Witt equivalence of rings of regular functions

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Abstract

In this paper we show that the rings of regular functions on two real algebraic functions fields over the same real closed field are Witt equivalent (i.e. their Witt rings are isomorphic) if and only if the curves have the same number of semi-algebraically connected components. Moreover, in the second part of the paper, we prove that every strong isomorphism of Witt rings of rings of regular functions can be extended to an isomorphism of Witt rings of fields of rational functions. This extension is not unique, in general.

The set of similarity classes of nonsingular bilinear forms over a fixed commutative ring A, equipped with operations induced by the orthogonal sum and the tensor product, has a natural structure of a ring. This ring is called the *Witt ring* WA of A. The Witt ring encodes numerous information of its ground ring. Unfortunately the complete theory of Witt rings is known only over fields (c.f. [8, 13]). The theory of Witt rings over integral domains has been intensively developed since 1970s by many authors (see e.g. [9]). This case is far more challenging than the previous one. So far the most progress has been done for Dedekind domains—hence of dimension one. The ultimate question in algebraic theory of quadratic forms is: when the Witt rings of two rings A and B are isomorphic? If this is the case we say that rings A, B are *Witt equivalent*. This problem is difficult even over fields and has been investigated in more than 40 scientific papers. So far it has been solved only in a very few cases. The three main are: fields having no more than 32 square classes (see [2]), global fields — this area has been most actively investigated in previous years (see e.g. [10, 15, 14]) and fields of rational functions on algebraic curves (see [6, 5, 7]). The pursue for criteria of Witt equivalence of rings has started only recently (see e.g. [11, 12]).

In this paper we cope with the problem of Witt equivalence of rings of regular functions on two smooth complete real curves. We prove that (see Theorem 3.1) two such rings are Witt equivalent if and only if the underlying curves consist of equal numbers of semi-algebraically connected components. If this is the case, the isomorphism of the Witt rings of rings of regular functions is obtained by restricting the tame Harrison map (see [6] for the definition) of their fields of fractions. Moreover we show (see Theorem 3.3) that every *strong* isomorphism of Witt rings of two rings of regular functions can be extended (in a non-unique way) to an isomorphism of Witt rings of fields of rational functions. The proof of the latter result occupies the subsection 3.1. The paper is organized as follows. In Section 1 we introduce all the necessary terminology and gather the needed tools. In Section 2 we present a number of results concerning the structure of the Witt ring of the ring of regular functions on a real curve. These results can hardly be considered new, since the similar ones (but formulated for a coarser object) may be found in [4]. Hence, we omits most of the proofs in this section. Finally, Section 3 constitutes the kernel of this paper and is completely devoted to our main results mentioned above.

1 Preliminaries

Let $\Bbbk$ be a real closed field. It will silently remain fixed throughout this whole paper. The letters K, L will always denote the formally real algebraic function fields over $\Bbbk$. Let $\Omega(K)$ be a set of all points of K trivial on $\Bbbk$. The completion of K with respect to a point $\mathfrak{p} \in \Omega(K)$ is denoted by $K_\mathfrak{p}$ while its residue field by $K(\mathfrak{p})$. The associated valuation is denoted by $\operatorname{ord}_\mathfrak{p}^K$. Among all the points $\mathfrak{p} \in \Omega(K)$ we select those having the formally real (hence isomorphic to $\Bbbk$) residue fields. Following [3, 4] we call such points *real* and we write $\gamma(K)$ for the set of all the real points of K. It is a real algebraic curve over $\Bbbk$. The field K can be treated as the field of rational functions on this curve.

On the curve $\gamma(K)$ we consider Euclidean topology (see [1]) induced by the ordering of $\Bbbk$ (note that in [3, 4] this topology is called "strong topology"). The curve $\gamma(K)$ consists of a finite number of semi-algebraically connected components $\gamma_1^K, \dots, \gamma_N^K$. With every real point $\mathfrak{p} \in \gamma(K)$ we associate two orderings $P_+(\mathfrak{p})$ and $P_-(\mathfrak{p})$ of the field K compatible with $\mathfrak{p}$:

$$P_+(\mathfrak{p}) = \Big\{ f \in K : \bigvee_{\mathfrak{p}' \in \gamma(K)} \bigwedge_{\mathfrak{q} \in (\mathfrak{p}, \mathfrak{p}')} f(\mathfrak{q}) > 0 \Big\}$$
$$P_-(\mathfrak{p}) = \Big\{ f \in K : \bigvee_{\mathfrak{p}' \in \gamma(K)} \bigwedge_{\mathfrak{q} \in (\mathfrak{p}', \mathfrak{p})} f(\mathfrak{q}) > 0 \Big\}$$

(Note that the left/right neighborhoods $(\mathfrak{p}', \mathfrak{p})$, $(\mathfrak{p}, \mathfrak{p}')$ are relative to an orientation of the curve $\gamma(K)$, which we assume to be fixed.) This permits us o introduce the notion of a signature of a square class. Namely for any square class $f \in \dot{K}/\dot{K}^2$ (to simplify the notation we use the same symbol f to denote both an element of the field and its square class) we define:

$$\operatorname{sgn}^K_{\mathfrak{p}} f := \begin{cases} 1, & \text{if } f \in P_+(\mathfrak{p}) \cap P_-(\mathfrak{p}),\\ 0, & \text{if either } f \in P_+(\mathfrak{p}) \cap \bigl(-P_-(\mathfrak{p})\bigr) \text{ or } f \in \bigl(-P_+(\mathfrak{p})\bigr) \cap P_-(\mathfrak{p}),\\ -1, & \text{if } -f \in P_+(\mathfrak{p}) \cap P_-(\mathfrak{p}). \end{cases}$$

(In [3, 4] this function is denoted by $\tau_{\mathfrak{p}}$.) We have an immediate observation which we formulate explicitly for future references.

Observation 1.1. $\operatorname{sgn}^K_{\mathfrak{p}} f = 0$ *if and only if* $\operatorname{ord}^K_{\mathfrak{p}} f \equiv 1 \pmod 2$.

Now let $R_K := \{f \in K : \operatorname{ord}^K_{\mathfrak{p}} f \geq 0 \text{ for every } \mathfrak{p} \in \gamma(K)\}$ be the subring of the field K consisting of all the functions regular on $\gamma(K)$. It is a Dedekind domain, hence its Witt ring WR_K injects into the Witt ring WK of its field of fractions (see [9, Corollary IV.3.3]). In fact we know that the following result is true:

Theorem 1.2 ([4, Theorem 11.2])**.** *If the curve $\gamma(K)$ consists of N semi-algebraically connected components $\gamma^K_1, \dots, \gamma^K_N$, then the following sequence is exact:*

$$0 \to WR \xrightarrow{i_K} WK \xrightarrow{\partial_K} \bigoplus_{\mathfrak{p}\in\gamma(K)} WK(\mathfrak{p}) \xrightarrow{\lambda_K} \mathbb{Z}^N \to 0.$$

Here $i_K : WR_K \hookrightarrow WK$ is the canonical injection induced by the inclusion $R_K \subset K$. Next, for every $\mathfrak{p} \in \gamma(K)$ the map $\partial_{\mathfrak{p}} : WK \to WK(\mathfrak{p}) \cong \mathbb{Z}$ is the second residue homomorphism and ∂_K denotes the compound map. Finally $\bigoplus WK(\mathfrak{p}) \cong \mathbb{Z}^{(\gamma(K))}$ and $\lambda_K : \mathbb{Z}^{(\gamma(K))} \to \mathbb{Z}^N$ is the epimorphism defined by

$$\lambda_K\bigl((a_{\mathfrak{p}})_{\mathfrak{p}\in\gamma(K)}\bigr) := \Bigl(\sum_{\mathfrak{p}\in\gamma^K_1} a_{\mathfrak{p}}, \dots, \sum_{\mathfrak{p}\in\gamma^K_N} a_{\mathfrak{p}}\Bigr).$$

In the rest of this paper we tend to identify WR_K with its image under the canonical injection $WR_K \hookrightarrow WK$ and so we conveniently treat WR_K as the subring of WK. Define the subgroup $\mathbb{E}_K < \dot{K}/\dot{K}^2$ by

$$\mathbb{E}_K := \bigl\{f \in \dot{K}/\dot{K}^2 : \operatorname{ord}^K_{\mathfrak{p}} f \equiv 0 \pmod 2 \text{ for all } \mathfrak{p} \in \gamma(K)\bigr\}.$$

Observation 1.1 allows us to rewrite this condition in the following way:

$$\mathbb{E}_K := \bigl\{f \in \dot{K}/\dot{K}^2 : \operatorname{sgn}^K_{\mathfrak{p}} f \neq 0 \text{ for all } \mathfrak{p} \in \gamma(K)\bigr\}.$$

Consequently, the signature of the square class f belonging to $\mathbb{E}_K$ is constant on every semi-algebraically connected component γ^K_i. So we denote it by $\operatorname{sgn}^K_i f$ (see also Proposition 2.1 below). Now, by Theorem 1.2, a unary form $\langle f\rangle$ lies in WR_K if and only if it belongs to the kernel of ∂_K and this is the case if and only if f is the class of a unit at the completion $K_{\mathfrak{p}}$ for every $\mathfrak{p} \in \gamma(K)$. This, in turn, means that f belongs to $\mathbb{E}_K$. So we have:

Corollary 1.3. *Let $f \in \dot{K}/\dot{K}^2$ be a square class, then $f \in \mathbb{E}_K$ if and only if the unary form $\langle f \rangle$ belongs to WR_K, i.e.*

$$f \in \mathbb{E}_K \iff \langle f \rangle \in WR_K.$$

The above corollary suggests that it may be fruitful to investigate the subset of WR_K consisting of all the classes of unary forms. Thus, we define

$$\langle \mathbb{E}_K \rangle := \big\{ \langle f \rangle \in WK : f \in \mathbb{E}_K \big\} \subset WR_K.$$

Notice that $\langle \mathbb{E}_K \rangle$ is closed under multiplication but not under addition hence it is not a subgroup of the "Witt group" WR_K.

Now, fix a single point $\mathfrak{p}_i$ in every component γ_i^K of $\gamma(K)$. Recall (c.f. [3, §6]) that for every two distinct points $\mathfrak{p}$, $\mathfrak{q}$ belonging to the same component γ_i^K there exists an element $\chi_{(\mathfrak{p},\mathfrak{q})}$ of K such that $\chi_{(\mathfrak{p},\mathfrak{q})}$ is definite on $\gamma \setminus \{\mathfrak{p},\mathfrak{q}\}$, positive definite on $\gamma \setminus \gamma_i$ and fulfills $\partial_\mathfrak{p}\chi = -1$, $\partial_\mathfrak{q}\chi = 1$. Following [3] we call χ an *interval function* for the pair $(\mathfrak{p},\mathfrak{q})$. An interval function is unique only upto multiplication by a totally positive element (see [3, §6]). Hence, in what follows, for every $\mathfrak{p}, \mathfrak{p}_i \in \gamma_i^K$, we assume that $\chi_{(\mathfrak{p},\mathfrak{p}_i)}$ is an arbitrarily chosen and fixed interval function associated with the pair $(\mathfrak{p},\mathfrak{p}_i)$. The group of square classes $\dot{K}/\dot{K}^2$ may be treated as a $\mathbb{F}_2$-vector space. The subgroup $\mathbb{E}_K$ is its subspace. We identify its completion:

Lemma 1.4. *The $\mathbb{F}_2$-vector space $\dot{K}/\dot{K}^2$ decomposes into*

$$\dot{K}/\dot{K}^2 = \mathbb{E}_K \oplus \operatorname{lin}_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{K}/\dot{K}^2 : \mathfrak{p} \in \gamma_i^K, 1 \le i \le N\}.$$

Proof. Take any square class $f \in \dot{K}/\dot{K}^2$ and let

$$\mathfrak{p}_{1,1}, \dots, \mathfrak{p}_{1,n_1}, \mathfrak{p}_{2,1}, \dots, \mathfrak{p}_{2,n_2}, \dots, \mathfrak{p}_{N,n_N}, \quad \mathfrak{p}_{i,j} \in \gamma_i^K$$

be all the points of $\gamma(K)$ where f has an odd valuation and so changes sign. Consider now a square class $\hat{f} \in \operatorname{lin}_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{K}/\dot{K}^2 : \mathfrak{p} \in \gamma_i^K, 1 \le i \le N\}$ given by the condition

$$\hat{f} := \prod_{\substack{1 \le i \le N \\ 1 \le j \le n_i \\ \mathfrak{p}_{i,j} \ne \mathfrak{p}_i}} \chi_{(\mathfrak{p}_{i,j},\mathfrak{p})}.$$

Then $\hat{f}$ changes sign precisely at the same points as f does. Hence the product $f \cdot \hat{f}$ has the constant sign on every semi-algebraically connected component. Thus $f \cdot \hat{f} \in \mathbb{E}_K$. □

Of course the notion of the notion of a real curve, semi-algebraically connected components, local signatures, ...etc. can be—in the same manner as above—defined over L. Therefore we denote $\Omega(L)$, $\gamma(L)$, γ_i^L, $L_\mathfrak{p}$, $L(\mathfrak{p})$, $\operatorname{ord}_\mathfrak{p}^L$, $\operatorname{sgn}_\mathfrak{p}^L$, WR_L, $\mathbb{E}_L$ to be the L-counterparts of the objects defined above for K. If it is clear from the context which field we discuss, we tend to omit the letters K and L. All the terminology and notation used in this paper and not introduced so far is standard and follows the convention established by [1, 8, 3, 4]. As it was mentioned before, in order to simplify the notation, we use the same symbol to denote both an element of the field and its square class. Likewise, we use the same symbol for a quadratic form and its Witt class. Throughout all this paper an orientation of $\gamma(K)$ (resp. $\gamma(L)$) is arbitrarily chosen and fixed. Intervals on both curves are silently defined with respect to this fixed orientation.

2 Structure of the Witt group WR

We discuss here the structure of the Witt group WR of the ring R of regular functions on γ. All the results presented in this section are fully analogous to the ones presented in [4]. In [4], however, they were formulated and proved for a coarser group. Thus, we feel obligated to state all of the results explicitly in our different set-up. On the other hand, since the proofs are completely analogous, we feel free to reduce some of them to only short sketches and to omit the rest, giving instead the references to the original theorems.

Proposition 2.1 ([4, Proposition 10.3]). *Let φ be an element of the Witt ring WK. If $\varphi \in WR$, then φ has a constant signature on every semi-algebraically connected component of γ.*

For φ denote by $\operatorname{sgn}_i \varphi$ the signature of φ on γ_i ($1 \le i \le N$).

Proposition 2.2 ([4, Theorem 10.4 (i)]). *Every element φ of WR is uniquely determined by its discriminant $\operatorname{disc}\varphi$ and its signatures $\operatorname{sgn}_1, \dots, \operatorname{sgn}_N$ on the components $\gamma_1, \dots, \gamma_N$ of γ.*

Consider now a subset $S \subseteq WR$ defined

$$S := \{\langle 1, -f\rangle : f \in \Sigma K^2\}.$$

Clearly S is a subgroup of WR. Moreover, $2 \cdot S = \{0\}$. We claim that S is the nil-radical of WR.

Proposition 2.3 ([4, Theorem 10.4 (ii)]). $S = \operatorname{Nil} WR$.

Sketch of the proof. The inclusion $S \subseteq \operatorname{Nil} WR$ follows from the fact that:

$$\operatorname{Nil} WR = WR \cap \operatorname{Nil} WK = \{\varphi \in WR : \operatorname{sgn}_i = 0 \text{ for } 1 \le i \le N\}.$$

As for the other inclusion, take any $\varphi \in \operatorname{Nil} WR$. Let $\langle f_1, \dots, f_n\rangle$ be a diagonalization of φ over the field K. Let further f be the discriminant of φ. For every $1 \le i \le N$ we have $\operatorname{sgn}_i \varphi = 0$ and hence at almost every point $\mathfrak{p} \in \gamma$ exactly half of f_j's are negative, the other half is positive. Consequently, $\operatorname{sgn}_{\mathfrak{p}} f = 1$ at every $\mathfrak{p} \in \gamma$. Thus the Witt classes (in WR) of φ and $\langle 1, -f\rangle$ are equal, but clearly $\langle 1, -f\rangle \in S$. □

Fix now $z_1, \dots, z_N \in \mathbb{E}$ such that $\operatorname{sgn}_i z_i = -1$ and $\operatorname{sgn}_j z_i = 1$ for $j \neq i$.

Proposition 2.4 ([4, Theorem 10.4 (iii)]). *The fundamental ideal $IR \lhd WR$ has a decomposition*

$$IR = \bigoplus_{i=1}^{N} \mathbb{Z}[\langle 1, -z_i\rangle] \oplus \operatorname{Nil} WR.$$

Sketch of the proof. Take any $\varphi \in IR$. If $\operatorname{sgn}_i \varphi = 0$ for all $1 \leq i \leq N$ then $\varphi \in \operatorname{Nil} WR$. Thus, assume that not all of $\operatorname{sgn}_i \varphi$ are null. We have $\operatorname{sgn}_i \varphi \equiv \dim \varphi \pmod 2$ and so $\operatorname{sgn}_i \varphi$ is even for $1 \leq i \leq N$. Take now $\hat\varphi_1, \dots, \hat\varphi_N$ defined by the formula

$$\hat\varphi_i := (\operatorname{sgn}_i \varphi / 2) \cdot \langle 1, -z_i \rangle.$$

Let further $\hat\psi := \hat\varphi_1 + \cdots + \hat\varphi_N$. For almost every $\mathfrak{p} \in \gamma_i$, if $\mathfrak{p} \in \gamma_i$, then we have

$$\operatorname{sgn}_{\mathfrak{p}} \operatorname{disc} \varphi = (-1)^{\operatorname{sgn}_i \varphi} \operatorname{sgn}_{\mathfrak{p}} \operatorname{disc} \hat\psi.$$

Therefore $\operatorname{disc} \varphi = g \cdot \operatorname{disc} \hat\psi$, for some $g \in \Sigma K^2$. Thus, by the Witt theorem we get

$$\varphi = \hat\psi + \langle 1, -g \rangle \in \bigoplus_{i=1}^{N} \mathbb{Z}[\langle 1, -z_i\rangle] \oplus \operatorname{Nil} WR. \qquad \square$$

The above proposition allows us to write down the structure of WR explicitely. We us the following result in the next section.

Corollary 2.5. *The Witt group WR has the decomposition:*

$$WR = \mathbb{Z}[\langle 1 \rangle] \oplus \bigoplus_{i=1}^{N} \mathbb{Z}[\langle 1, -z_i\rangle] \oplus \operatorname{Nil} WR.$$

Corollary 2.6. *The Witt ring WR is generated (as a ring) by the set $\{\langle f \rangle : f \in \mathbb{E}\}$.*

3 Main results

We are now ready to state our main result. Consider two formally real algebraic function fields K, L both having the same real closed field of constants $\Bbbk$. The associated curves will be denoted $\gamma(K)$, $\gamma(L)$ and the rings of regular functions R_K, R_L.

Theorem 3.1. *Let $\gamma(K)$, $\gamma(L)$ be two non-empty, smooth, complete real curves over a common real closed field $\Bbbk$. Then the rings R_K, R_L of functions regular on $\gamma(K)$, $\gamma(L)$ are Witt equivalent if and only if $\gamma(K)$ and $\gamma(L)$ has the same number of semi-algebraically connected components.*

Proof. Let $\Phi : WR_K \to WR_L$ be an isomorphism. Then Φ maps the nil-radical of R_K onto the nil-radical of R_L and preserves the rank of free $\mathbb{Z}$-modules. Decomposing now R_K and R_L according to Corollary 2.5 we see that $\gamma(K)$ and $\gamma(L)$ must have the same number of semi-algebraically connected components.

To prove the opposite implication, suppose that $\gamma(K)$, $\gamma(L)$ have the same number N of semi-algebraically connected components. Every component is Nash-diffeomorphic to a circle, thus there is a homomorphism $T : \gamma(K) \to \gamma(L)$ mapping components of $\gamma(K)$ onto the components of $\gamma(L)$. Using [6, Corollary 3.9] we may find an isomorphism $t : \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ such that:

a. t preserves minus one: $t(-1) = -1$;

b. the pair (T, t) preserves quaternion-symbols in the sense that for every $\mathfrak{p} \in \gamma(K)$ and every two rational functions f, g on γ one has:

$$\left(\frac{f, g}{K_{\mathfrak{p}}}\right) = 1 \iff \left(\frac{tf, tg}{L_{T\mathfrak{p}}}\right) = 1;$$

c. the pair (T, t) preserves the parity of valuations and local signatures, i.e. for every rational function $f \in K$ and every point $\mathfrak{p} \in \gamma$ one has:

$$\operatorname{ord}_{\mathfrak{p}} f \equiv \operatorname{ord}_{T\mathfrak{p}} tf \pmod 2 \qquad \text{and} \qquad \operatorname{sgn}_{\mathfrak{p}} f = \operatorname{sgn}_{T\mathfrak{p}} tf;$$

d. the map $\Psi_t : WK \to WL$ defined by the condition $\Psi_t\langle f_1, \ldots, f_n\rangle := \langle tf_1, \ldots, tf_n\rangle$ is an isomorphism of Witt rings.

We claim that the isomorphism Ψ_t of Witt rings of the fields K, L maps the Witt ring WR_K onto WR_L. Indeed, take $\varphi \in WR_K$. Fix a point $\mathfrak{q} \in \gamma(L)$ and let $\mathfrak{p} := T^{-1}\mathfrak{q}$ be its inverse image. It follows from the exactness of Knebusch–Milnor exact sequence (c.f. Theorem 1.2) that $\partial_{\mathfrak{p}}\varphi = 0$. Let $\langle f_1, \ldots, f_l, pf_{l+1}, \ldots, pf_n\rangle$ be a diagonalization of φ over K such that p is a $\mathfrak{p}$-uniformaizer and all f_i's are $\mathfrak{p}$-adic units. Then,

$$\operatorname{sgn}_{\mathfrak{p}}\langle f_{l+1}, \ldots, f_n\rangle = \partial_{\mathfrak{p}}\varphi = 0.$$

Now using condition c above, we see that $\operatorname{ord}_{\mathfrak{q}} tf_i \equiv \operatorname{ord}_{\mathfrak{p}} f_i \equiv 0 \pmod 2$ and $\operatorname{ord}_{\mathfrak{q}} tp \equiv \operatorname{ord}_{\mathfrak{p}} p \equiv 1 \pmod 2$. In other words,

$$\langle tf_1, \ldots, tf_l, tptf_{l+1}, \ldots, tptf_n\rangle$$

is a diagonalization of $\Psi_t\varphi$ over L with all tf_i's being $\mathfrak{q}$-adic units and tp being a $\mathfrak{q}$-uniformizer. Using c again, we see that

$$\partial_{\mathfrak{q}}\Psi_t\varphi = \operatorname{sgn}_{\mathfrak{q}}\langle tf_{l+1}, \ldots, tf_n\rangle = \operatorname{sgn}_{\mathfrak{p}}\langle f_{l+1}, \ldots, f_n\rangle = 0.$$

It follows that for every point $\mathfrak{q} \in \gamma(L)$ the Witt class $\Psi_t\varphi$ lays in the kernel of $\partial_{\mathfrak{q}}$. Hence the Knebusch–Milnor exact sequence (see Theorem 1.2) shows that $\Psi_t\varphi \in \ker \oplus\partial_L = WR_L$. □

It is worth to note that in the above prove we have shown that in the above prove we have shown that the isomorphism of Witt rings $WR_K \cong WR_L$ of rings of regular function on real curves implies the isomorphism of associated exact sequences. Indeed, consider Knebusch–Milnor exact sequences associated to WR_K and WR_L:

$$\begin{array}{lll} \mathcal{S}_K: & 0 \to WR_K \xrightarrow{i_K} WK \xrightarrow{\oplus\partial_K} \bigoplus WK_{\mathfrak{p}} \xrightarrow{\lambda_K} \mathbb{Z}^N \to 0 \\ \mathcal{S}_L: & 0 \to WR_L \xrightarrow{i_L} WL \xrightarrow{\oplus\partial_L} \bigoplus WL_{\mathfrak{q}} \xrightarrow{\lambda_L} \mathbb{Z}^M \to 0 \end{array}$$

It follows that if either ends of above sequences are isomorphic (i.e. either $WR_K \cong WR_L$ or $N = M$) then the whole sequences are isomorphic, as well.

Corollary 3.2. *Under the above assumptions, the following conditions are equivalent:*

- $N = M$;
- $WR_K \cong WR_L$;
- $\mathcal{S}_K \cong \mathcal{S}_L$.

Theorem 3.1 is existential in nature—the Witt equivalence of rings of regular functions implies that the curves have the same number of semi-algebraically connected components and this in turn, as we have shown in the proof of the theorem, implies the existence of an isomorphism Ψ_t of Witt rings WK, WL of the fields, which factors over the Witt rings WR_K, WR_L of rings of regular functions. The theorem does not say, however, if the restriction $\Psi_t\,|_{WR_K}$ is identical to the original isomorphism Φ or whether the two isomorphisms are at least correlated in any way. It is, thus, natural to ask the following question:

> *Can the isomorphism of the Witt rings WR_K, WR_L of rings of regular functions be extended to an isomorphism of Witt rings WK, WL of their quotient fields?*

We do not know the answer to this question in such a generality. However, if we assume that the isomorphism $WR_K \xrightarrow{\approxeq} WR_L$ is *strong* (in a sense which we will promptly define) the answer turns out to be affirmative.

Recall that the isomorphism of Witt rings of two *fields* is called strong if it maps classes of unary forms onto classes of unary forms. In our case, when we deal with projective modules, the notion of dimension of the form is a bit fuzzy. But the following notion seems to be justified by Corollary 1.3. We shall say that the isomorphism $\Phi : WR_K \xrightarrow{\approxeq} WR_L$ is *strong* if it maps $\langle \mathbb{E}_K \rangle$ onto $\langle \mathbb{E}_L \rangle$.

Theorem 3.3. *If an isomorphism $\Phi : WR_K \xrightarrow{\approxeq} WR_L$ is strong, then there exists a strong isomorphism $\Psi_t : LWK \xrightarrow{\approxeq} WL$ extending Φ (i.e. $\Psi_t\,|_{WR_K} \equiv \Phi$).*

3.1 Proof of Theorem 3.3

Let $\Phi : WR_K \xrightarrow{\cong} WR_L$ be a strong isomorphism. Using Theorem 3.1 we see that $\gamma(K)$ and $\gamma(L)$ have the same number of semi-algebraically connected components. Denote this number by N. The isomorphism Φ is assumed to map $\langle \mathbb{E}_K \rangle$ onto $\langle \mathbb{E}_L \rangle$ and so the conditions

$$tf := g \iff \Phi\langle f \rangle = \langle g \rangle$$

defines a group isomorphism $t : \mathbb{E}_K \to \mathbb{E}_L$. Observe that Φ, being an isomorphism of rings, maps nilradical $\operatorname{Nil} WR_K$ onto the nilradical $\operatorname{Nil} WR_L$. It follows from Proposition 2.3 that t maps $\Sigma K^2 \subseteq \mathbb{E}_K$ onto $\Sigma L^2 \subseteq \mathbb{E}_L$. Let $z_1, \ldots, z_N \in \mathbb{E}_K$ be fixed in the same way as in Proposition 2.4. The following lemma shows that they are mapped to their counterparts in $\mathbb{E}_L$.

Lemma 3.4. *For every $1 \leq i \leq N$ there is $1 \leq j \leq N$ such that*

$$\operatorname{sgn}_k^L tz_i = \begin{cases} -1, & \textit{if } k = j, \\ 1, & \textit{if } k \neq j. \end{cases}$$

Proof. Take a form $\langle 1, -z_i \rangle$. It belongs to IR_K and so $\Phi\langle 1, -z_i \rangle \in IR_L$. Thus, using Proposition 2.4, we have

$$\Phi\langle 1, -z_i \rangle = k_{i,1}\langle 1, -z_i'' \rangle + \cdots + k_{i,N}\langle 1, -z_N'' \rangle + \varepsilon 1, -g,$$

where $j_{i,1}, \ldots, k_{i,N} \in \mathbb{Z}$, $z_1'', \ldots, z_N'' \in \mathbb{E}_L$ are fixed in the same way as in Proposition 2.4 but for the curve γ_L this time, i.e.:

$$\operatorname{sgn}_k^L z_j'' = \begin{cases} -1, & \text{if } k = j, \\ 1, & \text{if } k \neq j. \end{cases}$$

further $\varepsilon \in \{0, 1\}$ and g is a sum of squares. First we show that every $k_{i,j}$ is either zero or one. Indeed, square the form $\langle 1, -z_i \rangle$ in WR_K. We have $\langle 1, -z_i \rangle^2 = 2\langle 1, -z_i \rangle$. Now Φ, as a *ring* homomorphism, preserves multiplication. Thus we obtain:

$$\begin{aligned} 2k_{i,1}&\langle 1, -z_i'' \rangle + \cdots + 2k_{i,N}\langle 1, -z_N'' \rangle \\ &= \Phi\big(2\langle 1, -z_i \rangle\big) = \Phi\big(\langle 1, -z_i \rangle^2\big) \\ &= 2k_{i,1}^2\langle 1, -z_i'' \rangle + \cdots + 2k_{i,N}^2\langle 1, -z_N'' \rangle. \end{aligned}$$

Therefore $k_{i,j}^2 = k_{i,j}$ for every $1 \leq j \leq N$ and so $k_{i,j} \in \{0, 1\}$.

Now, we show that all but one $k_{i,j}$'s are null. Suppose otherwise. Let for some $1 \leq i_1 \leq N$ the image $\Phi\langle 1, -z_{i_1} \rangle$ has two non-zero coordinates in the free $\mathbb{Z}$-module $\mathbb{Z}[\langle 1, -z_1'' \rangle] \oplus \cdots \oplus \mathbb{Z}[\langle 1, -z_N'' \rangle]$, then using Dirichlet's pigeonhole principle there is another element $\langle 1, -z_{i_2}\rangle$ such that $\Phi\langle 1, -z_{i_2}\rangle$ has at least one the same non-zero coordinate. Let for example

$$\Phi\langle 1, -z_{i_1}\rangle = \langle 1, -z''_a\rangle + \text{other terms}$$
$$\Phi\langle 1, -z_{i_2}\rangle = \langle 1, -z''_a\rangle + \text{other terms}.$$

Then

$$\begin{aligned} 0 = \Phi(0) = \Phi\big(\langle 1, -z_{i_1}\rangle \cdot \langle 1, -z_{i_2}\rangle\big) \\ = \Phi\langle 1, -z_{i_1}\rangle \cdot \Phi\langle 1, -z_{i_2}\rangle \\ = 2\langle 1, -z''_a\rangle + \text{other terms} \neq 0. \end{aligned}$$

This contradiction shows that indeed every $\langle 1, -z_i\rangle$ is mapped onto $\langle 1, -z''_j\rangle + \langle 1, -g\rangle$ for some $1 \leq j \leq N$ and $g \in \Sigma L^2$ (and form different i's, the corresponding j's differ too). Now, we have

$$0 = \langle 1, -z''_j\rangle \cdot \langle 1, -g\rangle = \langle 1, 1, -z''_j, -g\rangle - \langle 1, -z''_j g\rangle.$$

Therefore

$$\langle 1, -z''_j g\rangle = \langle 1, -z''_j\rangle + \langle 1, -g\rangle = \Phi\langle 1, -z_i\rangle$$

and so, by the definition of t, we have $tz_i = z''_j g$. □

It follows from the above lemma that t induces a permutation $i \mapsto j =: \tau(i)$ of the set $\{1, \ldots, N\}$. We may treat it as the bijection of $\gamma_i^K \mapsto \gamma_{\tau(i)}^L$ of the sets of components of $\gamma(K)$ and $\gamma(L)$. We may find a homeomorphism $T : \gamma(K) \to \gamma(L)$ such that $T\gamma_i^K = \gamma_{\tau(i)}^L$. Fix a single point $\mathfrak{p}_i$ in every component γ_i^K of $\gamma(K)$ and let $\mathfrak{q}_{\tau(i)} := T\mathfrak{p}_i \in \gamma_{\tau(i)}^L$. We may now define a $\mathbb{F}_2$-linear isomorphism $\hat{t} : \dot{K}/\dot{K}^2 \xrightarrow{\cong} \dot{L}/\dot{L}^2$ of a $\mathbb{F}_2$-vector spaces that extends t. Recall that by Lemma 1.4 we have a decomposition

$$\dot{K}/\dot{K}^2 = \mathbb{E}_K \oplus \operatorname{lin}_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{K}/\dot{K}^2 : \mathfrak{p} \in \gamma_i^K, 1 \leq i \leq N\},$$

likewise for $\dot{L}/\dot{L}^2$:

$$\dot{L}/\dot{L}^2 = \mathbb{E}_L \oplus \operatorname{lin}_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{L}/\dot{L}^2 : \mathfrak{p} \in \gamma_i^L, 1 \leq i \leq N\}.$$

Let $\hat{t}\,\big|_{\mathbb{E}_K} := t$ and define $\hat{t}$ on the basis of the other summand by the condition:

$$\hat{t}\chi_{(\mathfrak{p},\mathfrak{p}_i)} := \chi_{(T\mathfrak{p},\mathfrak{q}_i)}.$$

By linearity this define $\hat{t}$ on the whole $\mathbb{F}_2$-vector space $\dot{K}/\dot{K}^2$. Clearly $\hat{t}$ is a group isomorphism and $\hat{t}$ preserves local signatures in the sense that:

$$\bigwedge_{\mathfrak{p}\in\gamma(K)} \bigwedge_{f\in\dot{K}/\dot{K}^2} \operatorname{sgn}_{\mathfrak{p}}^K f = \operatorname{sgn}_{T\mathfrak{p}}^L \hat{t}f. \tag{3.5}$$

Consequently, $\hat{t}$ preserves the parity of valuation, as well

$$\bigwedge_{\mathfrak{p}\in\gamma(K)} \bigwedge_{f\in\dot{K}/\dot{K}^2} \operatorname{ord}^K_{\mathfrak{p}} f \equiv \operatorname{ord}^L_{T\mathfrak{p}} \hat{t}f \pmod 2.$$

We claim that $\hat{t}$ is a Harrison map (i.e. $\hat{t}(-1) = -1$ and $\langle f, g\rangle$ represents 1 over K if and only if $\langle \hat{t}f, \hat{t}g\rangle$ represents 1 over L). Indeed, $\Phi\langle -1\rangle = \Phi(-\langle 1\rangle) = \langle -1\rangle$ and so $\hat{t}(-1) = t(-1) = -1$. For the second condition, take $f, g \in \dot{K}/\dot{K}^2$ such that 1 is represented by $\langle f, g\rangle$ over K. This means that the form $\varphi := \langle 1, -f, -g, fg\rangle$ is hyperbolic over K and so it is hyperbolic over every completion $K_{\mathfrak{p}}$ for $\mathfrak{p} \in \gamma(K)$. Take $\mathfrak{p} \in \gamma(K)$ such that neither f nor g changes sign at $\mathfrak{p}$—almost every $\mathfrak{p}$ will do. The form φ is hyperbolic in $K_{\mathfrak{p}}$ if and only if $\operatorname{sgn}^K_{\mathfrak{p}} f = -\operatorname{sgn}^K_{\mathfrak{p}} g$. It follows from Eq. (3.5) that $\operatorname{sgn}^L_{\mathfrak{q}} \hat{t}f = -\operatorname{sgn}^L_{\mathfrak{q}} \hat{t}g$ for almost every $\mathfrak{q} \in \gamma(L)$. Thus, by the means of the Witt theorem (c.f. [4, Theorem 9.5]) the form $\langle 1, -\hat{t}f, -\hat{t}g, \hat{t}f\hat{t}g\rangle$ is hyperbolic over L. Consequently $\langle \hat{t}f, \hat{t}g\rangle$ represents 1 over L. This proves the claim. It is well known that if $\hat{t}$ is a Harrison map, then the mapping $\langle f_1, \dots, f_n\rangle \mapsto \langle \hat{t}f_1, \dots, \hat{t}f_n\rangle$ is a strong isomorphism of Witt rings of fields. Denote it Ψ_t. It is clear from the construction, that $\Psi_t \big|_{WR_K} \equiv \Phi$. □

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