ON THE IDEAL (v^0)

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Abstract. The σ-ideal (v^0) is associated with the Silver forcing, see [5]. Also, it constitutes the family of all completely doughnut null sets, see [9]. We introduce segment topologies to state some resemblances of (v^0) to the family of Ramsey null sets. To describe $add(v^0)$ we adopt a proof of Base Matrix Lemma. Consistent results are stated, too. Halbeisen's conjecture $cov(v^0) = add(v^0)$ is confirmed under the hypothesis $t = \min\{\mathrm{cf}(\mathfrak{c}), r\}$. The hypothesis $cov(v^0) = \omega_1$ implies that (v^0) has the ideal type $(\mathfrak{c}, \omega_1, \mathfrak{c})$.

1. Introduction

Our discussion focuses around the family $[\omega]^\omega$ of all infinite subsets of natural numbers. We are interested in some structures on $[\omega]^\omega$ which correspond to the inclusion $\subseteq$ and to the partial order $\subseteq^*$. Recall that, $A \subseteq^* X$ means that the set $A \setminus X$ is finite. We assume that the readers are familiar with some properties of the partial order $([\omega]^\omega, \subseteq^*)$. For instance, gaps of type (ω, ω^*) and ω-limits do not exist, see F. Hausdorff [10] or compare F. Rothberger [23]. We refer to books [8] and [12] for the mathematics used in this note. In particular, one can find basic facts about completely Ramsey sets and its applications to the descriptive set theory in [12] p. 129 - 136. Let us add, that E. Ellentuck (1974) was not the first one who considered properties of the topology which is called by his name. Non normality of this topology was established by V. M. Ivanowa (1955) and J. Keesling (1970), compare [8] p. 162 -163. We refer the readers to papers [3], [5], [11], [14], [15] and [19] for other applications of completely Ramsey sets, not discussed in [12].

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Let $\mathcal{W}$ be a family of sets such that $\cup\mathcal{W} \notin \mathcal{W}$. Recall that,

$$add(\mathcal{W}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{W} \text{ and } \cup\mathcal{F} \notin \mathcal{W}\}$$

is called the *additivity number* of $\mathcal{W}$. But

$$cov(\mathcal{W}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{W} \text{ and } \cup\mathcal{F} = \cup\mathcal{W}\}$$

is called the *covering number* of $\mathcal{W}$. Thus, $add(v^0)$ and $add(v)$ denote the additivity number of the ideal (v^0) and of the σ-field (v), respectively. But $cov(v^0)$ denotes the covering the ideal (v^0). For definitions of the tower number t and the reaping number r we refer to [4]. One can find there a thorough discussion of consistent properties of t and r, too.

J. Brendle [5] considered a few tree-like forcings with σ-ideals associated to them. The concept of these ideals is modeled on s^0-sets of Marczewski [25] and Morgan's category base [18]. One of these ideals is the ideal (v^0). It is associated with the Silver forcing. The ideal (v^0) is examined in papers [6], [9] and [13], too. L. Halbeisen [9] found some analogy with completely Ramsey sets and introduced so called completely doughnut sets, i.e. v-sets in our terminology. He introduced a pseudo topology - and called it the doughnut topology - such that X is a v-set iff X has the Baire property with respect to the doughnut topology. Using the method of B. Aniszczyk [1] and K. Schilling [24] we introduce segments topologies, each one corresponds to v-sets similarly as Halbeisen's pseudo topology. To describe $add(v)$ we adopt a proof of Base Matrix Lemma, compare [2] and [3]. The height $\kappa(v)$ of a base v-matrix equals to $add(v) = add(v^0)$. With a base v-matrix it is associated the increasing family of v^0-sets with the union outside the ideal (v^0). We can not confirm (in ZFC) that this union is $[\omega]^\omega$. Therefore, we get a few consistent results. For example, $cov(v^0) = \omega_1$ implies that (v^0) has the ideal type $(\mathfrak{c}, \omega_1, \mathfrak{c})$. The conjecture of Halbeisen $cov(v^0) = add(v^0)$ is confirmed under $t = \min\{\text{cf}(\mathfrak{c}), r\}$.

On the other hand, each maximal chain contained in a base v-matrix gives a $(\kappa(v), \kappa(v)^*)$-gap or a $\kappa(v)$-limit. If $cov(v^0) = add(v^0)$, then one can improve any base v-matrix such that each maximal chain, contained in a new one, gives a $(\kappa(v), \kappa(v)^*)$-gap, only. But, whenever $cov(v^0) \neq add(v^0)$, then there exist $\kappa(v)$-limits. Thus our's research continue Hausdorff [10] and Rothberger [23], too.

2. Segments and $*$-segments

In this section we consider segments and $*$-segments. The facts quoted here immediately arise from well known ones. A set

$$< A, B >= \{X \in [\omega]^\omega : A \subseteq X \subseteq B\}$$

is called a *segment*, whenever $A \subseteq B \subseteq \omega$ and $B \setminus A \in [\omega]^\omega$. By the definition any segment has the cardinality continuum. If $< A, B >$ and $< C, D >$ are segments, then the intersection

$$< A, B > \cap < C, D >=< A \cup C, B \cap D >$$

is finite or is a segment. It is a segment, whenever $A \cup C \subset B \cap D$ and $B \cap D \setminus A \cup C \in [\omega]^\omega$. Thus, the family of all segments is not closed under finite intersections.

Fact 1. *Any segment contains continuum many disjoint segments.*

Proof. Let $< A, B >$ be a segment. Consider a family $\mathcal{R}$ of almost disjoint subsets of $B \setminus A$ of the cardinality continuum. Divide each set $C \in \mathcal{R}$ into two infinite subsets D_C and $C \setminus D_C$. The family

$$\{< A \cup D_C, A \cup C >: C \in \mathcal{R}\}$$

is a desired one. □

For any set $S \subseteq [\omega]^\omega$ we put

$$S^* = \{Y : X \subseteq^* Y \subseteq^* X \text{ and } X \in S\}.$$

Thus, S^* is a countable union of copies of S, i.e. the union of sets $\{(X \setminus y) \cup (y \setminus X) : X \in S\}$, where $y \subset \omega$ runs over finite subsets. If $< A, B >$ is a segment, then the set

$$\{X : A \subseteq^* X \subseteq^* B\} =< A, B >^*$$

is called $*$-*segment*.

Fact 2. *If* $\{< A_n, B_n >: n \in \omega\}$ *is a sequence of segments decreasing with respect to the inclusion, then there exists a segment* $< C, D >$ *such that* $< C, D >\subseteq< A_n, B_n >^*$ *for each* $n \in \omega$.

Proof. Let $\{< A_n, B_n >: n \in \omega\}$ be a decreasing sequence of segments. We have

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq B_2 \subseteq B_1 \subseteq B_0.$$

Choose a set $C \in [\omega]^\omega$ such that $A_n \subseteq^* C \subseteq^* B_n$ for each $n \in \omega$. Additionally, we can assume that sets $C \setminus A_n$ and $B_n \setminus C$ are infinite, since there are no ω-limits and (ω, ω^*)-gaps. Then, choose a set $D \in [\omega]^\omega$ such that $D \setminus C$ is infinite and $C \subseteq D \subseteq^* B_n$ for each $n \in \omega$. $\square$

Occasionally segments show up in the descriptive set theory. For example, the work of G. Moran and D. Strauss [17] implies that any subset of $[\omega]^\omega$ having the property of Baire and of second category contains a segment. In other words, it has the doughnut property. One can prove this adopting the proof of Proposition 2.2 in [7], also. The work [17] implies that any subsets of $[\omega]^\omega$ with positive Lebesgue measure contains a segment, compare [22] and [13].

3. Segment topologies

C. Di Prisco and J. Henle [7] introduced so called doughnut property. Namely, a subset $S \subseteq [\omega]^\omega$ has the doughnut property, whenever S contains a segment or is disjoint with a segment. Afterwards, Halbeisen [9] generalized this property, considering so called completely doughnut sets and completely doughnut null sets. We feel that the use of "doughnut" is not appropriate. We swap it onto notations similar to that, which were used in [5] or [13]. A subset $S \subseteq [\omega]^\omega$ is called a *υ-set*, if for each segment $< A, B >$ there exists a segment $< C, D > \subseteq < A, B >$ such that

$$< C, D > \subseteq S \text{ or } < C, D > \cap S = \emptyset.$$

If always holds $< C, D > \cap S = \emptyset$, then S is called a *υ^0-set*. Any subset of a υ^0-set is a υ-set and a υ^0-set, too. Also, the complement of a υ-set is a υ-set. According to facts 1.3, 1.5 and 1.6 in Halbeisen [9], the family of all υ-sets is a σ-field and we denote this field (υ). The family of all υ^0-sets is a σ-ideal and we denote this ideal (υ^0). One can find many interesting results about (υ^0) in papers [5], [6] and [13].

We amplify the method of Aniszczyk [1] and Schilling [24] to introduce some topologies, which correspond to (υ). These topologies have the same features as the pseudo topology, which was considered by Halbeisen [9]. Fix a transfinite sequence $\{C_\alpha : \alpha < \mathfrak{c}\}$ consisting of all segments. Put $V_0 = C_0$. For every ordinal number $\alpha < \mathfrak{c}$, let M_α be the union of all intersections $C_{\beta_1} \cap C_{\beta_2} \cap \ldots \cap C_{\beta_n}$ such that

$$|C_{\beta_1} \cap C_{\beta_2} \cap \ldots \cap C_{\beta_n}| < \omega,$$

where $\beta_i \leq \alpha$ and $1 \leq i \leq n$. Put $V_\alpha = C_\alpha \setminus M_\alpha$. The topology generated by all (just defined) sets V_α is called a *segment topology.* There are many segment topologies, since any one depends on an ordering $\{C_\alpha : \alpha < \mathfrak{c}\}$. We get $|M_\alpha| < \mathfrak{c}$, for any $\alpha < \mathfrak{c}$. Also, each V_α contains a segment. Therefore, if $S \subset [\omega]^\omega$ and $|S| < \mathfrak{c}$, then S is nowhere dense with respect to any segment topology. Moreover, we have.

Lemma 3. *Any family $\{V_\alpha : \alpha < \mathfrak{c}\}$ is a π-base and subbase for the segment topology (which it generates).*

Proof. The family $\{V_\alpha : \alpha < \mathfrak{c}\}$ is a subbase by the definition. Thus, the family of all intersections $V_{\beta_1} \cap V_{\beta_2} \cap \ldots \cap V_{\beta_n}$ constitutes a base. If a base set $V_{\beta_1} \cap V_{\beta_2} \cap \ldots \cap V_{\beta_n}$ is non-empty, then it has the form of a segment minus a set of the cardinality less than the continuum, exactly

$$C_{\beta_1} \cap C_{\beta_2} \cap \ldots \cap C_{\beta_n} \setminus (M_{\beta_1} \cup M_{\beta_2} \cup \ldots \cup M_{\beta_n}).$$

By Fact 1, it contains some segment C_α. Hence $V_{\beta_1} \cap V_{\beta_2} \cap \ldots \cap V_{\beta_n}$ contains some $V_\alpha \subseteq C_\alpha$. □

Immediately, one obtains that any two segment topologies determine the same family of nowhere dense sets. As a matter of fact, every element of the base contains a segment and vice versa. Consequently, the nowhere dense sets with respect to any segment topology are the v^0-sets. The next lemma amplifies the fact that there are no (ω, ω^*)-gaps. It corresponds to the result of Moran and Strauss [17], compare Proposition 2.2 in [7]. We need the following abbreviation

$$< A, B >_n = < A, B \setminus (\{0, 1, \ldots, n\} \setminus A) > .$$

Lemma 4. *Let $S_0, S_1, \ldots$ be a sequence of nowhere dense subsets. For any segment $< A, B >$ there exists a segment $< E, F > \subseteq < A, B >$ such that $S_n \cap < E, F > = \emptyset$ for each $n \in \omega$.*

Proof. Assume that the sequence $S_0, S_1, \ldots$ is increasing. We shall define points $e_0, e_1, \ldots, e_n$ and sets

$$A \subseteq A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq B_n \subseteq \ldots \subseteq B_1 \subseteq B_0 \subseteq B,$$

where $B_n \setminus A_n$ is infinite, $\{e_0, e_1, \ldots, e_n\} \subset B_n \setminus A_n$ and

$$e_n = \min(B_n \setminus (A_n \cup \{e_0, e_1, \ldots, e_{n-1}\});$$

and such that $< A_n \cup x, B_n >_{e_n} \cap S_n = \emptyset$, for each $x \subseteq \{e_0, e_1, \ldots, e_n\}$ and any $n < \omega$.

We proceed inductively with respect to n. Let $e_0 = \min(B \setminus A)$. Choose a segment $< A^0_0, B^0_0 > \subseteq < A, B >_{e_0} \setminus S_0$. Then, choose sets $A_0 \supseteq A^0_0$ and $B_0 \subseteq B^0_0 \cup \{e_0\}$ such that $e_0 \in B_0 \setminus A_0$ and the segment $< A_0 \cup \{e_0\}, B_0 >_{e_0}$ is disjoint with S_0. We get

$$(< A_0 \cup \{e_0\}, B_0 >_{e_0} \cup < A_0, B_0 >_{e_0}) \cap S_0 = \emptyset.$$

Assume that sets A_n and B_n are defined. Let

$$e_n = \min(B_n \setminus (A_n \cup \{e_0, e_1, \dots, e_{n-1}\})).$$

Enumerate all subsets of $\{e_0, e_1, \dots, e_n\}$ into a sequence $x_1, x_2, \dots, x_{2^{n+1}}$. Choose a segment

$$< A^1_n, B^1_n > \subseteq < A_n \cup x_1, B_n >_{e_n} \setminus S_n.$$

If a segment $< A^{k-1}_n, B^{k-1}_n >$ has been already defined, then choose sets $A^k_n \supseteq A^{k-1}_n$ and $B^k_n \subseteq B^{k-1}_n \cup \{e_0, e_1, \dots, e_n\}$ such that $\{e_0, e_1, \dots, e_n\} \subset B^k_n \setminus A^k_n$ and the segment $< A^k_n \cup x_k, B^k_n >_{e_n}$ is disjoint with S_n. Let B_{n+1} be the last B^k_n and A_{n+1} be the last A^k_n. By the definition, we get $\{e_k : k < \omega\} \subset B_n \setminus A_n$ and

$$\cup\{< A^k_n \cup x_k, B^k_n >_{e_n} : 0 < k \leq 2^{n+1}\} \cap S_n = \emptyset,$$

for any $n < \omega$. Finally, the segment

$$< E, F > = < \cup\{A_n : n \in \omega\}, \cup\{A_n : n \in \omega\} \cup \{e_n : n \in \omega\} >$$

is disjoint with each S_k. Indeed, suppose $C \in < E, F > \cap S_k$. Let $x = C \cap \{e_0, e_1, \dots, e_k\}$. Then $C \in < A_k \cup x, B_k >_{e_k}$. But this contradicts $< A_k \cup x, B_k >_{e_k} \cap S_k = \emptyset$. □

Corollary 5. *For any segment topology, the intersection of countable many open and dense sets contains an open and dense subset.* □

Corollary 6. *The ideal (v^0) coincides with the family of all sets of the first category with respect to any segment topology.* □

Recall that, a subset Y of a topological space X has the property of Baire whenever $Y = (G \setminus F) \cup H$, where G is open and F, H are of the first category. If $X = [\omega]^\omega$ is equipped with a segment topology, then $Y \subseteq X$ has the Baire property (i.e. the property of Baire with respect to this segment topology) whenever $Y = G \cup H$, where G is open and H is a v^0-set.

Theorem 7. *The σ-field (v) coincides with the family of all sets which have the Baire property with respect to a segment topology.*

Proof. Fix a segment topology and a v-set X. Let $U = \cup\{V_\beta : V_\beta \subseteq X\}$ and $W = \cup\{V_\beta : V_\beta \cap X = \emptyset\}$. The union $U \cup W$ is open and dense. Thus $X = U \cup F$, where $F \subseteq [\omega]^\omega \setminus (U \cup W)$ is nowhere dense.

We shall show that any open set is a v-set. Suppose a set X is open. Take an arbitrary segment $< A, B >$ and choose a subbase set $V_\alpha \subseteq< A, B >$. There exists $V_\beta \subseteq V_\alpha$ such that $V_\beta \subseteq X$ or $V_\beta \subseteq \text{Int}([\omega]^\omega \setminus X)$. Each segment $< C, D >\subseteq V_\beta$ witnesses that X is a v-set. □

Every classical analytic set belongs to (v). This is a counterpart of Mathias-Silver theorem - compare (21.9) or (29.8) in [12] - which arises from Halbeisen's paper [9]. In fact, one could conclude it similarly like in the paper by Pawlikowski [20]. This was noted by Brendle, Halbeisen and Löwe in [6]. We obtain the counterpart directly, using Theorem 6 and theorems (29.11), (29.13) in [12].

4. Base v-matrix

We shall adopt a proof of Base Matrix Lemma - see B. Balcar J. Pelant and P. Simon, compare [2] and [3]. There are known some generalizations of this theorem for some partial orders, e.g. compare [16]. For completeness, we prove our's version directly. If $< A, B >$ and $< C, D >$ are segments, then the intersection $< A, B >^* \cap < C, D >^*$ is countable or has the cardinality continuum. In the second case the intersection is a $*$-*segment.*

Whenever $< A, B >^* \cap < C, D >^*$ is countable, then $< A, B >^*$ and $< C, D >^*$ are called $*$-*disjoint.*

Lemma 8. *If S is a v^0-set, then for any segment $< A, B >$ there exists a segment $< C, D >\subseteq< A, B >$ such that $< C, D >^* \cap S^* = \emptyset$.*

Proof. By the definition, S^* is a countable union of elements of (v^0), hence $S^* \in (v^0)$. Thus, any segment $< C, D >\subseteq< A, B >$ disjoint with S^* is a desired one. □

A family $\mathcal{P}$ of $*$-segments is a v-*partition*, whenever any two distinct members of $\mathcal{P}$ are $*$-disjoint and $\mathcal{P}$ is maximal with respect to the inclusion. A collection of v-partitions is called v-*matrix.* A v-partition

$\mathcal{P}$ *refines* a v-partition $\mathcal{Q}$ (briefly $\mathcal{P} \prec \mathcal{Q}$), if for each $< A, B >^* \in \mathcal{P}$ there exists $< C, D >^* \in \mathcal{Q}$ such that $< A, B >^* \subseteq < C, D >^*$. A v-matrix $\mathcal{H}$ is called *shattering*, if for each $*$-segment $< A, B >^*$ there exists $\mathcal{P} \in \mathcal{H}$ and $< A_1, B_1 >^*, < A_2, B_2 >^* \in \mathcal{P}$ such that $< A_1, B_1 >^* \cap < A, B >^*$ and $< A_2, B_2 >^* \cap < A, B >^*$ are different $*$-segments. Denote by $\kappa(v)$ the least cardinality of a shattering v-matrix.

Lemma 9. *If a v-matrix $\mathcal{H}$ is of the cardinality less than $\kappa(v)$, then there exists a v-partition $\mathcal{P}$ which refines any v-partition $\mathcal{Q} \in \mathcal{H}$.*

Proof. Fix a segment $< A, B >$. Let $\mathcal{H}(A, B) = \{\mathcal{P}(A, B) : \mathcal{P} \in \mathcal{H}\}$ be the relative v-matrix such that each $\mathcal{P}(A, B)$ consists of all $*$-segments $< C, D >^* \cap < A, B >^*$, where $< C, D >^* \in \mathcal{P}$. Any segment $< C, D >$ is isomorphic to $[D \setminus C]^{\leqslant\omega}$ and $[\omega]^{\leqslant\omega}$, hence $\mathcal{H}(A, B)$ is not shattering relative to $< A, B >^*$. Choose a segment $< C, D > \subseteq < A, B >$ such that there exists $< E, F >^* \in \mathcal{P}$ with $< C, D >^* \subseteq < E, F >^*$ for every $\mathcal{P} \in \mathcal{H}$. Any v-partition $\mathcal{P}$ consisting of above defined $*$-segments $< C, D >^*$ is a desired one. □

Let h be the height of the base matrix . See [2] and [3] for rudimentary properties of the cardinal number h.

Theorem 10. *$\omega_1 \leq \kappa(v) \leq h$ and $\kappa(v)$ is a regular cardinal number.*

Proof. Suppose $h < \kappa(v)$. Take a base matrix $\{\mathcal{H}_\alpha : \alpha < h\}$ such as in 2.11 Base Matrix Lemma in [2]. Let $\mathcal{P}_\alpha$ be a v-partition such that for any $< A, B >^* \in \mathcal{P}_\alpha$ there exists $V \in \mathcal{H}_\alpha$ with $B \setminus A \subseteq^* V$. The v-matrix $\{\mathcal{P}_\alpha : \alpha < h\}$ contradicts Lemma 9.

Consider a shattering v-matrix $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$. By Lemma 9, we can assume that $\alpha < \beta$ implies $\mathcal{P}_\beta \prec \mathcal{P}_\alpha$. Any cofinal family of v-partitions from $\mathcal{H}$ constitutes a shattering v-matrix. Hence $\kappa(v)$ has to be regular. It is uncountable by Fact 2. □

Theorem 11. *There exists a v-matrix $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ which is well ordered by the inverse of $\prec$. Moreover, for each $*$-segment $< A, B >^*$ there is $< C, D >^* \in \cup\mathcal{H}$ such that $< C, D >^* \subseteq < A, B >^*$.*

Proof. Build a shattering v-matrix $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ such that $\alpha < \beta$ implies $\mathcal{P}_\beta \prec \mathcal{P}_\alpha$. Let $J^c(\mathcal{P}_\alpha)$ be the family of all $*$-segments $< A, B >^*$ for which there are continuum many elements of $\mathcal{P}_\alpha$ not $*$-disjoint with $< A, B >^*$. Let $F : J^c(\mathcal{P}_\alpha) \to \mathcal{P}_\alpha$ be a one-to-one function such that $F(G) \cap G$ is a $*$-segment, for every $G \in J^c(\mathcal{P}_\alpha)$. Choose a v-partition

$$\mathcal{Q} \supseteq \{F(G) \cap G : G \in J^c(\mathcal{P}_\alpha)\}.$$

Having these, one can improve $\mathcal{H}$ to obtain $\mathcal{P}_{\alpha+1} \prec \mathcal{Q}$ and $\mathcal{P}_{\alpha+1} \prec \mathcal{P}_\alpha$. One obtains that, if $< A, B >^* \in J^c(\mathcal{P}_\alpha)$, then there is $< C, D >^* \in \mathcal{P}_{\alpha+1}$ with $< C, D >^* \subseteq < A, B >^*$.

For each $*$-segment $< A, B >^*$ there exists $\alpha < \kappa(v)$ such that $< A, B >^* \in J^c(\mathcal{P}_\alpha)$. Indeed, fix a $*$-segment $< A, B >^*$. Let $B^0_{\alpha_0}$ and $B^1_{\alpha_0}$ be two different $*$-segments belonging to $\mathcal{P}_{\alpha_0}$ such that $D^0_{\alpha_0} = < A, B >^* \cap B^0_{\alpha_0}$ and $D^1_{\alpha_0} = < A, B >^* \cap B^1_{\alpha_0}$ are $*$-segments. Thus, $D^{i_0}_{\alpha_0} \subseteq < A, B >^*$ for $i_0 \in \{0, 1\}$. Inductively, let $B^{i_0 i_1 \ldots i_{n-1} 0}_{\alpha_n}$ and $B^{i_0 i_1 \ldots i_{n-1} 1}_{\alpha_n}$ be two different $*$-segments belonging to $\mathcal{P}_{\alpha_n}$ such that $D^{i_0 i_1 \ldots i_{n-1} 0}_{\alpha_n} = < A, B >^* \cap B^{i_0 i_1 \ldots i_{n-1} 0}_{\alpha_n}$ and $D^{i_0 i_1 \ldots i_{n-1} 1}_{\alpha_n} = < A, B >^* \cap B^{i_0 i_1 \ldots i_{n-1} 1}_{\alpha_n}$ are $*$-segments. We get

$$D^{i_0 i_1 \ldots i_n}_{\alpha_n} \subset D^{i_0 i_1 \ldots i_{n-1}}_{\alpha_{n-1}} \subset < A, B >^* .$$

Put $\beta = \sup\{\alpha_n : n \in \omega\}$. By the construction and Fact 2, we get $< A, B >^* \in J^c(\mathcal{P}_{\beta+1})$. Therefore, for each $*$-segment $< A, B >^*$ there exists $\alpha < \kappa(v)$ and $< C, D >^* \in \mathcal{P}_\alpha$ such that $< C, D >^* \subseteq < A, B >^*$ □

Let $\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ be a v-matrix as in the Theorem 11. In general, any two members of the union $\cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ are $*$-disjoint or one is included in the other. One could remove a set M_C of cardinality less than $\mathfrak{c}$ from each $*$-segment $C \in \cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ such that any two members of the family

$$\mathcal{Q} = \{C \setminus M_C : C \in \cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}\}$$

are disjoint or one is included in the other. Any $\mathcal{Q}$ as above is called *a base v-matrix*. Thus, $\kappa(v)$ is the height of a base v-matrix. The next theorem yields analogy to nowhere Ramsey sets, compare [21] p. 665.

Theorem 12. *The ideal (v^0) coincides with the family of all nowhere dense subsets with respect to the topology generated by a base v-matrix.*

Proof. Let $S \subseteq [\omega]^\omega$ be a v^0-set and $\mathcal{Q}$ a base v-matrix. Any set $W \in \mathcal{Q}$ is a $*$-segment minus a set of cardinality less than $\mathfrak{c}$. By Fact 1 and Lemma 8, there is a $*$-segment $< A, B >^* \subseteq W$ such that $< A, B >^* \cap S = \emptyset$, for each $W \in \mathcal{Q}$. By Theorem 11 there exists a $*$-segment $V \in \cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ such that $V \subseteq < A, B >^*$. Sets $V \setminus M_V \in \mathcal{Q}$ witnesses that S is nowhere dense.

Let S be a nowhere dense set. Take a segment $< A, B >$. Choose a $*$-segment $W \in \cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ such that $W \subseteq < A, B >^*$. Then choose $V \in \mathcal{Q}$ such that $V \subseteq W \setminus S$. Any segment $< C, D > \subseteq V$ witnesses that S is a v^0- set. □

In ZFC, Hausdorff [10] proved that there exists a (ω_1, ω_1^*)-gap. This suggests that the height of a base v-matrix could be ω_1. We do not know:

$$\text{Is it consistent that } \omega_1 \neq \kappa(v)?$$

Without loss of generality, one can add to the definition of a base v-matrix that $\mathcal{P}_\beta \prec \mathcal{P}_\alpha$ means that for each $< C, D >^* \in \mathcal{P}_\beta$ there exists $< A, B >^* \in \mathcal{P}_\alpha$ such that $< C, D > \subset < A, B >$ and sets $C \setminus A$, $B \setminus D$ are infinite. This yields that each maximal chain contained in a such base v-matrix produces a $(\kappa(v), \kappa(v)^*)$-gap or a $\kappa(v)$-limit. We need $add(v^0) = cov(v^0)$ to obtain a base v-matrix such that each maximal chain contained in it produces a $(\kappa(v), \kappa(v)^*)$-gap, only. So, we consider additivity and covering numbers of the ideal (v^0).

5. Additivity and covering numbers

Foreseeing a counterpart of Plewik's result that the additivity number of completely Ramsey sets equals to the covering number of Ramsey null sets - compare [3] p. 352 - 353 - Halbeisen set the following question at the end of [9]: Does

$$add(v^0) = cov(v^0)?$$

The answer is obvious under the Continuum Hypothesis. We add another consistent hypotheses which confirm this equality.

Lemma 13. *If $\mathcal{P}$ is a v-partition, then the complement of the union $\cup\mathcal{P}$ is a v^0-set.*

Proof. Take a segment $< A, B >$. Since $\mathcal{P}$ is maximal, there exists $< C, D >^* \in \mathcal{P}$ such that $< A \cup C, B \cap D >^*$ is a $*$-segment contained in $\cup\mathcal{P}$. □

Lemma 14. *If $S \subseteq [\omega]^\omega$ is a v^0-set, then there exists a v-partition $\mathcal{P}$ such that $\cup\mathcal{P} \cap S = \emptyset$.*

Proof. If S is a v^0-set, then S^* is a v^0-set, too. Thus, for any segment $< A, B >$ there exists a segment $< C, D > \subseteq < A, B >$ such that $< C, D >^* \cap S^* = \emptyset$. Any v-partition $\mathcal{P}$ consisting of a such $< C, D >^*$ is a desired one. □

Theorem 15. $\kappa(v) = add(v^0)$.

Proof. Consider a family $\mathcal{F}$ of v^0-sets such that $|\mathcal{F}| < \kappa(v)$. Using Lemma 14, fix a v-partition $\mathcal{P}_W$ such that $\cup\mathcal{P}_W \cap W = \emptyset$ for each $W \in \mathcal{F}$. Let $\mathcal{P}$ be a v-partition refining any $\mathcal{P}_W$, which exists by Lemma 9. The v^0-set $[\omega]^\omega \setminus \cup\mathcal{P}$ contains $\cup\mathcal{F}$.

Take a base v-matrix $\mathcal{Q} = \{C \setminus M_C : C \in \cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}\}$. Without loss of generality one can assume that for every $C \in \mathcal{P}_\alpha$ the difference $C \setminus \cup\mathcal{P}_{\alpha+1}$ is not empty. Then, no segment is disjoint with the union of all sets $[\omega]^\omega \setminus \cup\mathcal{P}_\alpha$. In other words, this union is not a v^0-set. Therefore, $\kappa(v) \geq add(v^0)$. □

There are σ-fields with additivity strictly less than additivity of its natural σ-ideal. For example, consider a collection $\mathcal{F}$ of ω_1 pairwise disjoint sets, each of the cardinality ω_2. Let $\mathcal{S}$ be the σ-field generated by $\mathcal{F}$ and all subsets of $\cup\mathcal{F}$ of cardinality at most ω_1. Then $add(\mathcal{S}) = \omega_1$ and $add(\{X \in \mathcal{S} : |X| < \omega_2\}) = \omega_2$. This is not a case for the field (v).

Theorem 16. $add(v^0) = add(v)$.

Proof. Take a family $\mathcal{W}$ witnesses $add(v)$ and fix a segment topology. Each set $W \in \mathcal{W}$ is a v set, hence has the form $W = V_W \cup H_W$ where V_W is open and H_W is a v^0-set. The union $\cup\{H_W : W \in \mathcal{W}\}$ witnesses $add(v^0)$.

To prove the opposite inequality, take a set $\mathcal{B} \subseteq [\omega]^\omega$ which is dense and co-dense in a segment topology. One can construct $\mathcal{B}$ analogously to the classical construction of a Bernstein set. Let $\mathcal{Q} = \{C \setminus M_C : C \in \cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}\}$ be a base v-matrix . Then, the union of all sets $[\omega]^\omega \setminus \cup\mathcal{P}_\alpha$ is not a v^0-set. If also, it is not a v-set, then it witnesses $\kappa(v) \geq add(v^0)$. But if this union is a v-set, then sets $\mathcal{B} \setminus \cup\mathcal{P}_\alpha$ constitute the family which witnesses $\kappa(v) \geq add(v^0)$. □

Brendle observed that $cov(v^0) \leq r$, see Lemma 3 in [5] at page 21. Therefore, we get the following.

Theorem 17. $\omega_1 \leq \kappa(v) = add(v^0) = add(v) \leq cov(v^0) \leq \min\{\mathrm{cf}(\mathfrak{c}), r\}$.

Proof. Suppose $[\omega]^\omega = \cup\{\mathcal{A}_\alpha : \alpha < cf(\mathfrak{c})\}$, where always $|\mathcal{A}_\alpha| < \mathfrak{c}$. So, $cov(v^0) \leq cf(\mathfrak{c})$, since each $\mathcal{A}_\alpha$ is a v^0-set. Theorems 10, 15, 16 and Brendle's observation imply the rest inequalities. $\square$

Immediately, we infer the following: If $\kappa(v) = \min\{\mathrm{cf}(\mathfrak{c}), r\}$, then

$$\kappa(v) = add(v) = cov(v^0) = add(v^0).$$

But, if $\kappa(v) < t$, then there are no κ-limits, see [23], and for any base v-matrix $\mathcal{Q} = \{C \setminus M_C : C \in \cup\{\mathcal{P}_\alpha : \alpha < \kappa(v)\}\}$ the intersection $\cap\{\cup\mathcal{P}_\alpha : \alpha < \kappa(v)\}$ is empty. This yields $add(v) = cov(v^0)$. Therefore, $t = \min\{\mathrm{cf}(\mathfrak{c}), r\}$ implies $add(v) = cov(v^0)$, too.

6. Ideal type of (v^0)

The notion of an ideal type (λ, τ, γ) was introduced in [21], where it was obtained some consistent isomorphisms, applying the ideal type $(\mathfrak{c}, h, \mathfrak{c})$ to families of Ramsey null sets. Recall the notion of ideal types at two steps. To present it in a organized manner we enumerate conditions which are used in the definition.

Firstly, we adapt Base Matrix Lemma [3]. Suppose $\mathcal{I}$ is a proper ideal on $\cup\mathcal{I}$. A collection of families $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(\mathcal{I})\}$ is called a *base $\mathcal{I}$-matrix* whenever:
(1) Each family $\mathcal{P}_\alpha$ consists of pairwise disjoint subsets of $\cup\mathcal{I}$;
(2) If $\beta < \alpha$, then $\mathcal{P}_\alpha$ refines $\mathcal{P}_\beta$;
(3) Always $\cup\mathcal{I} \setminus \cup\mathcal{P}_\alpha$ belongs to $\mathcal{I}$;
(4) $\mathcal{I}$ is the ideal of nowhere dense sets with respect to the topology generated by $\cup\mathcal{H}$.

Secondly, we prepare the notions for applications with Ramsey null sets and v^0-sets. The ideal $\mathcal{I}$ has the ideal type $(\lambda, \kappa(\mathcal{I}), \gamma)$ whenever there exists a base $\mathcal{I}$-matrix $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \kappa(\mathcal{I})\}$ such that:
(5) Each $\mathcal{P}_\alpha$ has the cardinality λ;
(6) If $\beta < \alpha$ and $X \in \mathcal{P}_\beta$, then $X \setminus \cup\mathcal{P}_\alpha$ has the cardinality γ;
(7) If $\beta < \alpha$ and $Y \in \mathcal{P}_\beta$, then Y contains λ many members of $\mathcal{P}_\alpha$;
(8) There are no short maximal chains in $\cup\mathcal{H}$, i.e. if $\mathcal{C} \subseteq \cup\mathcal{H}$ is a

maximal chain, then $\mathcal{C} \cap \mathcal{P}_\alpha$ is nonempty for each $\alpha < \kappa(\mathcal{I})$;
(9) The intersection $\cap\{\cup\mathcal{P}_\alpha : \alpha < \kappa(\mathcal{I})\}$ is empty.

To describe the ideal type of (v^0) we have to assume that $cov(v^0) = \omega_1$. We do not know:

$$\text{Is it consistent that } \omega_1 \neq cov(v^0)?$$

If $\omega_1 = \min\{\text{cf}(\mathfrak{c}), r\}$, then Theorem 17 yields $\omega_1 = cov(v^0)$.

Theorem 18. *If $\omega_1 = cov(v^0)$, then (v^0) has the ideal type $(\mathfrak{c}, \omega_1, \mathfrak{c})$.*

Proof. Let $\mathcal{H} = \{\mathcal{P}_\alpha : \alpha < \omega_1\}$ be a base v-matrix. Since $\omega_1 = cov(v^0)$ one can inductively change $\mathcal{H}$ such that $\cap\{\cup\mathcal{P}_\alpha : \alpha < \kappa(v) = \omega_1\} = \emptyset$. If one considers families $\mathcal{P}_\alpha$ for limit ordinals, the one obtains a base v-matrix which witnesses that (v^0) has the ideal type $(\mathfrak{c}, \omega_1, \mathfrak{c})$. □

Thus, by [21] Theorem 2, if $h = \omega_1 = cov(v^0)$, then the ideal (v^0) is isomorphic with the ideal of all Ramsey null sets. This isomorphism clarify resemblances between definitions of completely Ramsey sets and v-sets. However, the σ-field (v) and the σ-field of all completely Ramsey sets are different. Some Ramsey null sets can be no v-sets, e.g. any intersection of a segment with a set which is dense and co-dense in a segment topology. Conversely, some v^0-sets can be no completely Ramsey sets. Indeed, if $\mathcal{H}$ is a base matrix, see [2], then $(\cup\mathcal{H})^*$ is not a completely Ramsey set and one can check that $(\cup\mathcal{H})^*$ is a v^0-set, compare Brendle [5].

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