BOLZANO-WEIERSTRASS PRINCIPLE OF CHOICE EXTENDED TOWARDS ORDINALS

by

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Abstract. The Bolzano-Weierstrass principle of choice is the oldest method of the set theory, traditionally used in mathematical analysis. We are extending it towards transfinite sequences of steps indexed by ordinals. We are introducing the notions: hiker's tracks, hiker's maps and statements $P_n(X, Y, m)$; which are used similarly in finite, countable and uncountable cases. New proofs of Ramsey's theorem and Erdös-Rado theorem are presented as some applications.

I - Introduction. The Bolzano-Weierstrass principle of choice is based on succeeding divisions of a segment onto disjoint subsegments and on choice of a subsegment which has some desired quality. Our extension towards transfinite sequences of steps indexed by ordinals imitates a hiker's track. Any hiker's step corresponds to dividing. It is uniquely determined by his previous steps and by his destination point. To express our extension we consider statements $P_n(X, Y, m)$. If some of these statements depend on uncountable parameters, then we use them for a new proof of the Erdös-Rado partition theorem. If some others depend on countable parameters, then we use them for a proof of Ramsey's theorem. But, if any one depends on finite parameters, then we could introduce numbers $p(k, r, n)$ which are very similar to the so called Ramsey's numbers.

For a given set X denote its cardinality by $|X|$. If n is a natural number, then $[X]^1 = X$, and $[X]^n = \{b \subseteq X : |b| = n\}$. Infinite ordinals are usually denoted by Greek letters. Sometimes we write $\alpha \in \beta$, instead of $\alpha < \beta$. The remaining notations are standard.

II - Hiker's track, hiker's map. Let X be a well ordered set, i.e. X is an ordinal number. Fix a function $f : [X]^{n+1} \to Y$. Let $x_0 < x_1 < \ldots < x_{n-1}$ be the first n points in X. For any point $x \in X$ the increasing sequence

$$\{x_\beta : \beta \leq \delta(x)\} \subseteq X$$

is called the *x-hiker's track*, if any x_β where $\beta \geq n$, is defined us follows. Suppose that $\{x_\gamma : \gamma < \beta\}$ has been defined. For any $s \in [\beta]^n$ consider the subset

$$\delta(x,s) = \{y \in X : \quad f(\{x_\gamma : \gamma \in s\} \cup \{y\}) = f(\{x_\gamma : \gamma \in s\} \cup \{x\})\} \subset X.$$

Let x_β be the first point in the intersection $\cap\{\delta(x,s) : s \in [\beta]^n\} \subset X$. Our construction stops when $x_\beta = x$. This β is denoted $\delta(x)$. Any x-hiker's track is uniquely determined by the increasing sequence $\{x_\gamma : \gamma \leq \delta(x)\} \subseteq X$ and $x_{\delta(x)} = x$ holds.

If $s \in [\delta(x)]^n$, then put $f_x(s) = f(\{x_\zeta : \zeta \in s\} \cup \{x\})$. A function f_x is called *x-hiker's map.*

Theorem 1. *Any hiker's map $f_x : [\delta(x)]^n \to Y$ is uniquely determined by the subsequence $\{x_\gamma : \gamma < \delta(x)\}$ of the x-hiker's track. In other words, the map which for each point $x \in X$ assigns f_x is one-to-one.*

Proof. We shall prove that if $x \neq y$, then functions f_x and f_y are different. Indeed, if $\delta(x) \neq \delta(y)$, then the functions have different domains. If $\delta(x) = \delta(y)$, then the x-hiker's track $\{x_\beta : \beta \leq \delta(x)\}$ and the y-hiker's track $\{y_\beta : \beta \leq \delta(y)\}$ are different. Suppose β is the first ordinal such that $x_\beta \neq y_\beta$. Without the loss of generality, assume $x_\beta < y_\beta$. Thus there exists $s \in [\beta]^n$ such that $x_\beta \notin \delta(y,s)$. Hence

$$f(\{y_\zeta : \zeta \in s\} \cup \{x_\beta\}) \neq f(\{y_\zeta : \zeta \in s\} \cup \{y\}).$$

But

$$f(\{x_\zeta : \zeta \in t\} \cup \{x\}) = f(\{x_\zeta : \zeta \in t\} \cup \{x_\beta\}),$$

for each $t \in [\beta]^n$. In consequence,

$$f(\{y_\zeta : \zeta \in s\} \cup \{y\}) \neq f(\{x_\zeta : \zeta \in s\} \cup \{x\}).$$

For such s the following $f_x(s) \neq f_y(s)$ holds. In others words, the hiker's maps f_x and f_y are different. $\square$

In the literature, when X is a countable set, the above construction is called the Bolzano-Weierstrass principle of choice. In this way the method of indicating a monotone sequence which is contained in an infinite set of real numbers is honored. In this note we are extending this principle onto an arbitrary ordinal number instead of a countable set X.

III - Statements $P_n(X,Y,m)$. Fix some sets X, Y and m. Assume that the set X is well ordered and m is an ordinal number. For any natural number n consider the following statement.

For each function $f : [X]^{n+1} \to Y$ *there exists an increasing sequence* $\{x_\beta : \beta < m\} \subseteq X$ *such that if* $\beta_0 < \beta_1 < \ldots < \beta_n < \beta_{n+1} < m$*, then*

$$f(\{x_{\beta_0}, x_{\beta_1}, \ldots, x_{\beta_{n-1}}, x_{\beta_n}\}) = f(\{x_{\beta_0}, x_{\beta_1}, \ldots, x_{\beta_{n-1}}, x_{\beta_{n+1}}\}).$$

Denote this statement by $P_n(X, Y, m)$. In other words, $P_n(X, Y, m)$ means that for each function $f : [X]^{n+1} \to Y$ there exists a hiker's track which as a sequence has the length $m + 1$. Note that in this definition X does not need to be well ordered. But in applications we usually assume that X is well ordered.

If $n = 0$, then one has the pigeonhole principle. Then $P_0(m^+, m, m^+)$ holds for any infinite cardinal number m. Because of $m < m^+$, then for any function $f : m^+ \to m$ there exists a point $x \in m$ such that the preimage $f^{-1}(x)$ has the cardinality m^+. For a such x one can write $f^{-1}(x) = \{a_\alpha : \alpha < m^+\}$, i. e. $x = f(a_\beta) = f(a_\alpha)$, whenever $\alpha < \beta < m^+$.

Theorem 2. *If* $n > 0$ *and* λ *is an infinite cardinal number, then the statement* $P_n(|2^\lambda|^+, 2^\lambda, \lambda^+ + 1)$ *holds.*

Proof. Fix a function $f : [|2^\lambda|^+]^{n+1} \to 2^\lambda$ and put $X = |2^\lambda|^+$ and $Y = 2^\lambda$ and $x_0 = 0, \quad x_1 = 1, \quad \ldots, \quad x_{n-1} = n - 1$. For any ordinal number $\alpha \in |2^\lambda|^+$ consider α-hiker's track $\{a_\beta : \beta \leq \delta(\alpha)\}$. Suppose that $|\delta(\alpha)| \leq \lambda$ for any $\alpha \in |2^\lambda|^+$. There are at most λ^+ different ordinals of the form $\delta(\alpha)$ and at most 2^λ different hiker's maps $f_\alpha : [\delta(\alpha)]^n \to 2^\lambda$. Because of $\lambda^+ \cdot 2^\lambda = 2^\lambda$ one obtains a contradiction, since Theorem 1 says that there has to be $(2^\lambda)^+$ different hiker's maps. So, there exists $\alpha \in |2^\lambda|^+$ such that $|\delta(\alpha)| = \lambda^+$. $\square$

Theorem 3. *Let* $n > 0$ *and* λ *be an infinite cardinal number. If* m *is a cardinal number such that* $2^m \leq 2^\lambda$*, then* $P_n(|2^\lambda|^+, 2^\lambda, m + 1)$ *holds.*

Proof. Fix a function $f : [|2^\lambda|^+]^{n+1} \to 2^\lambda$ and put $X = |2^\lambda|^+$ and $Y = 2^\lambda$ and $x_0 = 0, \quad x_1 = 1, \quad \ldots, \quad x_{n-1} = n - 1$. For any ordinal number $\alpha \in |2^\lambda|^+$ consider α-hiker's track $\{a_\beta : \beta \leq \delta(\alpha)\}$. Suppose that $|\delta(\alpha)| < m$ for any $\alpha \in |2^\lambda|^+$. There are at most m different ordinals of the form $\delta(\alpha)$ and at most $2^\lambda = (2^\lambda)^m$ different functions $f_\alpha : [\delta(\alpha)]^n \to 2^\lambda$. Because of $m \cdot 2^\lambda = 2^\lambda$ one obtains a contradiction, since by Theorem 1, there are $(2^\lambda)^+$ different hiker's maps. So, there exists $\alpha < |2^\lambda|^+$ such that $|\delta(\alpha)| = m$. $\square$

Theorem 4. *Let* $n > 0$ *and* λ *be an infinite cardinal number. If* $2^m \leq 2^\lambda$ *for any cardinal number* $m < 2^\lambda$*, then* $P_n(|2^\lambda|^+, 2^\lambda, 2^\lambda + 1)$ *holds.*

Proof. Fix a function $f : [|2^\lambda|^+]^{n+1} \to 2^\lambda$ and put $X = |2^\lambda|^+$ and $Y = 2^\lambda$ and $x_0 = 0, \quad x_1 = 1, \quad \ldots, \quad x_{n-1} = n - 1$. For any ordinal number $\alpha \in |2^\lambda|^+$

consider α-hiker's track $\{a_\beta : \beta \leq \delta(\alpha)\}$. Suppose that $|\delta(\alpha)| < 2^\lambda$ for any $\alpha \in |2^\lambda|^+$. There are at most 2^λ different ordinals of the form $\delta(\alpha)$ and at most

$$(2^\lambda)^{|\delta(\alpha)|} = 2^{\lambda \times |\delta(\alpha)|} = 2^{|\delta(\alpha)| \times \lambda} = (2^\lambda)^\lambda = 2^\lambda$$

different functions $f_\alpha : [\delta(\alpha)]^n \to 2^\lambda$. One obtains another contradiction with Theorem 1. This follows that there exists $\alpha < |2^\lambda|^+$ such that $|\delta(\alpha)| = 2^\lambda$. $\square$

IV - Applications to some proofs of Erdös–Rado partition theorems. To give some applications of statements $P_n(X, Y, m)$ we start with a proof of P. Erdös and R. Rado theorem [2]. Let $\exp^{(0)}(X) = X$ and $\exp^{(n+1)}(X) = 2^{\exp^{(n)}(X)}$. We need statements

$$P_k(|\exp^{(k)}(\zeta)|^+, \zeta, |\exp^{(k-1)}(\zeta)|^+),$$

where $0 < k \leq n$. One can deduce these statements from Theorem 2 putting $\lambda = \exp^{(k-1)}(\zeta)$ and restricting the second parameter to $\zeta < |2^\lambda|$.

Theorem (P. Erdös and R. Rado [2]). *Let n be a natural number, but ζ and κ be infinite cardinal numbers, and assume that $\kappa > |\exp^{(n)}(\zeta)|$. Then for any function $f : [\kappa]^{n+1} \to \zeta$ there exist an ordinal $\varphi < \zeta$ and a subset $Z \subseteq \kappa$ such that $|Z| > \zeta$ and $[Z]^{n+1} \subseteq f^{-1}(\varphi)$.*

Proof. We proceed by induction on n. For $n = 0$ we have assumed $\kappa > \zeta$ and $P_0(\zeta^+, \zeta, \zeta^+)$ holds by the same argumentat as this before Theorem 2. Let $|\exp^{(n-1)}(\zeta)| = \lambda$, and assume that the theorem holds for a natural number $n - 1 \geq 0$. Since $|2^\lambda|^+ \leq \kappa$ the statement $P_n(|2^\lambda|^+, \zeta, \lambda^+)$ yields a hiker's track $\{a_\beta : \beta < \lambda^+\}$ such that if $\beta_0 < \beta_1 < \ldots < \beta_n < \beta_{n+1} < \lambda^+$, then

$$f(\{a_{\beta_0}, a_{\beta_1}, \ldots, a_{\beta_{n-1}}, \alpha_{\beta_n}\}) = f(\{a_{\beta_0}, a_{\beta_1}, \ldots, a_{\beta_{n-1}}, a_{\beta_{n+1}}\}).$$

Consider the notion of a hiker's map, i.e. if $\beta_0 < \beta_1 < \ldots < \beta_{r-1} < \beta < \lambda^+$, then put

$$F(\{\beta_0, \beta_1, \ldots, \beta_{n-1}\}) = f(\{a_{\beta_0}, a_{\beta_1}, \ldots, a_{\beta_{n-1}}, a_\beta\}).$$

This hiker's map is a function $F : [\lambda^+]^n \to \zeta$. By the induction hypothesis there exist an ordinal $\varphi < \zeta$ and a subset $S \subseteq \lambda^+$ such that $[S]^n \subseteq F^{-1}(\varphi)$ and $|S| > \zeta$. The subset $Z = \{\alpha_\beta : \beta \in S\} \subseteq \kappa$ has cardinality greater than ζ, and if $\{\beta_0, \beta_1, \ldots, \beta_{k-1}, \beta\} \subseteq S$ and $\beta_0 < \beta_1 < \ldots < \beta_{k-1} < \beta$, then

$$f(\{\alpha_{\beta_0}, \alpha_{\beta_1}, \ldots, \alpha_{\beta_{n-1}}, \alpha_\beta\}) = F(\{\beta_0, \beta_1, \ldots, \beta_{n-1}\}) = \varphi.$$

This clearly implies $[Z]^{n+1} \subseteq f^{-1}(\varphi)$. $\square$

In the literature there are many proofs of the of the Erdös–Rado partition theorems. Our proof looks most similar to that of J.D.Monk, see [3] p. 1230. This similarity could be understood such that the notion of "pre-homogeneous" is replaced by suitable statements $P_n(X, Y, \lambda)$. This gives some reasons to consider $P_n(X, Y, \lambda)$ as self-made notions.

Add two corollaries which are similar to some results which are in the book [1], pages 8 - 11.

Corollary 5. *Let m, ξ, λ and κ be infinite cardinal numbers. Assume that $\kappa > |2^\lambda|^+$ and $\xi < m$ and $2^m \leq 2^\lambda$. Then for any function $f : [\kappa]^2 \to \xi$ there exist an ordinal $\varphi < \xi$ and a subset $Z \subseteq \kappa$ such that $|Z| \geq m$ and $[Z]^2 \subseteq f^{-1}(\varphi)$.*

Proof. By Theorem 3 the statemnet $P_1(\kappa, \xi, m + 1)$ holds. This yields a hiker's track $\{a_\beta : \beta \leq m\}$ such that if $\beta < \gamma < m$, then

$$f(\{a_\beta, a_\gamma\}) = f(\{a_\beta, a_m\}).$$

In consequence one obtains a hiker's map $F : m \to \xi$, where $F(\beta) = f(\{a_\beta, a_m)$ for any $\beta < m$. By the pigeonhole principle and since $\xi < m$ there exist an ordinal $\varphi < \xi$ and a subset $Z \subseteq \{a_\beta : \beta \leq m\}$ such that $[Z]^2 \subseteq f^{-1}(\varphi)$ and $|Z| = m$. □

Corollary 6. *Let ξ, λ and κ be infinite cardinal numbers. Assume that $\kappa > |2^\lambda|^+$ and $\xi < 2^\lambda$ and $m < 2^\lambda$ always implies that $2^m \leq 2^\lambda$. Then for any function $f : [\kappa]^2 \to \xi$ there exist an ordinal $\varphi < \xi$ and a subset $Z \subseteq \kappa$ such that $|Z| \geq |2^\lambda|$ and $[Z]^2 \subseteq f^{-1}(\varphi)$.*

Proof. By Theorem 4 the statement $P_1(\kappa, \xi, 2^\lambda + 1)$ holds. This yields a hiker's track $\{a_\beta : \beta \leq 2^\lambda\}$ such that if $\beta < \gamma < 2^\lambda$, then

$$f(\{a_\beta, a_\gamma\}) = f(\{a_\beta, a_{2^\lambda}\}).$$

In consequence one obtains a hiker's map $F : 2^\lambda \to \xi$, where $F(\beta) = f(\{a_\beta, a_{2^\lambda}\})$ for any $\beta < 2^\lambda$. By the pigeonhole principle and since $\xi < 2^\lambda$ there exist an ordinal $\varphi < \xi$ and a subset $Z \subseteq \{a_\beta : \beta \leq 2^\lambda\}$ such that $[Z]^2 \subseteq f^{-1}(\varphi)$ and $|Z| = 2^\lambda$. □

V - On a proof of Ramsey's theorem. In this part we give applications of statements $P_n(\omega, r, \omega)$, where ω denotes the set natural numbers and r is a natural number. To do this we present a proof of the Ramsey's theorem, see [5]. In [4] p. 5 there is a proof of Ramsey's theorem which contains some aspects of the Bolzano–Weierstrass principle of choice.

Theorem 7. *If $r > 0$ and n are natural numbers but ω is the first infinite ordinal, then the statement $P_n(\omega, r, \omega)$ holds.*

Proof. Fix a function $f : [\omega]^{n+1} \to r$. Any natural number $k \geq n$ uniquely determines the hiker's map $f_k : [\delta(k)]^n \to r$. All maps f_k forms an infinite tree. By the König infinity lemma this tree possesses an infinite path. Any such infinite path marks a desired hiker's track. $\square$

Now, using the reduction from our proof of the Erdös–Rado partition theorem: the reduction from $P_n(\omega, r, \omega)$ to $P_{n-1}(\omega, r, \omega)$); we obtain a proof of the Ramsey's theorem.

Ramsey's theorem. (F. P. Ramsey [5]). *If $r > 0$ and n are natural numbers, then for any function $f : [\omega]^{n+1} \to r$ there exist a natural number m and an infinite subset $Z \subseteq \omega$ such that $[Z]^{r+1} \subseteq f^{-1}(m)$.*

Proof. We proceed by induction on n. For $n = 0$ we have the pigeonhole principle. Assume that Ramsey's theorem holds for a natural number $n - 1 \geq 0$. The statement $P_n(\omega, r, \omega)$ yields an infinite hiker's track $\{a_k : k < \omega\}$ such that if $k_0 < k_1 < \ldots < k_n < k_{n+1}$, then

$$f(\{a_{k_0}, a_{k_1}, \ldots, a_{k_{n-1}}, \alpha_{k_n}\}) = f(\{a_{k_0}, a_{k_1}, \ldots, a_{k_{n-1}}, a_{k_{n+1}}\}).$$

To the hiker's map $F : [\omega]^r \to n$, where

$$F(\{k_0, k_1, \ldots, k_{n-1}\}) = f(\{a_{k_0}, a_{k_1}, \ldots, a_{k_{n-1}}, a_{k_n}\})$$

one applies the induction hypothesis. $\square$

Corollary 8. *If $r > 0$ and n are natural numbers but ω is the first infinite ordinal, then the statement $P_n(\omega, r, \alpha)$ holds for any countable ordinal number α.* $\square$

VI - Numbers $p(k, r, n)$. Consider statements $P_n(X, Y, \lambda)$ for cases when X, Y and λ are natural numbers. Fix positive natural numbers k, r and n. Similar to the definition of Ramsey's numbers - compare [4] p. 13 - let $p(k, r, n) > r$ be the least natural number such that the statement $P_r(p(k, r, n), n, k)$ holds. This means that $p(k, r, n)$ is the least natural number such that for any function $f : [p(k, r, n)]^{r+1} \to n$ there exists a hiker's track $\{a_i : i < k\}$ such that

$$f(\{a_{i_0}, a_{i_1}, \ldots, a_{i_{r-1}}, \alpha_{i_r}\}) = f(\{a_{i_0}, a_{i_1}, \ldots, a_{i_{r-1}}, a_{i_{r+1}}\}),$$

whenever $i_0 < i_1 < \ldots < i_r < i_{r+1}$. Numbers $p(k, r, n)$ are well defined since the following holds.

Theorem 9. *If n, r and k are positive natural numbers, then*

$$p(k,r,n) - r < n^{\binom{r+0}{r}} + n^{\binom{r+1}{r}} + \ldots + n^{\binom{r+k-2}{r}} + 1.$$

Proof. Use again the Bolzano-Weierstrass principle of choice. If $r \leq k$, then the function $f_k : [\delta(k)]^r \to n$ is uniquely determined. Also, if $\delta(k) = r + i$, then there are $n^{\binom{r+i}{r}}$ possibilities for any f_k with the domain of cardinality i. Therefore $p(k,r,n) - r - 1 \geq n^{\binom{r+0}{r}} + n^{\binom{r+1}{r}} + \ldots + n^{\binom{r+k-2}{r}}$ implies that a function f_k with the domain of cardinality $k-1$ has to be defined . Any such a function f_k designs a desired sequence. $\square$

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