DISCONTINUITY AND INVOLUTIONS ON COUNTABLE SETS

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ABSTRACT. For any infinite subset X of the rationals and a subset $F \subseteq X$ which has no isolated points in X we construct a function $f : X \to X$ such that $f(f(x)) = x$ for each $x \in X$ and F is the set of discontinuity points of f.

In the literature one finds a few algorithms that can produce any given subset of the rationals as the set of discontinuity points of a function. Probably Wacław Sierpiński [2] was the first to publish the algorithm, of the kind that is best known. In [1] this algorithm was reduced to following: let $X = A \cup B$ be a topological space, where sets A and B are dense and disjoint; assume that $Y = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{\frac{-1}{n} : n = 1, 2, \ldots\}$; suppose that $X \setminus C$ is the intersection of a decreasing sequence of open sets $F_n \subseteq X$ with $F_1 = X$; if $x \in X \setminus C$, then put $f(x) = 0$; if $x \in A \cap F_n \setminus F_{n+1}$, then put $f(x) = \frac{1}{n}$; if $x \in B \cap F_n \setminus F_{n+1}$, then put $f(x) = \frac{-1}{n}$; the set C is the set of discontinuity points of the defined function $f : X \to Y$. In this note we are suggesting an algorithm that works with involutions.

Let us assume that F and Q are disjoint subsets of the rationals.

Theorem 1. *If F is infinite and F has no isolated point in $Q \cup F$, then there is a bijection $f : Q \cup F \to Q \cup F$ such that: Q is the set of continuity points of f; f is the identity on Q; for any $x \in F$ we have $f(x) \neq x$ and $f(f(x)) = x$.*

Proof. Enumerate all points of Q as a sequence $y_0, y_1, \ldots$; enumerate all points of F as a sequence $x_0, x_1, \ldots$; choose an irrational number g such that $F \cap (-\infty, g)$ is empty or infinite, and $F \cap (g, +\infty)$ is empty or infinite; put $G_0 = \{(-\infty, g), (g, +\infty)\}$.

Take x_0 and choose $f(x_0) \in F \cap A$ such that $f(x_0) \neq x_0 \in A \in G_0$. Put $f(f(x_0)) = x_0$ and $F_0 = \{-\infty, +\infty, g, x_0, f(x_0), y_0\}$. Let G_1 be a family of all open intervals with endpoints which are succeeding points of F_0. Suppose that the set F_n has been defined and let G_{n+1} be consisted of all intervals with endpoints which are succeeding points of F_n. Let $x_{k_n} \in F \setminus F_n$ be the point with the least possible index such that $f(x_{k_n})$ has not been defined, but $f(x_i)$ has been defined for any $i < k_n$. Choose $f(x_{k_n}) \in F \cap A \setminus F_n$ such that $f(x_{k_n}) \neq x_{k_n} \in A \in G_j$, where $j \leq n+1$ is the greatest natural number for which a suitable $f(x_{k_n})$ could be chosen. Put $f(f(x_{k_n})) = x_{k_n}$ and $F_{n+1} = F_n \cup \{x_{k_n}, f(x_{k_n}), y_{n+1}\}$. The bijection f also requires that we set $f(y_n) = y_n$ for every n. The combinatorial properties of f follow directly from the definition. However, it remains to examine the continuity and discontinuity of f.

Suppose $x \in F_m \cap F$ and $\{a_0, a_1, \ldots\} \subseteq Q \cup F$ is a monotone sequence which converges to x. Choose a natural number $i \geq m$ such that for any $k \geq i$ there is some $I \in G_{k+1}$ and we have: x is an endpoint of I; $a_n \in I$ for all but finite many n; $f(x)$ is not an endpoint of I. By the definition $f(a_n) \in I$ for all but finite many n. It follows that $\lim_{n\to\infty} f(a_n) \neq f(x)$. Therefore f is discontinuous at any point $x \in F$.

Note that if $y \in Q$ is an isolated point in $Q \cup F$, then there is nothing to prove about the continuity of f at y. Suppose $y_m \in Q$ and $\{a_0, a_1, \ldots\} \subseteq Q \cup F$ is a monotone sequence which converges to y_m. Then for any $k \geq m$ there is some $I \in G_{k+1}$ and we have: y_m is an endpoint of I; $a_n \in I$ for all but finite many n. By the definition $f(a_n) \in I$ for all but finite many n. It follows that $\lim_{n\to\infty} a_n = y_m = f(y_m) = \lim_{n\to\infty} f(a_n)$. Therefore f is continuous at any point $x \in Q$. □

References

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