VALUATIVE MAHLER'S INEQUALITY

PRZEMYSŁAW KOPROWSKI

Abstract. We derive a Mahler-type upper bound for the Krasner's constant of a monic polynomial.

One of the classical problems in the theory of polynomials is to estimate the minimal distance between two distinct roots of a given polynomial. This problem can be traced back essentially to Cauchy (see e.g. [3, Eq. (48), p. 398]) and it has been extensively studied in the last half century by numerous authors.

Let $f = f_0 + f_1x + \cdots + f_nx^n$ be a (non-constant) polynomial with complex coefficients and $\xi_1, \ldots, \xi_n \in \mathbb{C}$ be all its roots, It is customary to denote

$$\operatorname{sep}(f) := \min\bigl\{|\xi_i - \xi_j| : i \neq j\bigr\}, \quad \operatorname{rsep}(f) := \min\bigl\{|\xi_i - \xi_j| : i \neq j,\ \xi_i, \xi_j \in \mathbb{R}\bigr\}$$

respectively: the minimal distance between roots and the minimal distance between real roots. In early 1960s Mahler derived a lower bound for $\operatorname{sep}(f)$. Namely he proved (see [6]):

$$\operatorname{sep}(f) > \frac{\sqrt{3|\Delta_f|}}{n^{\frac{n+2}{2}} \cdot \|f\|^{n-1}}, \tag{1}$$

where $n = \deg f$ is the degree, $\Delta_f = (-1)^{n(n-1)/2} \cdot f_n^{-1} \cdot \operatorname{res}(f, f')$ denotes the discriminant and $\|f\| = |f_0| + \cdots + |f_n|$ is the 1-norm of the vector of coefficients of f. A similar formula for $\operatorname{rsep}(f)$ was proved by Rump in [9]. Recently Herman, Hong and Tsigaridas in [4] improved Mahler's result finding a formula which is well-behaved with respect to a linear scaling of roots. Some other formulas were developed in [1, 7, 8]. Beside distances between isolated roots, another problem being investigated is a bound for a distance between *clusters* of roots. We refer the reader to [2] for a survey of recent results.

The goal of this short note is to show that the arguments of Mahler—after only minor adjustments—continue to work if we replace $\mathbb{C}$ by a valued field. Let (K, v) be a valued field and $f \in K[x]$ a non-constant polynomial over K. Classical results mentioned above concern polynomials over $\mathbb{C}$, hence the roots of f lie in the field of constants for free. It is not so in an arbitrary valued field, hence we *assume*[1] that K contains all the roots of f. Say $f = \operatorname{lc}(f) \cdot (x - \xi_1) \cdots (x - \xi_n)$. A valuative counterpart of sep and rsep is called the *Krasner constant*. It is defined by a formula:

$$\operatorname{kras}(f) := \max\bigl\{v(\xi_i - \xi_j) : i \neq j\bigr\}.$$

2010 *Mathematics Subject Classification.* 11C08,11C20,65H04,12J10.

Key words and phrases. polynomial roots, Mahler measure, Hadamard's inequality, Mahler's inequality.

[1]Alternatively one may take an extension of v to the splitting field L of f and perform all the computations in L instead of K.

Our goal is to find an upper bound for $\operatorname{kras}(f)$ in spirit of (1). As in the original Mahler's paper, the main tool is Hadamard's inequality.

Proposition 1 (Valuative Hadamard's inequality)**.** *Let $M = (m_{ij})$ be a square matrix of size $n \times n$ with entries in a valued field (K, v). Then*

$$v(\det M) \geq \sum_{j\leq n} \min_{i\leq n}\bigl\{v(m_{ij})\bigr\}.$$

Proof. The proof of the valuative version of version of Hadamard's inequality is even simpler than the standard one. All we need is the Laplace expansion formula and the following basic property of minimum:

$$\min_{i\leq n}\Bigl\{\sum_{j\leq n} a_{ij}\Bigr\} \geq \sum_{j\leq n}\min_{i\leq n}\bigl\{a_{ij}\bigr\}.$$

In order to prove Hadamard's inequality we proceed by induction on the size of the matrix. For $n = 1$ the inequality (in fact an equality, in this case) holds trivially, since $\det M = m_{11}$.

Assume that the proposition is true for matrices of size $(n-1)\times(n-1)$. Compute

$$\begin{aligned}
v(\det M) &= v\Bigl(\sum_{i\leq n}(-1)^{i+n}\cdot m_{in}\cdot \det M_{in}\Bigr)\\
&\geq \min_{i\leq n}\bigl\{0 + v(m_{in}) + v(\det M_{in})\bigr\}\\
&\geq \min_{i\leq n}\bigl\{v(m_{in})\bigr\} + \min_{i\leq n}\bigl\{v(\det M_{in})\bigr\}\\
&\geq \min_{i\leq n}\bigl\{v(m_{in})\bigr\} + \min_{i\leq n}\Bigl\{\sum_{j\leq n-1}\min_{k\leq n-1} v(m_{k'j})\Bigr\}.
\end{aligned}$$

Here k' equals either k if $k < i$ or $k+1$ otherwise. Further we have:

$$\begin{aligned}
&\geq \min_{i\leq n}\bigl\{v(m_{in})\bigr\} + \sum_{j\leq n-1}\min_{i\leq n}\Bigl\{\min_{k\leq n-1}\bigl\{v(m_{k'j})\bigr\}\Bigr\}\\
&= \min_{i\leq n}\bigl\{v(m_{in})\bigr\} + \sum_{j\leq n-1}\min_{i\leq n}\bigl\{v(m_{ij})\bigr\}\\
&= \sum_{j\leq n}\min_{i\leq n}\bigl\{v(m_{ij})\bigr\}.
\end{aligned}$$

□

Let (K, v) be a valued field and $f = f_0 + f_1x + \cdots + f_nx^n \in K[x]$ be a non-constant polynomial. The *Gauss valuation* of f is

$$vf := \min_{0\leq i\leq n} vf_i.$$

We generalize the notion of the Mahler measure as follows. Assume that f factors into linear terms over K, say $f = f_n \cdot (x - \xi_1)\cdots(x - \xi_n)$ for some $\xi_1, \dots, \xi_n \in K$. Set

$$m_f := vf_n + \sum_{\substack{i\leq n\\ v\xi_i<0}} v\xi_i.$$

We shall call m_f the *valuative Mahler measure* of f. The roots of the polynomial do not change when the polynomial is multiplied by a nonzero constant. Since eventually we are interested in the valuation of a difference between two distinct roots of f, we will assume that f is monic. In this case the valuative Mahler measure is just the sum of valuations of the roots of f that fall outside $\mathcal{O}_v$, where $\mathcal{O}_v = \{a \in K : va \geq 0\}$, is the valuation ring associated with v. The next lemma is a valuative analog of [5, Eq. (6)].

Lemma 2. *Let f be a non-constant monic polynomial. If $m_f \neq 0$, then*

$$\deg f \cdot vf \leq m_f \leq vf_0.$$

Proof. It follows from the assumption $m_f \neq 0$, that there is at least one root ξ of f such that $\xi \notin \mathcal{O}_v$. We then have

$$-\xi^n = f_0 + f_1\xi + \cdots + f_{n-1}\xi^{n-1}.$$

Computing the valuations of both sides we obtain

$$\begin{aligned} n \cdot v\xi \geq \min_{i<n}\{v(f_i\xi^i)\} &= \min_{i<n}\{vf_i + i \cdot v\xi\} \\ &\geq \min_{i<n}\{vf_i\} + \min_{i<n}\{i \cdot v\xi\} = \min_{i<n}\{vf_i\} + (n-1)v\xi \end{aligned}$$

The polynomial f is monic, hence $vf_n = 0$ and so we have

$$v\xi \geq \min_{i\leq n}\{vf_i\} = vf$$

for every root ξ of f not in $\mathcal{O}_v$. Let $d \geq 0$ be the number of such roots. It follows that

$$0 > m_f = \sum_{\xi_i \notin \mathcal{O}_v} v\xi_i \geq d \cdot vf \geq n \cdot vf.$$

This proves one inequality. The other one is an immediate consequence of Viète's formulas. □

A well known formula for the discriminant of a monic polynomial says that $\Delta_f = \prod_{i<j}(\xi_i - \xi_j)^2$, thus we have

$$v\Delta_f = 2\sum_{i<j} v(\xi_i - \xi_j).$$

Consequently, if we *a priori* know that $v(\xi_i - \xi_j)$ is non-negative for every $i \neq j$, then we instantly obtain an estimate

$$\operatorname{kras}(f) \leq \frac{1}{2}v\Delta_f.$$

The following theorem shows that a similar inequality holds unconditionally.

Theorem 3 (Valuative Mahler's inequality)**.** *Let f be a non-constant monic polynomial of degree $n \geq 2$ over a valued field (K, v). Then*

$$\operatorname{kras}(f) \leq \frac{1}{2}v\Delta_f - n \cdot (n-1) \cdot \min\{0, vf\},$$

where Δ_f is the discriminant of f.

Proof. The proof is very similar to the proof of the original Mahler's inequality, presented in [6]. If f has a multiple root, then $\operatorname{kras}(f) = v\Delta_f = \infty$ and the assertion holds trivially. Thus, we may freely assume that f is square-free. Fix two distinct roots $\xi, \zeta \in K$ of f. Without loss of generality we may assume that

$v\xi \le v\zeta$. We need to consider two cases. First assume that $v\xi < 0$, i.e. $\xi \notin \mathcal{O}_v$. Renumber the roots of f in the following manner:

$$\xi_1 := \xi, \quad \xi_2, \dots, \xi_m \notin \mathcal{O}_v \qquad \text{and} \qquad \xi_{m+1}, \dots, \xi_n \in \mathcal{O}_v.$$

Let $\mathrm{V}_{\xi_1,\dots,\xi_n}$ be the Vandermonde matrix associated to the roots of f. Subtraction of rows does not alter the determinant, hence we have

$$\det \mathrm{V}_{\xi_1,\dots,\xi_n} = \det \begin{pmatrix} 0 & \xi - \zeta & \xi^2 - \zeta^2 & \cdots & \xi^{n-1} - \zeta^{n-1} \\ 1 & \xi_2 & \xi_2^2 & \cdots & \xi_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \xi_n & \xi_n^2 & \cdots & \xi_n^{n-1} \end{pmatrix}.$$

Divide the first m rows of $\mathrm{V}_{\xi_1,\dots,\xi_n}$ (i.e. the rows corresponding to the roots of f not in $\mathcal{O}_v$) by $\xi_1^{n-1}, \dots, \xi_m^{n-1}$, respectively. Denote the resulting determinant by Q. We have

$$Q = \frac{\det \mathrm{V}_{\xi_1,\dots,\xi_n}}{(\xi_1 \cdots \xi_m)^{n-1}} = \det \begin{pmatrix} 0 & \frac{\xi-\zeta}{\xi^{n-1}} & \cdots & \frac{\xi^{n-2}-\zeta^{n-2}}{\xi^{n-1}} & \frac{\xi^{n-1}-\zeta^{n-1}}{\xi^{n-1}} \\ \xi_2^{-(n-1)} & \xi_2^{-(n-2)} & \cdots & \xi_2^{-1} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \xi_m^{-(n-1)} & \xi_m^{-(n-2)} & \cdots & \xi_m^{-1} & 1 \\ 1 & \xi_{m+1} & \cdots & \xi_{m+1}^{n-2} & \xi_{m+1}^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \xi_n & \cdots & \xi_n^{n-2} & \xi_n^{n-1} \end{pmatrix}.$$

Finally, divide the first row of the above matrix by $\xi - \zeta$ and denote the resulting entries by $q_0, \dots, q_{n-1}$. This way we have

$$\frac{Q}{\xi - \zeta} = \det \begin{pmatrix} q_0 & q_1 & \cdots & q_{n-2} & q_{n-1} \\ \xi_2^{-(n-1)} & \xi_2^{-(n-2)} & \cdots & \xi_2^{-1} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \xi_m^{-(n-1)} & \xi_m^{-(n-2)} & \cdots & \xi_m^{-1} & 1 \\ 1 & \xi_{m+1} & \cdots & \xi_{m+1}^{n-2} & \xi_{m+1}^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \xi_n & \cdots & \xi_n^{n-2} & \xi_n^{n-1} \end{pmatrix}, \tag{2}$$

where $q_0 = 0$ and

$$q_k = \frac{\xi^k - \zeta^k}{(\xi - \zeta) \cdot \xi^{n-1}} = \xi^{k-n} + \xi^{k-1-n}\zeta + \cdots + \xi^{1-n}\zeta^{k-1}$$

for $k \ge 1$. All the entries of the above matrix, except possibly the ones in the first row, lie in $\mathcal{O}_v$. Therefore the valuative Hadamard's inequality (applied to the transposed matrix) yields

$$\begin{aligned} vQ - v(\xi - \zeta) &\geq \min_{0\leq k\leq n-1} vq_k + 0 \\ &= \min_{0\leq k\leq n-1} v\left(\sum_{i=0}^{k-1} \xi^{k-i-n}\zeta^i\right) \\ &\geq \min_{0\leq k\leq n-1} \Big\{ \min_{0\leq i\leq k-1} \big\{(k-i-n)\cdot v\xi + i\cdot v\zeta\big\}\Big\} \\ &\geq \min_{0\leq k\leq n-1} \big\{(k-n)v\xi\big\} = -v\xi > 0. \end{aligned}$$

The polynomial f is monic by assumption, consequently its discriminant equals $(\det \mathrm{V}_{\xi_1,\dots,\xi_n})^2$. Combining these two facts with the definition of Q we obtain

$$v(\xi - \zeta) < vQ = \frac{1}{2}v\Delta_f - (n-1)\cdot m_f.$$

Now Lemma 2 yields

$$v(\xi-\zeta) < \frac{1}{2}v\Delta_f - n\cdot(n-1)\cdot\min\{0, vf\}.$$

In the case $v\xi \geq 0$ the proof runs along the same lines except that—when computing Q—we do not divide the first row of the matrix by ξ^{n-1}. It follows that the elements $q_0, \dots, q_{n-1}$ in Eq. (2) are given by the formula

$$q_0 = 0 \qquad \text{and} \qquad q_k = \frac{\xi^k - \zeta^k}{\xi - \zeta} = \xi^{k-1} + \xi^{k-2}\zeta + \cdots + \zeta^{k-1}.$$

Consequently $vq_k \geq (k-1)\cdot v\xi \geq 0$ for every $k \geq 1$ and $vq_0 = \infty$. Therefore

$$vQ - v(\xi-\zeta) \geq \min_{1\leq k\leq n-1} vq_k \geq \min_{1\leq k\leq n-1}(k-1)v\xi = 0.$$

Like in the previous case we obtain

$$v(\xi-\zeta) \leq \frac{1}{2}v\Delta_f - n\cdot(n-1)\cdot\min\{0, vf\}.$$

The inequality holds for every two distinct roots ξ, ζ of f and so its right-hand-side provides an upper bound for $\operatorname{kras}(f)$. □

Remark. If the valuation $v(\xi_i - \xi_j)$ between two roots ξ_i, ξ_j of a given polynomial f is constant (i.e. independent of i, j), then clearly

$$\operatorname{kras}(f) = v(\xi_i - \xi_j) = \frac{v\Delta_f}{2\deg f} \qquad \text{for all } i \neq j.$$

It is an interesting question, whether for an arbitrary polynomial f of degree $n \geq 2$, there is an upper bound for $\operatorname{kras}(f)$ that depends on $v\Delta_f/2n$ rather than $v\Delta_f/2$.

References

[1] Yann Bugeaud and Maurice Mignotte. On the distance between roots of integer polynomials. *Proc. Edinb. Math. Soc. (2)*, 47(3):553–556, 2004.

[2] Yann Bugeaud and Maurice Mignotte. Polynomial root separation. *Int. J. Number Theory*, 6(3):587–602, 2010.

[3] Augustin-Louis Cauchy. Note III. Sur la résolution numérique des équations. In *Oeuvres complètes d'Augustin Cauchy*, volume 3 of *Série 2*. Gauthier-Villars et fils (Paris), 1847.

[4] Aaron Herman, Hoon Hong, and Elias Tsigaridas. Improving root separation bounds. *J. Symbolic Comput.*, 84:25–56, 2018.

[5] K. Mahler. An application of Jensen's formula to polynomials. *Mathematika*, 7:98–100, 1960.

[6] K. Mahler. An inequality for the discriminant of a polynomial. *Michigan Math. J.*, 11:257–262, 1964.
[7] M. Mignotte. An inequality about factors of polynomials. *Math. Comp.*, 28:1153–1157, 1974.
[8] Maurice Mignotte. On the distance between the roots of a polynomial. *Appl. Algebra Engrg. Comm. Comput.*, 6(6):327–332, 1995.
[9] Siegfried M. Rump. Polynomial minimum root separation. *Math. Comp.*, 33(145):327–336, 1979.

Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland
E-mail address: przemyslaw.koprowski@us.edu.pl