q}$. Our definition of ADR is equivalent to the one considered in [14],

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where one can find a comprehensive discussion about almost disjoint refinements.

We are going to compare constructions of Hausdorff and Luzin gaps. If $b > \omega_1$, then we describe how one can rebuilt a AD-family of the cardinality ω_1 onto a Hausdorff pre-gap. If a such AD-family is a Luzin gap, then we obtain a Hausdorff gap. Under $b = \omega_1$, a similar rebuilding is impossible. For the sake of completeness, we enclose a construction of a Hausdorff gap which use no form of so called the second interpolation theorem, compare [12], and needs the hypothesis $b = \omega_1$.

P. Simon indicated to us that Hausdorff gaps and Luzin gaps do not look compatible, September 2008 in Katowice. M. Scheepers discerned something similar in [12]. Albeit, he wrote that Luzin gaps are reminiscent of Hausdorff gaps. In [8], K. Kunen declared that "The easiest to construct are Luzin gaps" and that constructions of Hausdorff gaps need some stronger inductive hypotheses. Constructions of Hausdorff gaps and Luzin gaps are considered apart, usually. Hausdorff gaps have been examined via topological manner, through gap spaces associated with them, for example [2], [3] or [9]. Forcing methods yield other treads to examine variety of Hausdorff gaps, for example [1], [6], [4], [12] or [15].

2. AD-families of the cardinality b.

Recall that, b is the least cardinality of unbounded families of functions $f : \omega \to \omega$ with respect to the partial order $\leq^*$, where $f \leq^* g$ whenever $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A function h *dominates* a restriction $f|_D$, whenever $f(n) \leq h(n)$ for all but finitely many $n \in D$. If $D = \omega$, then h dominates f. It is well known that each of hypotheses $b = \omega_1$ or $b > \omega_1$ is consistent with ZFC. The hypothesis $b > \omega_1$ is equivalent with Proposition (1): *The family of all sets of n.n. does not contain any* (Ω, ω^*) *gaps*; by Rothberger [11]. Consider the following question.

Question. *Could a given almost disjoint family be an almost disjoint refinement for the union of some two uncountable and orthogonal towers?*

To answer the question, we start with a ZFC result. Then a Rothberger lemma is adapted in order to conclude some consistent results.

Theorem 1. *There exists an almost disjoint family of the cardinality b, which is not almost disjoint refinement for any union of two orthogonal towers, where both towers have the cardinality b.*

Proof. Let $\mathcal{Q} = \mathcal{F} \cup \{B_n : n < \omega\}$ be an AD-family such that always $B_n = \{(n,k) : k < \omega\}$ and $\mathcal{F} = \{f_\alpha : \alpha < b\}$ consists of almost disjoint and increasing functions $f_\alpha : \omega \to \omega$. Assume that, $\mathcal{F}$ is unbounded and increasing. So, $\mathcal{Q}$ consists of subsets of $\omega \times \omega$ and every $\mathcal{H} \subseteq \mathcal{F}$ of the cardinality b is an unbounded family with respect to $\leq^*$.

Suppose that $\mathcal{Q}$ is ADR of the union of orthogonal towers $\{A_\alpha : \alpha < b\}$ and $\{C_\alpha : \alpha < b\}$. Without loss of generality, one can fix α such that C_α almost contains infinitely many B_n. Thus the family

$$\mathcal{H} = \{f_\beta \in \mathcal{F} : f_\beta \subset^* \omega \times \omega \setminus C_\alpha\}$$

contains a subfamily $\mathcal{P}$ of the cardinality b such that $\mathcal{P}$ is an ADR of some subfamily of $\{A_\alpha : \alpha < b\}$. So, the family $\mathcal{H}$ is unbounded. On the other hand, put $h(n) = \max\{k : (n,k) \notin C_\alpha\}$ whenever $B_n \subset^* C_\alpha$. Thus the function h dominates each restriction $f_\beta | D$, where $f_\beta \in \mathcal{H}$ and $D = \{n : B_n \subset^* C_\alpha\}$. Let $k_0, k_1, \ldots$ be an increasing enumeration of all elements of D. Put $g(i) = h(k_n)$ whenever $k_{n-1} < i \leq k_n$. Because of $\mathcal{H}$ consists of increasing functions, one can check that g dominates any function from $\mathcal{H}$; a contradiction. $\square$

The following lemma can be derived from Rothberger's Lemma 5 stated in [11].

Lemma 2. *Suppose a countable family $\mathcal{Q}$ consists of almost disjoint infinite subsets of natural numbers, and let $\mathcal{H}$ consists of sets almost disjoint with members of $\mathcal{Q}$. If $|\mathcal{H}| < b$, then families $\mathcal{Q}$ and $\mathcal{H}$ are separated.*

Proof. Without loss of generality, assume that members of $\mathcal{H}$ and $\mathcal{Q}$ are subsets of $\omega \times \omega$ such that

$$\mathcal{Q} = \{\{(n,i) : i \in \omega\} : n \in \omega\}.$$

Put $f_B(n) = \max\{i : (n,i) \in B\}$ for each $B \in \mathcal{H}$ (here $\max \emptyset = 0$). Functions $f_B : \omega \to \omega$ are well defined since members of $\mathcal{H}$ are almost disjoint with elements of $\mathcal{Q}$. The family of all functions f_B has the cardinality less than b, so there exits a function h which dominates each f_B. The set

$$\{(n, i) : i > h(n) \text{ and } n \in \omega\}$$

separates $\mathcal{Q}$ from $\mathcal{H}$. $\square$

Below, $A \subseteq^* B$ means that A is almost contained in B.

Theorem 3. *Assume that $b > \omega_1$. If $\{E_\alpha : \alpha < \omega_1\} \cup \{F_\alpha : \alpha < \omega_1\}$ is an AD-family, then there exists a Hausdorff pre-gap*

$$[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$$

such that $E_\alpha \subseteq^* A_{\alpha+1} \setminus A_\alpha \subseteq^* E_\alpha$ and $F_\alpha \subseteq^* B_\alpha \setminus B_{\alpha+1} \subseteq^* F_\alpha$, whenever $\alpha < \omega_1$.

Proof. We shall construct a desired Hausdorff pre-gap, defining by induction sets A_α and B_α such that

(1) If $\beta < \alpha$, then $A_\beta \subset^* A_\alpha \subset^* B_\alpha \subset^* B_\beta$;
(2) If $\alpha = \beta + 1$, then $E_\beta \cup A_\beta = A_\alpha$ and $B_\alpha = B_\beta \setminus F_\beta$;
(3) Each member of the union $\{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$ is almost disjoint with A_α;
(4) Each member of $\{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$ is almost contained in B_α.

Put $A_0 = \emptyset$ and $B_0 = \omega$ and $A_{\alpha+1} = E_\alpha \cup A_\alpha$ and $B_{\alpha+1} = B_\alpha \setminus F_\alpha$. It remains to define sets A_α and B_α for limit ordinals α. Take a sequence of ordinals $\gamma_0, \gamma_1, \ldots$ which is increasing and has the limit α. Assume that $\gamma_0 = 0$.

At the first step, let $\mathcal{Q} = \{A_{\gamma_{n+1}} \setminus A_{\gamma_n} : n \in \omega\}$ and $\mathcal{H} = \{B_{\gamma_n} \setminus B_{\gamma_{n+1}} : n \in \omega\} \cup \{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$. Families $\mathcal{Q}$ and $\mathcal{H}$ are orthogonal and $\mathcal{Q}$ is a countable AD-family. By Lemma 2, let A_α be a set which separates $\mathcal{Q}$ from $\mathcal{H}$. Observe that $\beta < \alpha$ implies $A_\beta \subset^* A_\alpha \subset^* B_\beta$. Indeed, $\emptyset = A_{\gamma_0} \subset^* A_\alpha \subset^* B_{\gamma_0} = \omega$. Inductively, $A_{\gamma_n} \subseteq^* (A_{\gamma_n} \setminus A_{\gamma_{n-1}}) \cup A_{\gamma_{n-1}} \subset^* A_\alpha$, since A_α separates $\mathcal{Q}$ from $\mathcal{H}$. There exists $\gamma_n > \beta$, hence $A_\beta \subset^* A_{\gamma_n} \subset^* A_\alpha$. Also, one can assume that $A_\alpha \subset^* B_{\gamma_m}$. But sets A_α and $B_{\gamma_m} \setminus B_{\gamma_{m+1}}$ are almost disjoint, hence $A_\alpha \subset^* B_{\gamma_{m+1}}$. This gives that $A_\alpha \subset^* B_\beta$.

At the second step, apply Lemma 2 to families $\mathcal{Q} = \{B_{\gamma_n} \setminus B_{\gamma_{n+1}} : n \in \omega\}$ and $\mathcal{H} = \{A_\alpha\} \cup \{E_\beta : \alpha \leq \beta\} \cup \{F_\beta : \alpha \leq \beta\}$. Let B_α be the complement of a set which separates $\mathcal{Q}$ from $\mathcal{H}$, i.e. B_α separates $\mathcal{H}$ from $\mathcal{Q}$. The union $\{B_\alpha\} \cup \{B_{\gamma_n} \setminus B_{\gamma_{n+1}} : n \in \omega\}$ is an AD-family, hence $\beta < \alpha$ implies $B_\alpha \subset^* B_\beta$. $\square$

Thus, one can reconstruct a Hausdorff gap from a Luzin gap, under $b > \omega_1$. Indeed, let $\{E_\alpha : \alpha < \omega_1\}$ and $\{F_\alpha : \alpha < \omega_1\}$ be AD-families which are orthogonal and not separated. Then any Hausdorff pre-gap like in the Theorem 3, i.e. $[\{A_\alpha : \alpha < \omega_1\}; \{B_\alpha : \alpha < \omega_1\}]$ such that $E_\alpha \subseteq^* A_{\alpha+1} \setminus A_\alpha \subseteq^* E_\alpha$ and $F_\alpha \subseteq^* B_\alpha \setminus B_{\alpha+1} \subseteq^* F_\alpha$, has to be a Hausdorff gap. If we assume that $\{E_\alpha : \alpha < \omega_1\} \cup \{F_\alpha : \alpha < \omega_1\}$ is a Luzin gap, then we have a construction of a Hausdorff gap with some additional properties.

Let us recall Luzin's construction of a gap, see [7]. To convince the readers of Kunen's opinion, which is quoted in Introduction, we run as follows. Start with a family $\{A_n : n \in \omega\}$ which consists of disjoint and infinite subsets of ω. Assume that almost disjoint sets $\{A_\beta : \beta < \alpha\}$ are just defined for a countable ordinal number $\alpha < \omega_1$. Enumerate these sets A_β into a sequence $\{B_n : n \in \omega\}$. For every n, choose a set

$$\{d_1, d_2, \ldots d_n\} \subset B_n \setminus (B_0 \cup B_1 \cup \ldots \cup B_{n-1}),$$

with exactly n elements. Than, put A_α to be the union of all already chosen sets $\{d_1, d_2, \ldots d_n\}$. The family $\{A_\alpha : \alpha < \omega_1\}$ forms a Luzin gap. Indeed, consider a partition of $\{A_\alpha : \alpha < \omega_1\}$ into two uncountably subfamilies $\mathcal{D}$ and $\mathcal{E}$. Suppose that a set set C separates $\mathcal{D}$ from $\mathcal{E}$. Fix a natural number n and uncountable subfamilies $\mathcal{F} \subseteq \mathcal{D}$ and $\mathcal{H} \subseteq \mathcal{E}$ such that $\cup\mathcal{F} \setminus n \subseteq C$ and $\cup\mathcal{H} \cap C \subseteq n$. Take $\alpha < \omega_1$ such that the intersection $\{A_\beta : \beta < \alpha\} \cap \mathcal{H}$ is infinite. Finally, for each $\gamma > \alpha$ with $A_\gamma \in \mathcal{F}$ there exist $\beta < \alpha$ and $A_\beta \in \mathcal{H}$ such that the intersection $A_\beta \cap A_\gamma$ is a set $\{d_1, d_2, \ldots d_m\}$, where $m > n$. This is in conflict with $\cup\mathcal{F} \setminus n \subseteq C$ and $\cup\mathcal{H} \cap C \subseteq n$.

If $b > \omega_1$ and there exists a Lebesgue non-measurable set of the cardinality ω_1, then there exist AD-families of the cardinality ω_1 which are non-measurable sets with respect to some Borel measures on $[\omega]^\omega$. But, any family of sets which consists of a Hausdorff gap has to be universally measure zero, see [10]. Thus, Hausdorff gaps and Luzin gaps could have consistently different measurable properties.

3. On constructions of Hausdorff gaps under $b = \omega_1$

It is consistent that any AD-family of the cardinality ω_1 is ADR of the union of some two orthogonal towers of the cardinality ω_1 because of Theorem 3. It is also clear that this statement implies $b > \omega_1$, since Theorem 1 points out a suitable AD-family. So, we obtain a characterization of the hypothesis $b = \omega_1$.

Corollary 4. *$b = \omega_1$ is equivalent with the existence of AD-family of the cardinality ω_1 which is not an ADR of the union of any two orthogonal towers each of the cardinality ω_1.* □

All known to us constructions of a Hausdorff gap use some forms of so called *The second interpolation theorem*, compare [2], [5], [13],[12] or [15]. In the previous part we do not use this principle in inductive hypotheses. So, we should add constructions which use no form of the second interpolation theorem. We use the following abbreviations: $\Delta = \{(n,k) \in \omega \times \omega : k < n\}$ and $\int f = \{(n,k) \in \omega \times \omega : k \leq f(n)\}$.

Assume that $b = \omega_1$. Let $\{\omega \setminus T_\alpha : \alpha < \omega_1\}$ be a maximal tower and let $\{f_\alpha : \alpha < \omega_1\}$ be a unbounded family of functions, where $f_\alpha : \omega \to \omega$. Let $A_0 = \emptyset$ and $B_0 = \omega \times \omega$ and fix a function $g_0 : \omega \to \omega$ such that $f_0 \leq^* g_0$. Suppose that sets A_β and B_β and functions g_β are defined for $\beta < \alpha$. We should define sets A_α and B_α and a function g_α such that

- $A_\beta \subset^* A_\alpha \subset B_\alpha \subset^* B_\beta$ for each $\beta < \alpha$;
- $g_\alpha \subseteq A_\alpha \subseteq \int g_\alpha$, where a function g_α is such that $f_\alpha \leq^* g_\alpha$;
- $B_\alpha = A_\alpha \cup (\omega \times T_\alpha) \setminus \Delta$.

To do this, take X such that $A_\beta \subset^* X \subset^* B_\beta$ for each $\beta < \alpha$. Fix a function $g_\alpha \subset \omega \times T_\alpha \setminus \Delta$ such that g_α dominates every function from $\{g_\beta : \beta < \alpha\} \cup \{f_\alpha\}$. Eventually, put $A_\alpha = g_\alpha \cup (X \cap \int g_\alpha)$ and $B_\alpha = A_\alpha \cup (\omega \times T_\alpha) \setminus \Delta$. Above defined sets A_α and B_α constitute a Hausdorff pre-gap. The tower $\{\omega \setminus T_\alpha : \alpha < \omega_1\}$ is maximal. Hence, whenever $A_\alpha \subset^* C \subset^* B_\alpha$ for any $\alpha < \omega_1$, then there exists a function h such that $C \subset \int h$. But this means that h dominates each f_α, a contradiction.

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