GAME APPROACH TO UNIVERSALLY KURATOWSKI-ULAM SPACES

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Abstract. We consider a version of the open-open game, indicating its connections with universally Kuratowski-Ulam spaces. From [2] and [3] topological arguments are extracted to show that: *Every I-favorable space is universally Kuratowski-Ulam*, Theorem 8; *If a compact space Y is I-favorable, then the hyperspace* $\exp(Y)$ *with the Vietoris topology is I-favorable, and hence universally Kuratowski-Ulam*, Theorems 6 and 9. Notions of uK-U and uK-U* spaces are compared.

1. Introduction

The following theorem was proved (in fact) by K. Kuratowski and S. Ulam, see [7] and compare [6] p. 246:

Let X and Y be topological spaces such that Y has countable π-weight. If $E \subseteq X \times Y$ is a nowhere dense set, then there is $P \subseteq X$ of first category such that the section $E_x = \{y : (x, y) \in E\}$ is nowhere dense in Y for any point $x \in X \setminus P$.

In [8] one can find less general formulation of the Kuratowski Ulam Theorem:

If E is a plane set of first category, then E_x is a linear set of first category for all x except a set of first category.

In the literature a set of the first category is usually called a meager set. The Kuratowski Ulam Theorem holds for any meager (nowhere dense) set $E \subseteq X \times Y$, where the Cartesian product $X \times Y$ is equipped with the Tychonov topology and π-weight of Y is less than additivity of meager sets in X, compare [3], [6] or [8].

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The above formulations of the Kuratowski Ulam Theorem suggests two notions of universally Kuratowski-Ulam spaces which one could study.

A space Y is *universally Kuratowski-Ulam* (for short, *uK-U space*), whenever for any topological space X and a meager set $E \subseteq X \times Y$, the set

$$\{x \in X : \{y \in Y : (x,y) \in E\} \text{ is not meager in } Y\}$$

is meager in X, see D. Fremlin, T. Natkaniec and I. Recław [3]. The class of uK-U spaces has been investigated in [3], [4] and [15].

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is meager in X, see D. Fremlin [4].

Any uK-U* space is uK-U space. A proof of this is standard. Indeed, suppose that a space Y is uK-U*, and X is a topological space. If $E \subseteq X \times Y$ is a meager set, then there exist nowhere dense sets $E^n \subseteq X \times Y$ such that $E^0 \cup E^1 \cup \ldots \supseteq E$. Put

$$P_n = \{x \in X : \{y \in Y : (x,y) \in E^n\} \text{ is not nowhere dense in } Y\}.$$

Each set P_n is meager, hence $P = P_0 \cup P_2 \cup \ldots \subseteq X$ is meager. Since $E_x \subseteq E^0_x \cup E^1_x \cup \ldots$ (recall that $E^n_x = \{y \in Y : (x,y) \in E^n\}$), then E_x is meager for each $x \in X \setminus P$.

The converse is not true: *There is a dense in itself and countable Hausdorff space which is not uK-U**; see "6. Examples (b)" in [4]. Any countable and dense in itself space is meager in itself, and hence has to be uK-U. The space $C[\omega^\omega]$ of all compact non-empty subsets of the irrationals equipped with the Pixley-Roy topology has a separable compactification, see A. Szymański [13]. One can check that $\omega^\omega \times C[\omega^\omega]$ does not satisfy the Kuratowski Ulam Theorem, hence $C[\omega^\omega]$ is not uK-U*, and any dense subspace of a compactification of $C[\omega^\omega]$ is not uK-U*, too. So, some compactification of $C[\omega^\omega]$ contains a countable Hausdorff space which is uK-U and not uK-U*. Natural examples of countable spaces which are not uK-U* are spaces of type *Seq*, compare [14]. They are not uK-U* by similar arguments which work with $C[\omega^\omega]$, or with Example 1 in [3].

The open-open game and I-favorable spaces were introduced by P. Daniels, K. Kunen and H. Zhou [2]. A space is I-favorable if, and only if it has a club filter, see [2]. Topics of almost the same kind like I-favorable spaces were considered by E. V. Shchepin [11], L. Heindorf and L. Shapiro [5], and by B. Balcar, T. Jech and J. Zapletal [1]. In [11] were introduced κ-metrizable spaces; in [5] were considered regularly filtered algebras; in [1] were considered semi-Cohen algebras. A Boolean algebra $\mathbb{B}$ is semi-Cohen (regularly filtered) if, and only if $[\mathbb{B}]^\omega$ has a closed unbounded set of countable regular subalgebras (contains a club filter). Semi-Cohen algebras and I-favorable spaces are corresponding classes, compare [1] and [5].

Every dyadic space is uK-U space, see [3]. We extend this fact by showing that any I-favorable space is uK-U*, Theorem 8. Additionally, we show that any hyperspace $\exp(D^\lambda)$ is uK-U* space, Corollary 10.

2. The game

The following game was invented by P. Daniels, K. Kunen and H. Zhou [2]. Two players take turns playing with a topological space X. A round consists of Player I choosing a non-empty open set $U \subseteq X$; and Player II choosing a non-empty open set $V \subseteq U$. Player I wins if the union of all open sets which have been chosen by Player II is dense in X. This game was called the *open-open game*. If the open-open game of uncountable length is being played with a space of countable cellularity (for example, some *Seq* spaces), then Player II could be forced to choose disjoint sets at each round. In consequence, Player I wins any such game. Thus, any open-open game is not trivial under some restrictions which imply that Player I can not win always. For example, rounds are played for each ordinal less than some given ordinal α. From here, we consider cases when games have the least infinite length i.e. $\alpha = \omega$.

Let us consider the following game. Player I chooses a finite family $\mathcal{A}_0$ of non-empty open subsets of X. Then Player II chooses a finite family $\mathcal{B}_0$ of non-empty open subsets of X such that for each $U \in \mathcal{A}_0$ there exists $V \in \mathcal{B}_0$ with $V \subseteq U$. Similarly at the n-th round Player I chooses a finite family $\mathcal{A}_n$ of non-empty open subset of X. Then Player II chooses a finite family $\mathcal{B}_n$ of non-empty open subsets of X such that for each $U \in \mathcal{A}_n$ there exists $V \in \mathcal{B}_n$ with $V \subseteq U$. If for any natural number k the union $\bigcup\{\mathcal{B}_k \cup \mathcal{B}_{k+1} \cup \ldots\}$ is a dense subset of X, then Player I wins; otherwise Player II wins.

The space X is I-*favorable* whenever Player I can be insured, by choosing his families $\mathcal{A}_n$ judiciously, that he wins no matter how Player II plays. In this case we say that Player I has a winning strategy. Player I has a winning strategy whenever any finite family of open and disjoint subsets of X he can consider as $\mathcal{B}_n$, and then Player I knows his $(n+1)$-th round, i.e. he knows how to define $\mathcal{A}_0 = \sigma(\emptyset)$ and $\mathcal{A}_{n+1} = \sigma(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n)$. Any winning strategy would be defined as function

$$(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n) \mapsto \sigma(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n),$$

where all families $\mathcal{B}_n$ and $\sigma(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n)$ are finite and consists of non-empty open sets; and for any game with succeeding rounds $\sigma(\emptyset)$, $\mathcal{B}_0$, $\sigma(\mathcal{B}_0)$, $\mathcal{B}_1, \sigma(\mathcal{B}_0, \mathcal{B}_1), \ldots, \mathcal{B}_n, \sigma((\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n)$ each union $\bigcup\{\mathcal{B}_k \cup \mathcal{B}_{k+1} \cup \ldots\}$ is a dense subset of X.

Our definition of I-favorable space is equivalent to the similar definition stated in [2, p. 209]. In fact, if $\mathcal{A}_n = \{U_1, U_2, \ldots, U_k\}$, then Player I should play k-rounds choosing $U_1, U_2, \ldots, U_k$, successively. If Player I has a strategy σ which forced Player II to choose families $\mathcal{B}_k$ such that $\bigcup\{\mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots\}$ is a dense subset of X, then Player I could divide the set of natural numbers onto infinite many of pairwise disjoint infinite pieces. Then Player I could play at each piece following σ, and he obtains the winning strategy. In consequence, for the definition of I-favorable spaces one can use the open-open game, or the topological version of the game G_4, see [2, p. 219].

Many cases when Player II can be insured that he wins no matter how Player I plays were considered in [2] or [13]. By Theorem 8 spaces *Seq* are not I-favorable. However, one can check directly that Player II could always win a game with any *Seq*: Any *Seq* has a tiny sequence, compare [13], and therefore Player II has winning strategy.

Let us recall a few comments according to [2]. Any space with countable π-weight is I-favorable. Indeed, if $\{W_0, W_1, \ldots\}$ is a π-base for X, then Player I chooses $\mathcal{A}_n$ such that always there exists $U \in \mathcal{A}_n$ and $U \subseteq W_n$. If a space X has uncountable cellularity, then X is not I-favorable. Indeed, there exists an uncountable family $\mathcal{W}$ of open and disjoint subsets of X, and Player II can choose $\mathcal{B}_n$ such that always $\bigcup \mathcal{B}_n$ intersects finite many members of $\mathcal{W}$. Another example is a regular Baire space X with a category measure μ such that $\mu(X) = 1$ (for more details see [8, p. 86 - 91]). Any such X is not I-favorable, since Player II can choose $\mathcal{B}_n$ such that always $\mu(\bigcup \mathcal{B}_n) < \frac{1}{2^{n+2}}$. This follows $\mu\left(X \setminus \left(\bigcup\{\mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots\}\right)\right) \geq \frac{1}{2}$. Therefore the complement $X \setminus \left(\bigcup\{\mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots\}\right)$ has to have non empty interior.

3. On I-favorable spaces

A topological characterization of I-favorable spaces is applied to describe direct proofs of some know facts. Moreover, we show that if a compact space X is I-favorable, then the hyperspace $\exp(X)$ with the Vietoris topology is I-favorable. We extract topological versions of arguments used in [2] and [3].

For any Cantor cube D^λ fix the following notation. Let λ be a cardinal number, $D = \{0, 1\}$, and let D^λ be equipped with the product topology. The product topology is generated by subsets $\{q \in D^\lambda : q(\alpha) = k\}$, where $\alpha \in \lambda$ and $k \in D$. If $f : Y \to D$ and $Y \in [\lambda]^{<\omega}$, then $W_f = \{q \in D^\lambda : f \subseteq q\}$. All sets W_f constitute an open base.

Example 1. *The Cantor cube D^λ is I-favorable.*

Proof. Player I put $\mathcal{A}_0 = \{D^\lambda\}$. If a family $\mathcal{B}_0$ is defined, then Player I chooses base open sets $W_q \subseteq Q$ for any $Q \in \mathcal{B}_0$ and put $\mathcal{A}_1 = \{W_f : f \in D^{J_1}\}$, where $J_1 = \bigcup\{\mathrm{dom}\,(q) : W_q \subseteq Q \in \mathcal{B}_0\}$. Player I wins, whenever at the n-th round he always chooses base sets $W_q \subseteq Q$ for any $Q \in \mathcal{B}_{n-1}$, and put $\mathcal{A}_n = \{W_f : f \in D^{J_n}\}$, where $J_n = \bigcup\{\mathrm{dom}\,(q) : W_q \subseteq Q \in \mathcal{B}_{n-1}\}$. Any such played game defined a sequence $J_1 \subseteq J_2 \subseteq \ldots$ of finite subsets of λ. Fix a base set W_f where $f \in D^J$, i.e. $J = \mathrm{dom}(f)$. Take a natural number n such that $J \cap J_n = J \cap J_{n+1}$, and next take $q \in D^{J_n}$ such that functions f and q are compatible on the set $J \cap J_n = \mathrm{dom}(f) \cap \mathrm{dom}(q)$. There exists $q^* \in D^{J_{n+1}}$ such that

$$\mathcal{A}_n \ni W_q \supseteq V \supseteq W_{q^*} \in \mathcal{A}_{n+1},$$

where $V \in \mathcal{B}_n$. Functions f and q^* are compatible on the set

$$J \cap J_n = \mathrm{dom}(f) \cap \mathrm{dom}(q^*) = J \cap J_{n+1}.$$

Therefore W_f meets W_{q^*}, and hence $\emptyset \neq W_f \cap W_{q^*} \subseteq W_f \cap V \subseteq V$. Since n could be arbitrarily large and $V \in \mathcal{B}_n$, then each $\bigcup\{\mathcal{B}_k \cup \mathcal{B}_{k+1} \cup \ldots\}$ has to be a dense subset of X. □

We have repeated a special case of Theorem 1.11, see [2]. Our proof of Example 1 explicitly defines a winning strategy. But if families $\mathcal{A}_0, \mathcal{A}_1, \ldots$ have been defined simultaneously, then Player I would lose.

This would not happen when X has countable π-base. However for $X = D^\lambda$, where λ is uncountable, this is possible. Indeed, if Player I fixes each family $\mathcal{A}_n$, then Player II could choose a finite family $\mathcal{B}^*_n$ such that for any $U \in \mathcal{A}_n$ there exists a base subset $W_q \in \mathcal{B}^*_n$ with $W_q \subseteq U$. Put $J_n = \bigcup\{\mathrm{dom}\,(q) : W_q \in \mathcal{B}^*_n\}$, and take an index $\alpha \in \lambda \setminus (J_0 \cup J_1 \cup \ldots)$. Afterwards Player II put

$$\mathcal{B}_n = \{V \cap \{q \in D^\lambda : q(\alpha) = 1\} : V \in \mathcal{B}^*_n\}.$$

No member of $\mathcal{B}_n$ meets $\{q \in D^\lambda : q(\alpha) = 0\}$. In fact, we get the following.

Remark 2. *For each sequence $(\mathcal{A}_0, \mathcal{A}_1, \ldots)$ consisting of finite non-empty families of open subsets of D^λ, there is a corresponding sequence $(\mathcal{B}_0, \mathcal{B}_1, \ldots)$ consisting of finite non-empty families $\mathcal{B}_n$ of non-empty open sets such that each $\mathcal{B}_n$ refines $\mathcal{A}_n$, and yet the union $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots$ is not dense.* □

Countable subsets of λ are important in our proof of Example 1. Any $J \in [\lambda]^\omega$ fixes the countable family of base sets

$$C_J = \{W_f : f : Y \to D \text{ and } Y \in [J]^{<\omega}\},$$

which fulfills the following condition:

For any open $V \subseteq D^\lambda$ there is $W \in C_J$ such that if $U \in C_J$ and $U \subseteq W$, then $U \cap V \neq \emptyset$.

This condition may be considered in an arbitrary topological space X with a fixed π-base $\mathcal{Q}$. According to definitions [2, p. 208] a family $\mathcal{C} \subset [\mathcal{Q}]^\omega$ is called *a club filter* whenever:

(1) *The family $\mathcal{C}$ is closed under ω-chains with respect to inclusion, i.e. if $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \ldots$ is an ω-chain which consists of elements of $\mathcal{C}$, then $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \in \mathcal{C}$;*

(2) *For any countable subfamily $\mathcal{A} \subseteq \mathcal{Q}$, where $\mathcal{Q}$ is the $\pi-$base fixed above, there exists $\mathcal{P} \in \mathcal{C}$ such that $\mathcal{A} \subseteq \mathcal{P}$;*

(3) *For any non-empty open set V and each $\mathcal{P} \in \mathcal{C}$ there is $W \in \mathcal{P}$ such that if $U \in \mathcal{P}$ and $U \subseteq W$, then U meets V, i.e. $U \cap V \neq \emptyset$.*

Conditions (1) – (3) are extracted from properties of Cantor cubes used in Example 1. The following two lemmas repeat Theorem 1.6, see [2].

Lemma 3. *If a topological space has a club filter, then it is I-favorable.*

Proof. Without lost of generality one can assume that any $\mathcal{B}_n$ will be contained in $\mathcal{Q}$. Let $\mathcal{A}_0 = \{X\}$. If $\mathcal{B}_0$ has been defined, then Player I chooses $\mathcal{P}_0 \in \mathcal{C}$ such that $\mathcal{B}_0 \subseteq \mathcal{P}_0$, by (2). Enumerate $\mathcal{P}_0 = \{V_0^0, V_1^0, \ldots\}$ and put $\mathcal{A}_1 = \{V_0^0\}$. If families $\mathcal{B}_n$ and $\mathcal{P}_{n-1}$ have been defined, then Player I chooses $\mathcal{P}_n \in \mathcal{C}$ such that $\mathcal{B}_n \cup \mathcal{P}_{n-1} \subseteq \mathcal{P}_n$, using (2) again. Let $\mathcal{P}_n = \{V_0^n, V_1^n, \ldots\}$, and put $\mathcal{A}_{n+1} = \{V_j^i : i \leqslant n \text{ and } j \leqslant n\}$. By Condition (1), let $\mathcal{P}_0 \cup \mathcal{P}_1 \cup \ldots = \mathcal{P}_\infty \in \mathcal{C}$. We shall show that any union $\bigcup\{\mathcal{B}_k \cup \mathcal{B}_{k+1} \cup \ldots\}$ is a dense subset of X. Suppose that V is a non-empty open set such that $V \cap \bigcup\{\mathcal{B}_k \cup \mathcal{B}_{k+1} \cup \ldots\} = \emptyset$. By (3) choose $V_j^i \in \mathcal{P}_\infty$ such that if $U \in \mathcal{P}_\infty$ and $U \subseteq V_j^i$, then $U \cap V \neq \emptyset$. Take $m \geqslant \max\{i, j, k\}$. There exists $W \in \mathcal{B}_{m+1} \subseteq \mathcal{P}_\infty$ such that $W \subseteq V_j^i$, hence $W \cap V \neq \emptyset$. But $W \in \mathcal{B}_k \cup \mathcal{B}_{k+1} \cup \ldots$, a contradiction. □

Lemma 4. *If a topological space is I-favorable, then it has a club filter such that any of its elements is closed under finite intersection.*

Proof. Let $\mathcal{Q}$ be a fixed π-base, which is closed under finite intersection, and let σ be a winning strategy for Player I. For each countable family $\mathcal{R} \in [\mathcal{Q}]^{\leqslant\omega}$ let $\mathcal{R}_1$ be the closure under finite intersection of $\mathcal{R}$ and the family

$$\bigcup\{\sigma(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k) : \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k\} \subset [\mathcal{R}]^{<\omega} \text{ and } k \in \omega\}.$$

By induction, let $\mathcal{R}_{n+1}$ be the closure under finite intersection of $\mathcal{R}_n$ and

$$\bigcup\{\sigma(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k) : \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k\} \subset [\mathcal{R}_n]^{<\omega} \text{ and } k \in \omega\}.$$

A desired club filter $\mathcal{C}$ consists of all unions $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \ldots$, where $\mathcal{R} \in [\mathcal{Q}]^{\leqslant\omega}$. By the definition any element of $\mathcal{C}$ is closed under finite intersection. Consider an ω-chain $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \ldots$ in $\mathcal{C}$. Let $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots = \mathcal{R}$. If $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k$ are finite families contained in $\mathcal{R}$, then there exists n such that $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k \subseteq \mathcal{P}_n$ and $\sigma(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k) \subseteq \mathcal{P}_{n+1}$. This follows $\mathcal{R} \in \mathcal{C}$, i.e. Condition (1) holds. Condition (2) follows directly from the definition of $\mathcal{C}$. Suppose that $\mathcal{P} \in \mathcal{C}$ and an open set V fulfill the negation of (3). Then, Player II chooses families consisting of sets disjoint with V. In consequence, he wins the game $\sigma(\emptyset), \mathcal{B}_0, \sigma(\mathcal{B}_0), \mathcal{B}_1, \ldots$, a contradiction. □

The next corollary was proved in [2, Corollary 1.7].

Corollary 5. *Any product of I-favorable spaces is I-favorable.*

Proof. Consider a product $\prod\{X_\alpha : \alpha \in T\}$, where any X_α is I-favorable. Let $\mathcal{C}_\alpha$ be a club filter which witnesses that X_α is I-favorable, where $\mathcal{Q}_\alpha$ is a π-base needed in Condition (2). Fix $\lambda \in [T]^\omega$ and $\mathcal{P}_\alpha \in \mathcal{C}_\alpha$ for each $\alpha \in \lambda$. Let $\mathcal{P}(\lambda)$ be the family of all $\prod\{W_\alpha : \alpha \in S\}$, where $S \in [\lambda]^{<\omega}$ and $W_\alpha \in \mathcal{P}_\alpha \in \mathcal{C}_\alpha$. The family $\mathcal{C} = \{\mathcal{P}(\lambda) : \lambda \in [T]^\omega\}$ is a desired club filter. □

In [2, p. 210] it was proved that dyadic spaces are I-favorable. But L. Shapiro [9] show that some hyperspaces over dyadic spaces can be non-dyadic. For example, $\exp(D^{\omega_2})$ is a non-dyadic space. For some facts and notions concerning a hyperspace with the Vietoris topology, which are not defined here, see [6]. Now, prove the following.

Theorem 6. *If a compact space X is I-favorable, then the hyperspace $\exp(X)$ with the Vietoris topology is I-favorable, too.*

Proof. Fix a π-base $\mathcal{Q}$ closed under finite intersection, and a club filter $\mathcal{C}$ for X. If n is a natural number and $V_1, V_2, \ldots, V_n$ are open subsets of X, then let $< V_1, V_2, \ldots, V_n >$ denotes the family of all closed sets $A \subseteq V_1 \cup V_2 \cup \ldots \cup V_n$ such that $A \cap V_i \neq \emptyset$ for $1 \leqslant i \leqslant n$. The family

$$\mathcal{Q}^* = \{< V_1, V_2, \ldots, V_n >: V_i \in \mathcal{Q} \text{ for } 1 \leqslant i \leqslant n\}$$

is a π-base for $\exp(X)$. For any $\mathcal{P} \in \mathcal{C}$, let

$$\mathcal{P}^* = \{< V_1, V_2, \ldots, V_n >: V_i \in \mathcal{P} \text{ for } 1 \leqslant i \leqslant n\}.$$

We shall check that the family $\mathcal{C}^* = \{\mathcal{P}^* : \mathcal{P} \in \mathcal{C}\}$ is a club filter for $\exp(X)$. Then the result follows from Lemma 3.

By definitions $\mathcal{C}^*$ fulfills conditions (1) and (2) and any family $\mathcal{P}^* \in \mathcal{C}^*$ is closed under finite intersection. Consider an open set $< V_1, V_2, \ldots, V_n > \subseteq \exp(X)$ and a family $\mathcal{P} \in \mathcal{C}$. For $1 \leqslant i \leqslant n$, by (3), choose $W_i \in \mathcal{P}$ such that if $U \in \mathcal{P}$ and $U \subseteq W_i$, then U meets V_i. If

$$< W_1, W_2, \ldots, W_n > \supseteq < U_1, U_2, \ldots, U_m > \in \mathcal{P}^*,$$

then fix $U_i^j \in \{U_1, U_2, \ldots, U_m\}$ with $U_i^j \subseteq W_i$. Since $\mathcal{P}$ is closed under finite intersection, then $U_i^j \cap W_i \in \mathcal{P}$. By (3) choose $x_i \in V_i \cap W_i \cap U_i^j$ for $1 \leqslant i \leqslant n$. Similarly, choose $y_i^j \in V_i \cap U_j \cap W_i$ whenever U_j meets W_i. The closed (finite) set

$$\{x_i : 1 \leqslant i \leqslant n\} \cup \{y_i^j : 1 \leqslant j \leqslant m \text{ and } 1 \leqslant i \leqslant n\} \subseteq X$$

belongs to the intersection $< V_1, V_2, \ldots, V_n > \cap < U_1, U_2, \ldots, U_m >$. It follows that (3) holds for $\exp(X)$. □

Special cases of Theorem 6 could be deduced in another way. L. Shapiro observed that $\exp(D^\lambda)$ is co-absolute with D^λ, see [10, Theorem 4] and [12, p.17-18]. Therefore one could obtain that $\exp(D^\lambda)$ is I-favorable by [2, Fact 1.3].

One can check that if there is a club filter $\mathcal{C}$ for $\exp(X)$ such that any $\mathcal{P} \in \mathcal{C}$ consists of base sets of the form $< V_1, V_2, \ldots, V_n >$, then families constitute all V_i such that $V_i \in \{V_1, V_2, \ldots, V_n\}$, where $< V_1, V_2, \ldots, V_n > \in \mathcal{P}$ consists of a club filter for X. This gives the converse of Theorem 6.

4. On uK-U* spaces

In this note the next theorem is main novelty. Closed nowhere dense sets are valid for uK-U* properties. Now, it will be convenient for us to use open and dense subsets of $X \times Y$, instead of nowhere dense ones. In the proof of Theorem 7 Player II uses an obvious fact: *If a dense subset $E \subseteq X \times Y$ is open, then for any non-empty open sets U of X and $V_1, V_2, \ldots V_n$ of Y there exist non-empty open sets $U^* \subseteq U$ and $V_1^* \subseteq V_1, V_2^* \subseteq V_2 \ldots V_n^* \subseteq V_n$ such that always $U^* \times V_i^* \subseteq E$.*

Theorem 7. *Suppose X and Y are topological spaces, where Y is I-favorable. If a set $E \subseteq X \times Y$ is open and dense with respect to the product topology, then there exists a meager subset $P \subseteq X$ such that the section*

$$E_x = \{y \in Y : (x, y) \in E\}$$

is dense in Y for all $x \in X \setminus P$.

Proof. If Player I has chosen a finite family $\mathcal{A}_0$ of open and disjoint subsets of Y, then Player II chooses an open set $Q_0 \subseteq X$ and a finite family $\mathcal{B}_0(Q_0)$ of open and disjoint subset of Y such that for each $U \in \mathcal{A}_0$ there exists $V \in \mathcal{B}_0(Q_0)$ with $V \subseteq U$ and $Q_0 \times V \subseteq E$.

Afterwards Player I chooses a finite family $\mathcal{A}_1(Q_0)$ of open and disjoint subsets of Y in accordance with to his winning strategy at the round following after $\mathcal{A}_0$, $\mathcal{B}_0(Q_0)$.

Assume that open sets $X \supseteq Q_0 \supseteq Q_1 \supseteq \ldots \supseteq Q_{n-1}$ and finite families $\mathcal{A}_0, \mathcal{B}_0(Q_0), \mathcal{A}_1(Q_0), \ldots, \mathcal{B}_{n-1}(Q_{n-1}), \mathcal{A}_n(Q_{n-1})$ are defined. Then Player II chooses an open set $Q_n \subseteq Q_{n-1}$ and a finite family $\mathcal{B}_n(Q_n)$ of open and disjoint subset of Y such that for each $U \in \mathcal{A}_n(Q_{n-1})$ there exists $V \in \mathcal{B}_n(Q_n)$ with $V \subseteq U$ and $Q_n \times V \subseteq E$.

Afterwards Player I chooses a finite family $\mathcal{A}_{n+1}(Q_n)$ of open and disjoint subsets of Y in accordance with his winning strategy in the round following after $\mathcal{A}_0, \mathcal{B}_0(Q_0), \ldots, \mathcal{B}_{n-1}(Q_{n-1}), \mathcal{A}_n(Q_{n-1}), \mathcal{B}_n(Q_n)$.

Let $\mathcal{W}_0$ be some maximal family of open and disjoint subsets of X from which Player II could choose at start as sets Q_0. Suppose that families $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_{n-1}$ are defined. Let $\mathcal{W}_n^Q$ be a maximal family of open and disjoint subsets of X which Player II could choose at the round following after $\mathcal{A}_0, \mathcal{B}_0(Q_0), \ldots, \mathcal{B}_{n-1}(Q_{n-1}), \mathcal{A}_n(Q_{n-1})$, where $Q_0 \supseteq Q_1 \supseteq \ldots \supseteq Q_{n-1}$ and $Q_i \in \mathcal{W}_i$, for $0 \leq i \leq n-1$. Put

$$\mathcal{W}_n = \bigcup \{\mathcal{W}_n^Q : Q \in \mathcal{W}_{n-1}\}.$$

By the induction families $\mathcal{W}_0, \mathcal{W}_1, \ldots$ are defined. Any $\bigcup \mathcal{W}_n$ is an open dense subset of X. If always $Q_n \in \mathcal{W}_n$ and $x \in Q_0 \cap Q_1 \cap \ldots$, then any union

$$\bigcup \{\mathcal{B}_k(Q_k) \cup \mathcal{B}_{k+1}(Q_{k+1}) \cup \ldots\}$$

is a dense subset of Y since the winning strategy of I forces moves $\mathcal{B}_0(Q_0), \mathcal{B}_1(Q_1), \ldots$ with a such property. But $V \in \mathcal{B}_n(Q_n)$ implies $Q_n \times V \subseteq E$. Therefore E_x should be dense in Y. Families $\mathcal{W}_n$ are maximal and consists of open sets, so $\bigcup \mathcal{W}_n$ is always open and dense in X. Hence for any

$$x \in \bigcup \mathcal{W}_0 \cap \bigcup \mathcal{W}_1 \cap \ldots$$

the set E_x should be dense in Y. Let $P = X \setminus (\bigcup \mathcal{W}_0 \cap \bigcup \mathcal{W}_1 \cap \ldots)$. $\square$

Apply the above theorem to indicate connections between games and universally Kuratowski-Ulam spaces.

Theorem 8. *Every I-favorable space is uK-U*.*

Proof. Suppose that a space Y is I-favorable, and X is a topological space. If $D \subseteq X \times Y$ is nowhere dense, then it's closure is nowhere dense, too. Apply Theorem 7 with $E = X \times Y \setminus \operatorname{cl} D$. $\square$

Thus, there has been given an argument which suggests that an adequate meaning of universally Kuratowski-Ulam spaces should be uK-U* spaces, compare [4]. There exist non-dyadic and compact spaces which are uK-U*.

Theorem 9. *If a compact space Y is I-favorable, then the hyperspace* $\exp(Y)$ *with the Vietoris topology is uK-U*.*

Proof. The hyperspace $\exp(Y)$ is I-favorable by Theorem 6. So, one could apply Theorem 8. □

Corollary 10. *If $\lambda > \omega_1$, then the hyperspace $\exp(D^\lambda)$ is uK-U* and non-dyadic.*

Proof. For any cardinal $\lambda > \omega_1$ the hyperspace $\exp(D^\lambda)$ is non-dyadic, by [9]. The Cantor cube D^λ is I-favorable and hence $\exp(D^\lambda)$ is I-favorable by Theorem 6. Theorem 9 implies that $\exp(D^\lambda)$ is uK-U*. □

5. Final remarks

In [10, Theorem 1] L. Shapiro showed that any dyadic space is co-absolute with a finite disjoint union of Cantor cubes or is co-absolute with the one point compactification of countable many Cantor cubes. Therefore, any dyadic space is co-absolute with some I-favorable space. One can check this using the definition of I-favorable space. So, one can reprove [2, Theorem 1.11] using [2, Fact 1.3]. In other words, any dyadic space is I-favorable since it is co-absolute with a I-favorable space. This and Theorem 8 give a proof that dyadic spaces are universally Kuratowski-Ulam. We have reproved Corollary 3 from [3]. Similarly, by Theorem 8, and Corollary 5.5.5 [5], and Proposition 5.5.6 [5] one obtains that any space which is co-absolute with a κ-metrizable space is uK-U*, compare [11], [5, p. 44]. However, we do not know: *Does there exist a compact universally Kuratowski-Ulam space which is not I-favorable*?

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