ON EXISTENCE OF TAME HARRISON MAP

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Abstract. We present here two new criteria for existence of a tame Harrison map of two formally real algebraic function fields over a fixed real closed field of constants. The first criterion (c.f. theorem 2.5) shows that a square class group isomorphism is a tame Harrison map if it induces an isomorphism of the coproduct rings of residue Witt rings. The other result (c.f. proposition 3.5) associates a tame Harrison map to an integral quaternion-symbol equivalence.

1. Introduction

The notion of Witt ring plays a central role in the algebraic theory of quadratic forms. It is natural to ask for a criteria for existence of an isomorphism of Witt rings. This subject has been studied by several authors for over 30 years (see e.g. [11, 2, 13, 14, 6, 5, 7]). One of the tools used here, called an *Harrison isomorphism*, is an isomorphism t of square class groups of the fields in question such that $t(-1) = -1$ and the element 1 is represented by a binary form $\langle f, g\rangle$ iff it is represented by $\langle tf, tg\rangle$. Any such Harrison map induces an isomorphism of the Witt rings of the fields involved (see [9, Theorem XII.1.8]).

Take now a fixed real closed field $\mathbb{k}$ and let K, L be two formally real algebraic function fields over $\mathbb{k}$. Denote $\Omega(K), \Omega(L)$ the sets of points of K, L trivial on $\mathbb{k}$. Among all the points of K (resp. L) we select those having the residue field isomorphic to $\mathbb{k}$. Following [4], we call such points *real* and denote the set of all real points γ^K (resp. γ^L). It is a real curve over $\mathbb{k}$. Fix an orientation (see [4, §5]) of γ^K, γ^L. With a point $\mathfrak{p} \in \gamma^K$ we associate two orderings of K, consisting of all elements of K (treated as functions on γ^K), positive in right/left neighbourhood of $\mathfrak{p}$. We denote those ordering $P_+(\mathfrak{p})$ and $P_-(\mathfrak{p})$ respectively (see [6, §2]).

Recall that a Harrison isomorphism $t\colon \dot K/\dot K^2 \to \dot L/\dot L^2$ is called *tame* (see [6, Definition 2.18]) if it maps 1-pt fans onto 1-pt fans. The tame Harrison map canonically induces a bijection $T\colon \gamma^K \to \gamma^L$ of the sets of real points of K and L. Namely, let $\mathfrak{p} \in \gamma^K$ be a real point of K and $P_+(\mathfrak{p}), P_-(\mathfrak{p})$ be the two orderings of K compatible with $\mathfrak{p}$ (for the notion of orderings compatible with valuation consult [8]), if t is tame, then it maps the fan $P_+(\mathfrak{p}) \cap P_-(\mathfrak{p})$ onto a 1-pt fan of L. Denote $\mathfrak{q} \in \gamma^L$ the point of L compatible with $t(P_+(\mathfrak{p}) \cap P_-(\mathfrak{p}))$. The bijection is then given by $T(\mathfrak{p}) := \mathfrak{q}$. It was shown in [6, Proposition 3.1 and Lemma 3.6] that T is a homeomorphism with respect to Euclidean topologies of γ^K, γ^L. It is, thus, natural to ask for a criteria for existence of a tame Harrison isomorphism. One such a criterion was given already in [6, Theorem 3.8]. It was shown there that any homeomorphism $\gamma^K \to \gamma^L$ gives rise to an appropriate tame Harrison map. In particular, if γ^K and γ^L have the same number of semi-algebraically connected components then such a map does exist.

In this paper we present two more conditions for an existence of a tame Harrison map: theorem 2.5 and proposition 3.5. To this end we need to utilise the notion

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of a quaternion-symbol equivalence. Recall (see [6, 5]) that a quaternion-symbol equivalence of fields K, L with respect to (γ^K, γ^L) is the pair of maps (t, T) such that $t\colon \dot K/\dot K^2 \to \dot L/\dot L^2$ is an isomorphism and $T\colon \gamma^K \to \gamma^L$ a bijection and such that a local quaternion algebra $\left(\frac{f,g}{K_{\mathfrak p}}\right)$ splits if and only if the local quaternion algebra $\left(\frac{tf,tg}{L_{T\mathfrak p}}\right)$ splits for all square classes $f, g \in \dot K/\dot K^2$ and every point $\mathfrak p \in \gamma^K$. In general a quaternion-symbol equivalence does not preserve -1 but it is easy to show (see [6, Proposition 2.6]) that there exists a quaternion-symbol equivalence iff there exists one that maps $-1 \in \dot K/\dot K^2$ to $-1 \in \dot L/\dot L^2$. In this paper we consider only those quaternion-symbol equivalences that preserve -1. Moreover, we implicitly assume a quaternion-symbol equivalence to be taken always with respect to (γ^K, γ^L). Hence in what follows, unless stated otherwise, the phrase 'a quaternion-symbol equivalence' actually means 'a quaternion-symbol equivalence (with respect to the sets of real points) that preserves -1'. It was shown in [7, Theorem 3.1] that any quaternion-symbol equivalence preserves the parity of a valuation in the sense that $\operatorname{ord}_{\mathfrak p} f \equiv \operatorname{ord}_{T\mathfrak p} tf \pmod 2$ for every $f \in \dot K/\dot K^2$ and every $\mathfrak p \in \gamma^K$.

A tame Harrison isomorphism together with the canonically induced bijection $T\colon \gamma^K \to \gamma^L$ is a quaternion-symbol equivalence (see [6, Proposition 3.1]). Conversely, if (t, T) is a quaternion-symbol equivalence (recall that we implicitly assume that t maps -1 to -1), then t is a tame Harrison map.

In the theorem 2.5 we prove that a Harrison map for which the diagram 2.6 commutes is tame. This result is a real counterpart of the analogous result proved for global fields in [14]. Next, in the proposition 3.4 we show that any tame Harrison map induces an integral equivalence. The converse of this is generally false. However in the proposition 3.5 we show that the *existence* of an integral equivalence implies the *existence* of a tame Harrison map.

2. Isomorphism of a coproduct ring

Paper [14] presents a criterion for a Hilbert-symbol equivalence of global fields to be tame (recall [11, 13] that a Hilbert-symbol equivalence is a predecessor of a quaternion-symbol equivalence, in particular those two terms coincide on global function fields). The theorem 2.5, that we present in this section, can be treated as a certain analogue of that result.

First we need the following lemma. Here, for a non-empty set I, we denote $\mathbb Z^{(I)}$ a coproduct of $\operatorname{card}(I)$ copies of $\mathbb Z$. Obviously, $\mathbb Z^{(I)}$ is a group but we can augment it with a ring structure (without a unit!) by considering a componentwise multiplication. In particular, if we denote $\delta_i \in \mathbb Z^{(I)}$ a 'Kronecker delta' (i.e. an element having 1 on i-th coordinate and zeros everywhere else), then we have

$$\delta_i \cdot \delta_j = \begin{cases} \delta_i, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.1}$$

Lemma 2.2. *If $\Phi\colon \mathbb Z^{(I)} \to \mathbb Z^{(I)}$ is a ring automorphism of $\mathbb Z^{(I)}$, then there exists such a permutation σ of the set I that*

$$\Phi\bigl((n_i)_{i\in I}\bigr) = \bigl(n_{\sigma(i)}\bigr)_{i\in I}$$

Proof. Fix $i \in I$. The element δ_i is then a unit of a subring $\mathbb Z_i$ of the ring $\mathbb Z^{(I)}$, defined by $\mathbb Z_i := \bigl\{(x_j)_{j\in I} : x_j = 0 \text{ for } j \neq i\bigr\} \cong \mathbb Z$. Moreover $\mathbb Z_i = \delta_i \cdot \mathbb Z^{(I)}$, hence

$$\Phi(\mathbb Z_i) = \Phi(\delta_i)\Phi(\mathbb Z^{(I)}) = \Phi(\delta_i)\cdot \mathbb Z^{(I)}. \tag{2.3}$$

Thus, we need only to show that there is $j \in I$ such that $\Phi(\delta_i) = \delta_j$. Let $\Phi(\delta_i) = \beta_1 + \cdots + \beta_n$ where $\beta_k \in \mathbb Z_{i_k} < \mathbb Z^{(I)}$ are the all nonzero coordinates of $\Phi(\delta_i)$. Now

$\delta_i^2 = \delta_i$ and so

$$\beta_1 + \cdots + \beta_n = \Phi(\delta_i^2) = \Phi(\delta_i)^2 = (\beta_1 + \cdots + \beta_n)^2 = \beta_1^2 + \cdots + \beta_n^2.$$

Here the last equality follows from (2.1). Consequently $\beta_k^2 = \beta_k \in \mathbb{Z}_{i_k} \cong \mathbb{Z}$, hence $\beta_k = \delta_{i_k}$.

Now $\mathbb{Z}_i$ is a free $\mathbb{Z}$-module of rank 1 and so is its image $\Phi(\mathbb{Z}_i)$. On the other hand

$$\Phi(\mathbb{Z}_i) = (\delta_{i_1} + \cdots + \delta_{i_n}) \cdot \mathbb{Z}^{(I)} = \mathbb{Z}_{i_1} \oplus \cdots \oplus \mathbb{Z}_{i_n}.$$

is a free $\mathbb{Z}$-module of rank n. Therefore, $n = 1$ and $\Phi(\delta_i) = \delta_{i_1}$. $\square$

Now, if instead of a single ring $\mathbb{Z}^{(I)}$ we consider two such rings, then the above lemma transcribes to:

Corollary 2.4. *Let A, B be two non-empty sets of the same cardinality. If $\Phi\colon \mathbb{Z}^{(A)} \to \mathbb{Z}^{(B)}$ is a ring isomorphism, then there exists a bijection $T\colon A \to B$ such that*

$$\Phi\big((n_i)_{i\in A}\big) = \big(n_{Ti}\big)_{i\in A}.$$

We are now ready to state our first main result. Recall that any Harrison map $t\colon \dot K/\dot K^2 \to \dot L/\dot L^2$ induces the isomorphism of Witt rings $WK \to WL$ sending $\langle f_1, \ldots, f_n\rangle$ to $\langle tf_1, \ldots, tf_n\rangle$. Such an isomorphism is called a *strong* Witt isomorphism associated to t.

Theorem 2.5. *Let $t\colon \dot K/\dot K^2 \to \dot L/\dot L^2$ be a Harrison map and let $i_t\colon WK \to WL$ denote a strong Witt isomorphism associated to t. If the diagram*

$$\begin{array}{ccc} WK & \xrightarrow{\partial_K} & \bigoplus_{\mathfrak{p}\in\gamma^K} WK(\mathfrak{p}) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ WL & \xrightarrow[\partial_L]{} & \bigoplus_{\mathfrak{q}\in\gamma^L} WL(\mathfrak{q}), \end{array} \tag{2.6}$$

commutes, where the vertical arrows are ring isomorphisms, then t is tame.

Proof. For every point $\mathfrak{p} \in \gamma^K$ we have $K(\mathfrak{p}) \cong \Bbbk$, likewise for every point $\mathfrak{q} \in \gamma^L$. Hence $\bigoplus_{\mathfrak{p}\in\gamma^K} WK(\mathfrak{p}) \cong \mathbb{Z}^{(\gamma^K)}$ and $\bigoplus_{\mathfrak{q}\in\gamma^L} WL(\mathfrak{q}) \cong \mathbb{Z}^{(\gamma^L)}$. By the previous corollary the isomorphism Φ induces the bijection $T\colon \gamma^K \to \gamma^L$. Thanks to [6, Theorem 3.2] and [7, Theorem 3.1], all we need to do is to prove that (t, T) is a quaternion-symbol equivalence.

We claim that the pair (t, T) preserves a parity of a valuation in the sense that $\operatorname{ord}_{\mathfrak{p}} f \equiv \operatorname{ord}_{T\mathfrak{p}} tf \pmod 2$ for every $f \in \dot K/\dot K^2$ and every $\mathfrak{p} \in \gamma^K$. Indeed, take a point $\mathfrak{p} \in \gamma^K$, fix its uniformizer p and let $q := tp$. Consider the second residue homomorphism $\partial_{\mathfrak{p}}\colon WK_{\mathfrak{p}} \to WK(\mathfrak{p}) \cong \mathbb{Z}$. We have $\partial_{\mathfrak{p}}\langle p\rangle = 1$, so using the commutativity of (2.6) we obtain

$$1 = \big(\Phi\,\big|_{\mathbb{Z}_{\mathfrak{p}}} \circ \partial_K\big)(\langle p\rangle) = \big(\pi_{T\mathfrak{p}} \circ \partial_L\big)(\langle tp\rangle) = \partial_{T\mathfrak{p}}(\langle q\rangle),$$

here $\pi_{T\mathfrak{p}}\colon \mathbb{Z}^{(\gamma^K)} \to \mathbb{Z}_{T\mathfrak{p}}$ denotes the projection. Thus q is a uniformizer of $T\mathfrak{p}$. Similarly $-p$ is mapped to $-q$ and this proves our claim.

In particular this implies that t factors through $\dot K_{\mathfrak{p}}^2$ for every point $\mathfrak{p} \in \gamma^K$. Indeed, take a point $\mathfrak{p} \in \gamma^K$ and such a square class $f \in \dot K/\dot K^2$ that $f \in \dot K_{\mathfrak{p}}^2$. We then have $\partial_{\mathfrak{p}}\langle fp\rangle = 1$ and so the commutativity of (2.6) implies that also

$$\partial_{T\mathfrak{p}}(\langle tf \cdot q\rangle) = \big(\Phi\,\big|_{\mathbb{Z}_{\mathfrak{p}}} \circ \partial_K\big) = 1,$$

thus $tf \in L_{T\mathfrak{p}}^2$.

Take now two square classes $f, g \in \dot K/\dot K^2$ of K and a point $\mathfrak{p} \in \gamma^K$. Assume that a quaternion algebra $\left(\frac{f,g}{K_{\mathfrak{p}}}\right)$ splits. This is possible only if at least one of the elements: f, g, fg is a square in the completion $K_{\mathfrak{p}}$. As we have already noted, t preserves local squares, so tf, tg or $tftg$ must be a square in $L_{T\mathfrak{p}}$. Which means that a quaternion algebra $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ splits. All in all, [7, Theorem 3.1] implies that (t,T) is tame in every point of γ^K and so [6, Theorem 3.2] implies that t is a tame Harrison map. □

3. Integral equivalence

Let $R(K)$ (respectively $R(L)$) denote the ring of regular functions on γ^K (resp. γ^L), for details see [1]. It is a Dedekind domain (see [12, §III.2]) and can be explicitly written as

$$R(K) = \{f \in K : \operatorname{ord}_{\mathfrak{p}} f \geq 0 \text{ for all } \mathfrak{p} \in \gamma^K\}.$$

By [10, Corollary IV.3.3] and [3, Theorem 11.2] the following sequence is exact:

$$0 \to WR(K) \to WK \to \bigoplus_{\mathfrak{p}\in\gamma^K} WK(\mathfrak{p}) \to \mathbb{Z}^M \to 0, \tag{3.1}$$

where M is the number of semi-algebraically connected components of γ^K. Of course we can build a similar sequence for the field L, as well:

$$0 \to WR(L) \to WL \to \bigoplus_{\mathfrak{q}\in\gamma^L} WL(\mathfrak{q}) \to \mathbb{Z}^N \to 0. \tag{3.2}$$

Following the terminology of [2, 7] we shall say that a quaternion-symbol equivalence (t,T) is *integral* if the induced strong isomorphism of Witt rings $WK \to WL$ maps $WR(K)$ onto $WR(L)$. Here, as in the rest of this paper, we identify $WR(K)$ with its image in WK under (3.1). First let us note down an immediate observation.

Observation 3.3. *If the sequences* (3.1) *and* (3.2) *are isomorphic, then there exists a tame Harrison map* $\dot K/\dot K^2 \to \dot L/\dot L^2$.

Indeed, if (3.1) and (3.2) are isomorphic, then $M = N$, so γ^K and γ^L both have the same number of semi-algebraically connected components. Hence, [6, Corollary 3.9] implies the assertion.

Proposition 3.4. *Let* $t\colon \dot K/\dot K^2 \to \dot L/\dot L^2$ *be a tame Harrison map and let* $T\colon \gamma^K \to \gamma^L$ *be the associated bijection, then* (t,T) *is an integral quaternion-symbol equivalence.*

Proof. Fix a quadratic form φ over K and assume that $\varphi \in WR(K) \subset WK$. It follows from the exactness of (3.1) that $\partial_{\mathfrak{p}}\varphi = 0$ for every point $\mathfrak{p} \in \gamma^K$. Let $\varphi = \langle f_1,\dots,f_k\rangle + \langle p\rangle\langle f_{k+1},\dots,f_n\rangle$ with $\operatorname{ord}_{\mathfrak{p}} p \equiv 1 \pmod 2$ and $\operatorname{ord}_{\mathfrak{p}} f_i \equiv 0 \pmod 2$ for $1 \leq i \leq n$. Then $0 = \partial_{\mathfrak{p}}\varphi = \operatorname{sgn}\langle f_{k+1}(\mathfrak{p}),\dots,f_n(\mathfrak{p})\rangle$, now t being tame preserves local signs (see [6, Lemma 3.3]), hence

$$0 = \langle tf_{k+1}(T\mathfrak{p}),\dots,tf_n(T\mathfrak{p})\rangle = \partial_{T\mathfrak{p}} i_t(\varphi).$$

Thus, exactness of (3.2) implies that $i_t\varphi \in WR(L)$. □

It is natural to ask whether the opposite implication is also true: whether integrality implies tameness. Unfortunately, in general the answer is negative. To see this consider a bijection $T\colon \mathbb{P}^1(\mathbb{R})\setminus\{\pm1\} \to \mathbb{P}^1(\mathbb{R})\setminus\{\pm1\}$ of a projective line $\mathbb{P}^1(\mathbb{R})$ with points ±1 excluded, which inverts the interval $(-1,1)$. Namely,

$$T\,\big|_{(-1,1)} = -\operatorname{id}, \qquad T\,\big|_{\mathbb{P}^1(\mathbb{R})\setminus[-1,1]} = \operatorname{id}.$$

Now [6, §4] implies that T gives rise to an Harrison automorphism t of $\mathbb{R}(\dot{X})/\mathbb{R}(\dot{X})^2$. It is straightforward to check that (t,T) is integral but t maps

$$t\big(P_-(-1)\big) = P_-(-1) \qquad t\big(P_+(-1)\big) = P_-(1)$$
$$t\big(P_-(1)\big) = P_+(-1) \qquad t\big(P_+(1)\big) = P_+(1),$$

hence, it does not preserve 1-pt fans and so it is not tame.

Nevertheless, although we cannot expect an integral equivalence to be tame, the following existentional result does hold.

Proposition 3.5. *If there exists an integral quaternion-symbol equivalence of K and L, then there exists a tame Harrison map $\dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$.*

Before we proceed with the proof, let us first make a simple observation. Denote

$$\mathbb{E}(K) := \big\{ f \in \dot{K}/\dot{K}^2 : \operatorname{ord}_{\mathfrak{p}} f \equiv 0 \pmod 2 \quad \text{for every } \mathfrak{p} \in \gamma^K \big\}.$$

Now it follows from the exactness of (3.1):

Observation 3.6. *Let $f \in \dot{K}/\dot{K}^2$ be a square class. Then $f \in \mathbb{E}(K)$ if and only if $\langle f\rangle \in WR(K)$.*

Now equipped with the above observation we are ready to prove 3.5.

Proof. Assume that the asserted Harrison map does not exists. Hence, by [6, Corollary 3.9], the real curves γ^K and γ^L have different numbers of semi-algebraically connected components. Let $\gamma^K_1, \ldots, \gamma^K_M$ be the components of γ^K and $\gamma^L_1, \ldots, \gamma^L_N$ be the components of γ^L. Without loss of generality we may assume that $M > N$. By [6, §4] we can find finite subsets $S \subset \gamma^K, S' \subset \gamma^L$ and a quaternion-symbol equivalence (t,T) of K, L *with respect to* $(\gamma^K \setminus S, \gamma^L \setminus S')$. (This is the only part of this paper, where we consider a quaternion-symbol equivalence with respect to anything else than (γ^K, γ^L)).) It is tame at every point of $\gamma^K \setminus S$ (see [7, Theorem 3.1]), hence in particular it preserves local signs at those points: $\operatorname{sgn} f(\mathfrak{p}) = \operatorname{sgn}(tf)(T\mathfrak{p})$ for every $f \in \dot{K}/\dot{K}^2$ and $\mathfrak{p} \in \gamma^K \setminus S$.

By the means of [4, Theorem 2.10], for every $1 \le i \le M$, we can find a function $g_i \in K$ such that g_i is regular on γ^K_i, positive definite on $\gamma^K \setminus \gamma^K_i$ and negative definite on γ^K_i. In particular, by the above observation, $\langle g_i\rangle \in WR(K)$. Hence the integrality implies implies $\langle tg_i\rangle \in WR(L)$. Consequently, using again the above observation, we have $tg_i \in \mathbb{E}(L)$, which means that it has a constant sign on every component of γ^L.

Now, since γ^K has more components than γ^L, thus at least one component of γ^K must be mapped into more than one component of γ^K. To express this precisely: we can find $1 \le j,k \le M$ and $1 \le l \le N$ such that both $T\gamma^K_j \cap \gamma^L_l$ and $T\gamma^K_k \cap \gamma^L_l$ are open and non-empty. Therefore, tg_j is positive definite on $T\gamma^K_k \cap \gamma^L_l$ and negative definite on $T\gamma^K_j \cap \gamma^L_l$, so it changes sign on γ^L_l — a contradiction □

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