POLYNOMIALS WITH PRESCRIBED CYCLE LENGTHS

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Abstract. We discuss here the problem of the existence of polynomials with prescribed cycle lengths. From Šarkovskii's theorem we know that for every real polynomial f, the set of lengths of all cycles generated by f is of the form $\mathrm{Cycl}(f) = \{m \in \mathbb{N} \colon m \succcurlyeq n\}$ for some $n \in \mathbb{N}$, where $\succcurlyeq$ denotes Šarkovskii's ordering. We show that for every odd integer $n \geq 3$ there exist a polynomial f for which $\mathrm{Cycl}(f) = \{m \in \mathbb{N} \colon m \succcurlyeq n\}$. Moreover our proof works not only over $\mathbb{R}$ but over any arbitrary real closed field.

1. Preliminaries

The theory of polynomial cycles is an interesting inter-mixture of number theory and iteration theory, both having their roots in quite different disciplines. It has been studied recently by several authors. Recall that for any ring A, a finite subset $\{c_0, c_1, \dots, c_{n-1}\} \subseteq A$ is called a *cycle* if there exist a polynomial $f \in A[X]$ such that

$$f(c_0) = c_1,\ f(c_1) = c_2, \dots,\ f(c_{n-1}) = c_0.$$

The number n is called the *length* of the cycle. For basic results and terminology we refer the reader to [2]. One of fundamental questions in the theory of polynomial cycles is the following one: having a given polynomial f what are all lengths of cycles generated by f. This question has been answered over the reals by Šarkovskii. Order the set $\mathbb{N}$ of positive integers as follows:

$$3 \prec 5 \prec 7 \prec \dots \prec 2\cdot 3 \prec 2 \cdot 5 \prec \dots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec \dots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Theorem 1.1 (Šarkovskii)**.** *Let $f \in \mathbb{R}[X]$ and $a, b \in \mathbb{R}$, $a < b$. Assume that f maps the interval $[a, b]$ into itself. If f generates in $[a, b]$ a cycle of length n and $n \preccurlyeq m$, then f generates in $[a, b]$ a cycle of length m, as well.*

It has been pointed out by Professor Andrzej Sładek that the proof of this theorem presented in [1] remains valid when $\mathbb{R}$ is substituted by any real closed field $\mathbb{K}$. This leads us to the following extended version of Šarkovskii's theorem:

Theorem 1.2. *Let $\mathbb{K}$ be a real closed field, $f \in \mathbb{K}[X]$ and $a, b \in \mathbb{K}$, $a < b$. Assume that f maps the interval $[a, b]$ into itself. If f generates in $[a, b]$ a cycle of length n and $n \preccurlyeq m$, then f generates in $[a, b]$ a cycle of length m, as well.*

The above result provoked two questions, asked by Professor Sładek at the 4th Czech and Polish Conference on Number Theory:

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- Let $\mathbb{K}$ be a real closed field and $n \in \mathbb{N}$. Does a polynomial $f \in \mathbb{K}[X]$ such that $\operatorname{Cycl}(f) = \{m \in \mathbb{N} : \ m \succcurlyeq n\}$ exist?
- Is the degree of f above correlated to n?

We show in this paper (c.f. theorem 2.1) that the answer to the first question is affirmative at least for odd n, while the latter one is negative. Over the reals the answer to the first question is well known (see e.g. [1, Example I.2.12]) when f is allowed to be an arbitrary continuous function (in this case the second question does not make any sense). We are not aware, however, of any paper that would deal with this problem in the above sharper sense (i.e. requiring f to be a polynomial).

Let us introduce some notation we use extensively through the rest of this paper. For a polynomial f and an integer n, f^n always denotes the n-th iteration of f. For $a, b \in \mathbb{K}$ denote

$$\langle a, b\rangle := \operatorname{conv}\{a, b\} = \begin{cases} [a, b] & \text{if } a \le b \\ [b, a] & \text{if } a > b \end{cases}$$

Take a cycle $\{c_0, c_1, \dots, c_{n-1}\}$. We can find a permutation $\sigma \in S(n)$ such that $c_{\sigma(0)} < c_{\sigma(1)} < \cdots < c_{\sigma(n-1)}$. Construct a digraph in the following way.

- the vertices are the intervals $I_j := [c_{\sigma(j)}, c_{\sigma(j+1)}]$;
- the vertices are connected by an arc $I_j \to I_k$ if $I_k \subseteq \langle f(c_{\sigma(j)}), f(c_{\sigma(j+1)})\rangle$.

Choose once and for all the real closed field $\mathbb{K}$. Take $\Bbbk$ to be the real closure of $\mathbb{Q}$ in $\mathbb{K}$. The field $\Bbbk$ is isomorphic to the field $\overline{\mathbb{Q}} \cap \mathbb{R}$ of real algebraic numbers. The following observation allows us to reduce the whole discussion to $\Bbbk$.

Observation 1.3. *If $\Bbbk \subseteq K$ are formally real fields, $\Bbbk$ is real closed and $f \in \Bbbk[X]$, then every cycle of f lying in K must lie already in $\Bbbk$.*

With the above observation we can specify to our context two results form [1]:

Lemma 1.4 ([1, Lemma I.1.6])**.** *Suppose $f \in \Bbbk[X]$ generates in the interval $[0, 1]$ a cycle of length $n > 1$. If the associated digraph contains a primitive cycle $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$ of length m, then there exists $y \in [0, 1]$ such that $f^m(y) = y$ and $f^k(y) \in J_k$ for $0 \le k < n$.*

Lemma 1.5 ([1, Proposition I.1.8])**.** *Suppose $f \in \Bbbk[X]$ generates in the interval $[0, 1]$ a cycle of an odd length $n > 1$ but not of any odd length strictly between 1 and n. If c is the midpoint of the cycle of odd length n, then the points of this cycle have the order*

$$f^{n-1}(c) < f^{n-3}(c) < \ldots < f^2(c) < c < f(c) < \ldots < f^{n-2}(c)$$

or the reverse order

$$f^{n-2}(c) < \ldots < f(c) < c < f^2(c) < \ldots < f^{n-3}(c) < f^{n-1}(c).$$

In either case the associated digraph is given by:

$$\begin{tikzcd} J_1 \arrow[loop left] \arrow[r] & J_2 \arrow[r] & J_3 \arrow[r] & \cdots \arrow[r] & J_{n-3} \arrow[r] & J_{n-2} \arrow[r] & J_{n-1} \arrow[llllll, bend left] \arrow[llll, bend left] \arrow[l, bend left] \end{tikzcd}$$

where $J_1 = \langle c, fc\rangle$ and $J_k = \langle f^{k-2}(c), f^k(c)\rangle$ for $1 < k < n$.

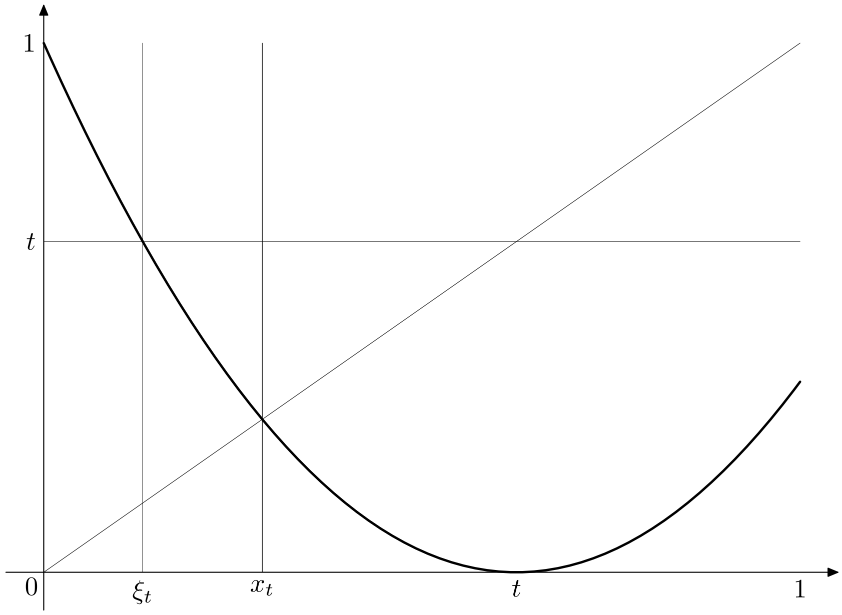

FIGURE 1. Graph of f_t

2. Main result

Our main result is the following theorem.

Theorem 2.1. *Let $\mathbb{K}$ be a real closed field. Then for every odd integer $n \geq 5$ and any even $k \geq 2$ there exist a polynomial $f \in \mathbb{K}[X]$ of degree $\deg(f) = k$ such that f generates in the interval $[0,1]$ a cycle of length n and not of length $n-2$.*

Fix an even degree $k > 0$. Using Observation 1.3 we need only to prove this theorem for the field $\Bbbk$. In the proof we consider polynomials of a special form. Denote $f_t(X) := \left(\frac{X}{t} - 1\right)^k \in \Bbbk[X]$ for $t \in (\frac{1}{2}, 1) \cap \Bbbk$. Figure 1 shows the graph of f_t. We enumerate some obvious observations concerning f_t.

(i) f_t maps $[0,1]$ onto $[0,1]$;
(ii) f_t is decreasing on $[0,t]$ and increasing on $[t,1]$;
(iii) $f_t(0) = 1,\ f_t(t) = 0,\ f_t(1) < 1$;
(iv) there exist exactly one $x_t \in [0,1]$ such that $f_t(x_t) = x_t$; moreover $x_t < t$;
(v) there exist exactly one $\xi_t \in [0,t]$ such that $f_t(\xi_t) = t$; moreover $\xi_t < x_t$;
(vi) in every cycle generated by f_t at least one element lies in $(t,1)$.

Let us start with the following simple lemma giving a sufficient condition for the existence of cycles of a given length.

Lemma 2.2. *Let $n \geq 3$ be an odd integer. If $f_t^n(t) \geq t$, then f_t generates a cycle of length n.*

Proof. Denote $\Delta(x) := f_t^n(x) - x$. We have $\Delta(t) \geq 0$, but on the other hand $\Delta(x_t) = 0$ since x_t is a fixed point of f_t. In the view of $(f_t^n)'(x_t) = f_t'(x_t) \cdot (f_t^{n-1})'(x_t)$ we have $(f_t^n)'(x_t) \cdot (f_t^{n-1})'(x_t) < 0$, and since $f_t'(x_t)$ is negative, for odd n the polynomial f_t^n decreases in a neighborhood of x_t. Therefore, there exist $x > x_t$ for which $\Delta(x) < 0$.

The polynomial $\Delta \in \mathbb{k}[X]$ has the Darboux property in $\mathbb{k}$, hence $\Delta(y) = 0$ for some $y \in (x, t]$. It follows that f_t generates a cycle of length $m > 1$, where m divides n. Šarkovskii's theorem implies that f_t generates a cycle of length n. $\square$

Lemma 2.3. *If $f_t(1) > x_t$ for some $t \in (\frac{1}{2}, 1) \cap \mathbb{k}$, then f_t generates a cycle of an odd length.*

Proof. If for some odd $n \geq 3$ one has $f_t^n(t) \geq t$, then Lemma 2.2 is applicable, hence we may assume that for such n one has $f_t^n(t) < t$. In particular one has $f_t^3(t) = f_t(1) < t$. Observe now that for $i \geq 3$ we have

$$\begin{cases} f_t^i(t) \in (\xi_t, x_t), & \text{if } i \text{ is even} \\ f_t^i(t) \in (x_t, t), & \text{otherwise.} \end{cases}$$

It is straightforward to check that $f_t^2(x) - x$ is positive and increasing on (x_t, t), and it follows that the sequence $c_i = f_t^{i+2}(t) - f_t^i(t)$ increases and is bounded from below. Since $\mathbb{k}$ is archimedean, thus for some odd i we get $f_t^i(t) > t$, contrary to our assumption. $\square$

Lemma 2.4. *Suppose that f_t generates a cycle of an odd length. Denote by $n > 1$ the length of the shortest odd cycle (i.e. n is the smallest length in the sense of Šarkovskii's ordering), finally, let c be the midpoint of a cycle of length n. Then $f_t(c) < x_t < c$. Moreover, c is the smallest element of the cycle laying in $(x_t, 1)$.*

Proof. By means of Lemma 1.5 we have a loop in the digraph of f_t of the form $\langle c, f_t(c)\rangle \to \langle c, f_t(c)\rangle$. Lemma 1.4 implies that there is a fixed point in $\langle c, f_t(c)\rangle$ and since x_t is the only fixed point of f_t in $[0, 1]$, therefore, either $f_t(c) < x_t < c$ or $c < x_t < f_t(c)$. We show that the latter case is impossible. Indeed, suppose that $c < x_t < f_t(c)$, again by Lemma 1.5 we have

$$f_t^{n-1}(c) < f_t^{n-3}(c) < \cdots < f_t^2(c) < c < f_t(c) < \cdots < f_t^{n-2}(c).$$

We know that at least one of the above elements is greater than t (cf. (vi)). Hence, in particular $f_t^{n-2}(c) > t$. Consequently, $\xi_t > f_t^{n-3}(c) > f_t^{n-1}(c)$ and so $c = f_t^n(c) > t$. But $x_t < t$ by (iv) and this contradicts our assumption that $c < x_t$. $\square$

Corollary 2.5. *If $t \in \mathbb{k}$, then the polynomial f_t generates a cycle of an odd length if and only if $f_t(1) > x_t$.*

Proof. The sufficiency of the stated condition has already been proved in Lemma 2.3. We show the other implication. Assume that f_t generates a cycle of an odd length and let n, c be as in the above lemma. Then $x_t < c = f_t^n(c)$. Now Lemma 1.5 implies that $f_t^{n-1}(c) \in (t, 1)$ where f_t is increasing, hence $x_t < f_t\big(f_t^{n-1}(c)\big) < f_t(1)$. $\square$

Observe that the proof of the sufficiency condition does not require t to belong to $\mathbb{k}$.

Corollary 2.6. *There exist $t_0 \in (\frac{1}{2}, 1)$ — depending only on the degree k — such that for every $t \in (\frac{1}{2}, t_0) \cap \mathbb{k}$, the polynomial f_t generates a cycle of an odd length.*

For $k = 2$ one can compute t_0 precisely. The following result was obtained by a computer algebra system [3]:

$$t_0 = \frac{1}{3} - \frac{2^{\frac{5}{3}}}{3\,\left(13 + 3\sqrt{33}\right)^{\frac{1}{3}}} + \frac{\left(13 + 3\sqrt{33}\right)^{\frac{1}{3}}}{3 \cdot 2^{\frac{2}{3}}} \simeq 0.6478$$

Corollary 2.7. *If for $t \in \Bbbk$ we put $n(t) := \min\{n \in 2\mathbb{N}+1\colon f_t^n(1) \geq t\}$, then*

$$\lim_{t \to t_0^-} n(t) = \infty.$$

Moreover, $n(t)$ is non-decreasing and assumes every odd integer ≥ 3.

Now add another obvious observation to our list:

(vii) if $t \in (\frac{1}{2}, t_0)$, then there exist exactly one $y_t \in (t, 1)$ such that $f_t(y_t) = x_t$;

Lemma 2.8. *Assume $t \in (\frac{1}{2}, t_0)$ and let n, c be as in Lemma 2.4. In the cycle $\{c, f_t(c), \ldots, f_t^{n-1}(c)\}$:*

- *$f_t^{n-1}(c)$ is the only element $> y_t$;*
- *there are no more than two elements $> t$, these are $f_t^{n-3}(c), f_t^{n-1}(c)$;*
- *there are no more than two elements $< \xi_t$, these are $f_t^{n-4}(c), f_t^{n-2}(c)$.*

Proof. The second assertion is equivalent with the third. Moreover, Lemma 2.4 together with Lemma 1.5 implies that $f_t^{n-1}(c) > y_t$. The fact that this is the only such element follows immediately from Lemma 2.4. Thus we need only to show that $f_t^{n-5}(c) < t$. Suppose $f_t^{n-5}(c) \geq t$. Lemma 1.5 implies that $f_t^{n-5}(c) < f_t^{n-3}(c)$ and since f_t is increasing on $[t, 1]$ we have

$$f_t^{n-4}(c) = f_t\Big(f_t^{n-5}(c)\Big) < f_t\Big(f_t^{n-3}(c)\Big) = f_t^{n-2}(c).$$

This, however, contradicts Lemma 2.4. □

Lemma 2.9. *For every odd integer $n \geq 3$ there exist such an element $t \in (\frac{1}{2}, t_0) \cap \Bbbk$ that $t \leq f_t^{n-2}(1) < y_t$. Moreover, if $n \geq 5$ then also $f_t^{n-4}(1) < t$.*

Proof. Corollary 2.7 implies that for every fixed odd n there exists $t' \in (\frac{1}{2}, t_0)$ such that n is the smallest odd integer for which $f_{t'}^{n-2}(1) \geq t'$. By Corollary 2.7 we can find $t'' \in (\frac{1}{2}, t_0)$, $t'' > t'$ such that n is the smallest odd integer having the property $f_{t''}^{n}(1) \geq t''$. In particular, $f_{t''}^{n-2}(1) < y_{t''}$. Thus there is some $t \in (t', t'')$ for which $f_t^{n-2}(1) < y_t$. Then also $f_t^{n-2}(1) \geq t$ and $f_t^{n-4}(1) < t$ since $n-2$ and n are minimal for t' and t'' respectively. □

Lemma 2.10. *For every odd integer $n \geq 3$ there exist such an element $t \in (\frac{1}{2}, t_0)$ that the polynomial f_t generates a cycle of length n but not of any odd length strictly between 1 and n.*

Proof. Fix an integer n and take t as in the previous lemma. Lemma 2.2 implies that f_t generates a cycle of length n. If $n = 3$ this ends the proof. Assume that $n > 3$. We show that f_t does not generate a cycle of length $n-2$. Indeed, suppose that it does. Let n_0 be the smallest, in the sense of Šarkovskii's ordering, length of the cycle generated by f_t. Thus $n_0 \leq n-2$. Take c to be the midpoint of a cycle of length n_0, hence $c > x_t$ by Lemma 2.4. On the other hand, (vi) implies that $c < f_t(1)$. Thus $c \in (x_t, f_t(1))$. Take $i \in 2\mathbb{N}$, $i \geq 2$ such that $f_t^i(c) \in (f_t(1), f_t^3(1))$, now for every j we have

$$f_t^{j+i}(c) \in \langle f_t^{j+1}(1), f_t^{j+3}(1) \rangle.$$

In particular

$$f_t^{n-5+i}(c) < f_t^{n-2}(1) < y_t.$$

Now Lemma 1.5 together with properties (ii) and (vi) imply that

$$f_t^{n_0-1}(c) > y_t,$$

consequently

$$n_0 > n - 4 + i \geq n - 2.$$

This is impossible since $n_0 \leq n - 2$. □

The proof of theorem 2.1 now follows immediately from the above lemma and Observation 1.3. For $\mathbb{K} = \mathbb{R}$ and small k's and n's one can compute appropriate t's explicitly. The following examples where obtained with Mathematica [3]:

degree k	minimal length n	parameter t
2	5	$19/32$
2	7	$5/8$
2	9	$41/64$
4	5	$9/16$
4	7	$19/32$
4	9	$153/256$

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