SPLITTING NATURAL INJECTION OF WITT RINGS OF GEOMETRIC RINGS

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Abstract. For a smooth real affine curve γ that is semi-algebraically connected and semi-algebraically compact we show that the Witt group WP of the ring P of polynomial functions is a direct summand of the Witt group WR of the ring R of regular functions on γ. In particular, this means that the canonical injection $WP \hookrightarrow WK$ splits, where K is the field of rational functions on γ.

1. Preliminaries

The notion of a Witt ring of symmetric bilinear forms plays a central role in the algebraic theory of quadratic forms. If the base ring is a Dedekind domain, then its Witt ring can be embedded into the Witt ring of its field of fractions (see [7, Corollary IV.3.3]). In general this injection does not split. There are, however, cases when it does. For algebraic integers appropriate criteria are described in [8], those were later used in [2] to describe the structure of the Witt ring of algebraic integers. In our previous paper [5] we show that the natural injection of the Witt ring of the ring of regular functions on a real curve into the Witt ring of the field of rational functions on that curve always splits. The aim of this paper is to show that for a smooth, real affine curve that is semi-algebraically compact and semi-algebraically connected the canonical injection of the Witt ring of the ring of polynomial functions into the Witt ring of the ring of regular functions splits, as well (see Theorem 3).

Throughout this paper $\Bbbk$ denotes a fixed real closed field. Let K be a formally real algebraic function field over $\Bbbk$. The set of all points of K trivial on $\Bbbk$ is a smooth algebraic curve, denote it C. Among all the points of C we consider those that have a formally real residue field (hence, in fact, equal to $\Bbbk$), following [3] we call such point *real*. The set of all real points is denoted by γ. Since K is assumed to be formally real, clearly $\gamma \neq \emptyset$. It is a *real* algebraic curve over $\Bbbk$. We may view γ as a subset of the projective plane $\mathbb{P}^2\Bbbk$. Fix an affine plane $\mathbb{A}^2\Bbbk \subset \mathbb{P}^2\Bbbk$. Two subrings of K are naturally associated to this curve. The first one is a ring R of regular functions on γ. If $\mathcal{O}_\mathfrak{p}$ denotes a valuation ring associated with $\mathfrak{p}$, then

$$R = \bigcap_{\mathfrak{p}\in\gamma} \mathcal{O}_\mathfrak{p}.$$

The other one is the ring of polynomial functions

$$P = \bigcap_{\substack{\mathfrak{p}\in C\\ \mathfrak{p}\ \text{finite}}} \mathcal{O}_\mathfrak{p}.$$

Both R and P are Dedekind domains. For a point $\mathfrak{p} \in C$ denote $K(\mathfrak{p})$ its residue field. If $\mathfrak{p}$ is real, then $K(\mathfrak{p}) = \Bbbk$, hence its Witt ring $WK(\mathfrak{p})$ is isomorphic to $\mathbb{Z}$. Otherwise, if $\mathfrak{p}$ is not real, then $K(\mathfrak{p}) = \Bbbk(\sqrt{-1})$ and $WK(\mathfrak{p}) \cong \mathbb{Z}/2\mathbb{Z}$. For every point $\mathfrak{p} \in \gamma$, respectively every finite point $\mathfrak{p} \in C$, consider the second residue homomorphism $\partial_\mathfrak{p} : WK \to WK(\mathfrak{p})$ associated to $\mathfrak{p}$. This way we construct two maps:

$$\partial_R : WK \to \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \quad \text{and} \quad \partial_P : WK \to \bigoplus_{\mathfrak{p} \text{ finite}} WK(\mathfrak{p}).$$

The curve γ is a disjoint union of finitely many semi-algebraically connected components $\gamma_1, \dots, \gamma_N$ (see [3]). The letter N will always denote the number of those components. We have another two maps

$$\lambda_R : \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \to \mathbb{Z}^N \quad \text{and} \quad \lambda_P : \bigoplus_{\mathfrak{p} \text{ finite}} WK(\mathfrak{p}) \to \mathbb{Z}^N.$$

assigning to the sequence $(n_\mathfrak{p})_{\mathfrak{p}\in\gamma}$ (respectively $(n_\mathfrak{p})_{\mathfrak{p} \text{ finite}}$) the N-tuple $(n_1, \dots, n_N)$ where $n_i = \sum_{\mathfrak{p}\in\gamma_i} n_\mathfrak{p}$ (see [4]). Using those maps we may express WP and WR in terms of exact sequences (called Knebusch-Milnor exact sequences):

Theorem 1. *The following two sequences are exact:*

$$0 \to WR \xrightarrow{i_R} WK \xrightarrow{\partial_R} \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \xrightarrow{\lambda_R} \mathbb{Z}^N \to 0$$

$$0 \to WP \xrightarrow{i_P} WK \xrightarrow{\partial_P} \bigoplus_{\mathfrak{p} \text{ finite}} WK(\mathfrak{p}) \xrightarrow{\lambda_R} \mathbb{Z}^N \to 0.$$

For the proof consult [4, Theorem 11.2], [7, Corollary IV.3.3] (see also [5, Lemma 3.1]). From now on we identify the rings WR, WP with their images in WK. Let $\mathbb{E}(R), \mathbb{E}(P) < \dot{K}/\dot{K}^2$ be two subgroups of the square class group of K consisting of those classes that have an even valuation in every point of γ (respectively, in every finite point of C):

$$\mathbb{E}(R) := \{f \in \dot{K}/\dot{K}^2 : \operatorname{ord}_\mathfrak{p} f \equiv 0 \pmod 2 \text{ for all } \mathfrak{p} \in \gamma\}$$
$$\mathbb{E}(P) := \{f \in \dot{K}/\dot{K}^2 : \operatorname{ord}_\mathfrak{p} f \equiv 0 \pmod 2 \text{ for all finite } \mathfrak{p} \in C\}.$$

The following corollary is an immediate consequence of exactness of Knebusch-Milnor sequences (see also [2, Proposition 3.1] and [5, Corollary 3.3]):

Corollary 2. *The following hold:*

1. $\langle f\rangle \in WR \iff f \in \mathbb{E}(R)$;
2. $\langle f\rangle \in WP \iff f \in \mathbb{E}(P)$;
3. $\varphi \in WR \implies \operatorname{dis}\varphi \in \mathbb{E}(R)$;
4. $\varphi \in WP \implies \operatorname{dis}\varphi \in \mathbb{E}(P)$.

It is known that the Knebusch-Milnor exact sequence for WR "splits" (see [5, Theorem 5.1]), in particular the injection $WR \hookrightarrow WK$ splits in a usuall sense (see [5, Theorem 3.9]). If C has no real points at infinity (i.e. γ is affine and semi-algebraically compact) then every polynomial function is regular in any real point, hence $P \subset R$. It is thus natural to ask if also $WP \hookrightarrow WR$ splits. In our main theorem (c.f. Theorem 3) we show that it is the case when γ is semi-algebraically connected (i.e. $N = 1$).

In this paper we use the standard notation and terminology for quadratic forms as in [6]. We also use the notion of: a topology on γ, an orientation and open intervals of a component as they were introduced in [3, 4]. We assume that an orientation of γ is arbitrarily chosen and fixed throughout this paper.

Moreover, to simplify the notation, we use the same symbol f to denote both an element of a field and a square class containing this element. Analogously, $\langle f_1, \dots, f_n\rangle$ is (depending on the context) either a quadratic form or its class in the Witt's ring.

2. Main result

Theorem 3. *Using the above notation, if the curve C has no real points at infinity (i.e. γ is affine semi-algebraically compact) and its real part γ is semi-algebraically connected, then the Witt group WP of the ring P of polynomial functions is a direct summand of the Witt group WR of the ring of regular functions. In particular WP is a direct summand of WK, as well.*

The "in particular" part of the above theorem follows from the main part by the means of [5, Theorem 3.9], but in fact we will prove it directly in the proceeding of the proof of the above theorem. Let us explicitly notice one immediate consequence of semi-algebraic connectedness of γ.

Observation 4. *Under the above assumptions $\mathbb{E}(R) = \pm\sum \dot{K}^2$.*

Our first step toward the proof is to define a group S. The group $\mathbb{E}(P)$ is a subgroup of $\mathbb{E}(R)$. Now, since the rank of every element of $\mathbb{E}(R) \subset \dot{K}/\dot{K}^2$ equals 2, $\mathbb{E}(P)$ is a direct summand of $\mathbb{E}(R)$. Let for example $\mathbb{E}(R) = \mathbb{E}(P) \oplus E'$. The square class of -1 belongs to $\mathbb{E}(P)$, hence without loss of generality we may assume that $E' \subset \sum \dot{K}^2$. Define S to be the subgroup of the Witt group WK generated by the set $\{\langle -1, f\rangle : f \in E'\}$.

Lemma 5. *The following hold*

(1) *$S \subset WR$;*
(2) *$2S \subset \{0\}$;*
(3) *S contains no classes of 1-dimensional forms.*

Proof. The first assertion follows immediately from Corollary 2(1). Further (3) follows from (2). Therefore we need only to prove (2). It is enough to show that every generator of S is of rank two. Take $f \in E' \subset \sum \dot{K}^2$. Now $\langle f, f\rangle = \langle 1, 1\rangle$ and so $\langle -1, f\rangle + \langle -1, f\rangle = 0$ in WK. Since WR injects into WK, thus the sum is null also in WR. □

Lemma 6. *$WP \cap S = \{0\}$.*

Proof. Suppose otherwise, let φ be a non-hyperbolic form over K such that its class belongs to the intersection $WP \cap S$. Since, in particular $\varphi \in S$ we write

$$\varphi = \pm\langle -1, f_1\rangle \pm \dots \pm \langle -1, fn\rangle.$$

Using Lemma 5(2), without loss of generality we may assume that $f_1, \dots, f_n$ are pair-wise distinct in $\dot{K}/\dot{K}^2$. Compute the discriminant of φ:

$$\operatorname{dis}\varphi = f_1 \cdots f_n,$$

hence $\operatorname{dis}\varphi \in E'$. On the other hand, Corollary 2(4) implies that $\operatorname{dis}\varphi \in \mathbb{E}(P)$. Therefore, $\operatorname{dis}\varphi \in \mathbb{E}(P) \cap E' = \{1\}$. But this is impossible as all f_i's were pair-wise distinct. □

It follows from the above two lemmas that $WP \oplus S$ is a subgroup of WR. We need to prove that it is a whole group WR. We shall first show that the classes of forms of dimensions ≤ 2 over K that belong to WR are in this direct sum (see propositions 7 and 8).

Proposition 7. *For every $f \in \mathbb{E}(R)$, $\langle f \rangle \in WP \oplus S$.*

Proof. Take $f \in \mathbb{E}(R) = \pm \sum \dot{K}^2$. First assume that $f \in -\sum \dot{K}^2$. Since $\mathbb{E}(R) = \mathbb{E}(P) \oplus E'$, therefore there exist unique $e \in \mathbb{E}(P)$ and $e' \in E'$ such that $e \cdot e' = f$. Now, $e' \in E' \subset \sum \dot{K}^2$ hence $e = f \cdot e' \in -\sum \dot{K}^2$. By the Witt theorem (see [4, Theorem 9.5]) the form $\langle e, e' \rangle$ represents 1 over K. Consequently $\langle e, e' \rangle = \langle 1, ee' \rangle = \langle 1, f \rangle$. and so

$$\langle f \rangle = \langle e \rangle + \langle -1, e' \rangle \in WP \oplus S.$$

Now assume that $f \in \sum \dot{K}^2$. By the previous part $\langle -f \rangle$ belongs to $WP \oplus S$ and so does $\langle f \rangle$. □

Proposition 8. *For every $f, g \in \dot{K}/\dot{K}^2$, if $\langle f, g \rangle \in WR$ then $\langle f, g \rangle \in WP \oplus S$.*

Proof. Assume that $\langle f, g \rangle \in WR$, therefore $fg \in \pm \sum \dot{K}^2$ by Corollary 2(3) and Observation 4. First suppose that $H := fg \in \sum \dot{K}^2$. We claim that f may not change a sign at any point of γ. Indeed, assume that f changes sign at $\mathfrak{p}$, then $\partial_{\mathfrak{p}} \langle f, fh \rangle = \langle 1, 1 \rangle \neq 0$. Thus $\langle f, g \rangle = \langle f, fh \rangle \notin \ker \partial_R = WR$ and this is a contradiction. Now, since γ is semi-algebraically connected, $f \in \pm \sum \dot{K}^2 = \mathbb{E}(R)$. Using Proposition 7 we show that $\langle f \rangle, \langle fh \rangle \in WP \oplus S$. Thus also $\langle f, g \rangle = \langle f \rangle + \langle fh \rangle \in WP \oplus S$.

On the other hand suppose that $fg \in -\sum \dot{K}^2$. Then $\operatorname{sgn}_{\mathfrak{p}} f = -\operatorname{sgn}_{\mathfrak{p}} g$ for every real point $\mathfrak{p}$. Hence, using the Witt theorem (see [4, Theorem 9.5]) we show that the forms $\langle f, g \rangle$ and $\langle 1, fg \rangle$ are isometric. Thus $\langle 1, fg \rangle \in WR$ and so $\langle fg \rangle \in WR$. Now, Proposition 7 implies that $\langle fg \rangle \in WP \oplus S$ and consequently $\langle f, g \rangle = \langle 1 \rangle + \langle fg \rangle \in WP \oplus S$. □

Now, when we are done with unary and binary forms, we may turn our attention to forms of higher dimensions. Those, however, require a more general approach. Fix a real point $\mathfrak{p}_0 \in \gamma$. For every $\mathfrak{p} \in \gamma \setminus \{\mathfrak{p}_0\}$ let $\chi_{(\mathfrak{p}_0, \mathfrak{p})}$ be a fixed function nonpositive on $(\mathfrak{p}_0, \mathfrak{p})$ and nonnegative everywhere else. Such a function is unique up to a multiplication by a sum of squares. Let further $\chi_{(\mathfrak{p}, \mathfrak{p}_0)} := -\chi_{(\mathfrak{p}_0, \mathfrak{p})}$. Take the subgroup D of WK generated by the set $\{\langle \chi_{(\mathfrak{p}_0, \mathfrak{p})} \rangle : \mathfrak{p} \neq \mathfrak{p}_0\}$. By [5, Theorem 3.9] we have a decomposition $WK = WR \oplus D$. Hence, if we show that $WP \oplus S \oplus D = WK$ it will follow that $WR = WP \oplus S$. The following two lemmas are analogous to [5, Corollary 4.3–Lemma 4.9].

Lemma 9. *Let $f, g \in \dot{K}/\dot{K}^2$ be two square classes of K such that for almost every point $\mathfrak{p} \in \gamma$ either $f(\mathfrak{p}) > 0$ or $g(\mathfrak{p}) > 0$. If $\langle f \rangle, \langle g \rangle \in WP \oplus S \oplus D$, then also $\langle fg \rangle \in WP \oplus S \oplus D$.*

Proof. Forms $\langle f, g \rangle$ and $\langle 1, fg \rangle$ are isometric by the Witt theorem (see [4, Theorem 9.5]), hence $\langle fg \rangle = \langle f \rangle + \langle g \rangle + \langle -1 \rangle \in WP \oplus S \oplus D$. □

Lemma 10. *Let $f \in \dot{K}/\dot{K}^2$ be a square class. Assume that f changes sign at exactly two points of γ. Then $\langle f\rangle \in WP \oplus S \oplus D$.*

Proof. First suppose that f changes sign at $\mathfrak{p}_0$. Therefore, there exists a sum of squares $g \in \sum \dot{K}^2$ such that $f = \pm\chi_{(\mathfrak{p}_0,\mathfrak{p})}\cdot g$. Now, by the definition $\langle\chi_{(\mathfrak{p}_0,\mathfrak{p})}\rangle \in D \subset WP \oplus S \oplus D$, on the other hand $\langle g\rangle \in WP \oplus S \subset WP \oplus S \oplus D$ by Proposition 7. Consequently, the previous lemma implies that $\langle f\rangle \in WP \oplus S \oplus D$.

Thus, suppose that f does not change a sign at $\mathfrak{p}_0$. Let $\mathfrak{p}, \mathfrak{q}$ be the two points where f changes sign. The curve γ is homeomorphic to a circle, hence without loss of generality we may assume that $\mathfrak{p}_0 \in (\mathfrak{p}, \mathfrak{q})$. We may once again write $f = \pm\chi_{(\mathfrak{p}_0,\mathfrak{p})} \cdot \chi_{(\mathfrak{p}_0,\mathfrak{q})} \cdot g$ for some sum of squares g. In a similar fashion as above, using the previous lemma twice, we show that $\langle f\rangle \in WP \oplus S \oplus D$. □

We are now ready to prove our main theorem.

Proof of Theorem 3. Take any square class $f \in \dot{K}/\dot{K}^2$ of K. We will show that $\langle f\rangle \in WP \oplus S \oplus D$ by induction on s being the half of the number of sign changes of f. If $s = 0$ then $f \in \pm\sum \dot{K}^2$ and so $\langle f\rangle \in WP \oplus S \subset WP \oplus S \oplus D$ by Proposition 7. Is $s = 1$ then $\langle f\rangle \in WP \oplus S \oplus D$ by Lemma 10. Assume that we have proved the assertion for every integer less than s. Take $\mathfrak{p}, \mathfrak{q}$ to be two consecutive points where f changes sign (i.e. f changes sign at $\mathfrak{p}$ and $\mathfrak{q}$ but not inbetween). Without loss of generality we may assume that $\operatorname{sgn}_{\mathfrak{r}} f = -1$ for $\mathfrak{r} \in (\mathfrak{p}, \mathfrak{q})$. Take now $\tilde{f} := f\cdot\chi_{(\mathfrak{p},\mathfrak{q})}$. Then $\tilde{f}$ changes sign at $2s-2$ point (i.e. in all the points where f does except $\mathfrak{p}$ and $\mathfrak{q}$), hence by the induction hypothesis $\langle\tilde{f}\rangle \in WP \oplus S \oplus D$. Moreover $\langle\chi_{(\mathfrak{p},\mathfrak{q})}\rangle \in WP \oplus S \oplus D$ by Lemma 10. Thus $\langle f\rangle = \langle\tilde{f} \cdot \chi_{(\mathfrak{p},\mathfrak{q})}\rangle \in WP \oplus S \oplus D$ by Lemma 9.

We have thus shown that class of every unary form over K belongs to the direct sum $WP \oplus S \oplus D$, but the Witt group of a field is generated by classes of unary forms, hence $WK = WP \oplus S \oplus D$. Now since $WK = WR \oplus D$ (see [5, Theorem 3.9]) and $WP \oplus S \subset WR$ we have $WP \oplus S = WR$. □

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