MATCHING LOCAL WITT INVARIANTS

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Abstract. The starting point of this note is the observation that the local condition used in the notion of a Hilbert-symbol equivalence and a quaternion-symbol equivalence — once it is expressed in terms of a Witt invariant — admits a natural generalisation. In this paper we show that for global function fields as well as the formally real function fields over a real closed field all the resulting equivalences coincide.

1. Introduction and notation

The ultimate question in the algebraic theory of quadratic-forms is whether Witt rings of two given fields are isomorphic. Such fields are then said to be Witt equivalent. This subject has been investigated by many authors. In particular the study of Witt equivalence of global fields resulted in the notion of a Hilbert-symbol equivalence (c.f. [6, 7, 8, 9, 10, 1]). A slight variation of it, called a quaternion-symbol equivalence, was used to investigate Witt equivalence of algebraic function fields. This two terms refer to a pair of maps (one being an isomorphism of square class groups, the other a bijection of sets of points) preserving a splitting of local Witt invariants of binary forms. It is thus natural to ask whether substituting binary forms in the definition leads to a different theory or not.

In this paper we show that for a class of global function fields of characteristic $\neq 2$ as well as for a class of formally real algebraic function fields over a fixed real closed field of constants the local condition can be changed to the one concerning n-dimensional forms for any $n \geq 2$ without changing the resulting theory. This is quite intuitive for global function fields, as it is well known that every anisotropic quadratic form over such a field is of dimension ≤ 4 (c.f. [5, Corollary VI.3.5]). The result may seem a bit more surprising for real function fields, though. Here it follows from the fact that any such an equivalence is tame (i.e. preserves the parity of a valuation).

Throughout the paper letters K, L are used to denote algebraic function fields of characteristics $\neq 2$ over either finite fields or a fixed real closed field. For a function field K we denote by $\Omega(K)$ the set of all points of K trivial on its field of constants. If K is a real function field we further denote γ^K the set of all points having a formally real residue field. Next, for a given point $\mathfrak{p} \in \Omega(K)$ symbols $K_\mathfrak{p}$, $\mathrm{ord}_\mathfrak{p}$, $\theta_\mathfrak{p}$ and $\Theta_\mathfrak{p}$ denote respectively: the completion of K at $\mathfrak{p}$, the valuation associated with $\mathfrak{p}$, the canonical epimorphism $\dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ and the canonical epimorphism $WK \to WK_\mathfrak{p}$. Moreover, in order to simplify the notation, we use the same symbol f to denote both: the element of the field as well as its square class. In a similar fashion $\left(\frac{f,g}{K}\right)$ denotes — depending on the context — either a quaternion algebra or its class in the Brauer group $\mathrm{Br}(K)$ of the field K. Finally $s(K)$ is the level of K (c.f. [5, ch. XI, sec. 2]) and $c(\varphi)$ is the Witt invariant of the form φ (c.f. [5, ch. VI, sec. 3]).

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2. Preliminaries

Recall (see [3, 2, 4]) that a quaternion-symbol equivalence of K and L (with respect to $A \subseteq \Omega(K)$ and $B \subseteq \Omega(L)$) is a pair of maps (t, T) in which $t\colon \dot{L}/\dot{L}^2 \to \dot{L}/\dot{L}^2$ is an isomorphism of square class groups, while $T\colon A \to B$ is a bijection and such that

$$\left(\frac{f,g}{K_\mathfrak{p}}\right) = 1 \in \mathrm{Br}(K_\mathfrak{p}) \iff \left(\frac{tf,tg}{L_T\mathfrak{p}}\right) = 1 \in \mathrm{Br}(L_{T\mathfrak{p}}) \tag{2.1}$$

for all square classes $f, g \in \dot{K}/\dot{K}^2$ and every point $\mathfrak{p} \in A$. In a special case, when K, L are global fields and $A = \Omega(K), B = \Omega(L)$ it is called a Hilbert-symbol equivalence (see [6, 7, 9]). Observe that a quaternion algebra $\left(\frac{f,g}{K_\mathfrak{p}}\right)$ is the value of the Witt invariant (see [5, Ch. V, sec. 3]) of the element $\langle f, g\rangle \in WK_\mathfrak{p}$ of the Witt ring of $K_\mathfrak{p}$. Therefore the condition (2.1) can be rewritten as

$$c_\mathfrak{p}\langle f, g\rangle = 1 \iff c_{T\mathfrak{p}}\langle tf, tg\rangle = 1.$$

To simplify the notation $c_\mathfrak{p}\colon WK \to \mathrm{Br}_2 K_\mathfrak{p}$ (resp. $c_{T\mathfrak{p}}\colon WL \to \mathrm{Br}_2 L_{T\mathfrak{p}}$) denotes here the composition of the Witt invariant with the canonical epimorphism $\Theta_\mathfrak{p}\colon WK \to WK_\mathfrak{p}$ (resp. $\Theta_{T\mathfrak{p}}\colon WL \to WL_{T\mathfrak{p}}$), so in fact $c_\mathfrak{p}$ should read $c \circ \Theta_\mathfrak{p}$. The above formula suggests a natural generalisation of the above condition to

$$c_\mathfrak{p}\langle f_1, \dots, f_n\rangle = 1 \iff c_{T\mathfrak{p}}\langle tf_1, \dots, tf_n\rangle = 1.$$

This leads us directly to the following definition:

Definition 2.2. Let $N \in \mathbb{N}$, $N \geq 2$ be a fixed integer and let $A \subseteq \Omega(K), B \subseteq \Omega(L)$ be two fixed sets of points of K, L. The pair of maps (t, T) in which $t\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ is an isomorphism such that $t(-1) = -1$ and $T\colon A \to B$ is a bijection is called an N-*equivalence* of K and L with respect to the pair (A, B) if it preserves local Witt invariants in the sense that

$$c_\mathfrak{p}\langle f_1, \dots, f_N\rangle = 1 \in \mathrm{Br}_2(K_\mathfrak{p}) \iff c_{T\mathfrak{p}}\langle tf_1, \dots, tf_N\rangle = 1 \in \mathrm{Br}_2(L_{T\mathfrak{p}})$$

for all square classes $f_1, \dots, f_N \in \dot{K}/\dot{K}^2$ and every point $\mathfrak{p} \in A$.

In this paper we are interested only in global and real function fields, hence we implicitly assume that if K, L are global function fields then $A = \Omega(K), B = \Omega(L)$, on the other hand if K, L are formally real function fields over a real closed field of constants, then $A = \gamma^K, B = \gamma^L$. Thus, in what follows we drop out the phrase 'with respect to the pair (A, B)'. Obviously a 2-equivalence is just a quaternion-symbol equivalence (Hilbert-symbol equivalence when K, L are global).

In general it is rather improbable that N-equivalences for different N's coincide. However it is so for the class of fields we study here. First observe that a "higher" equivalence implies a "lower" one. Namely, taking $\langle f_1, \dots, f_N, 1, -1\rangle$ we see:

Observation 2.3. *If (t, T) is an $(N + 2)$-equivalence than it is an N-equivalence.*

To examine other dependencies we however need to specify the field to work with. Thus, from here on we separately discuss global fields and function fields over a real closed field.

3. Global function fields

In this whole section K, L are global function fields of characteristic $\neq 2$. Recall (see [5, Theorem VI.2.2]) that a 2-torsion subgroup of a Brauer group of a $\mathfrak{p}$-adic local field consists of just two elements: the unit being a class of $\left(\frac{1,-1}{K_\mathfrak{p}}\right)$ and the non-unit element being the class of the unique non-split quaternion algebra $\left(\frac{u,p}{K_\mathfrak{p}}\right)$

with $\operatorname{ord}_{\mathfrak{p}} u \equiv 0 \pmod 2$ and $\operatorname{ord}_{\mathfrak{p}} p \equiv 1 \pmod 2$. In particular, this means that for every $\mathfrak{p} \in \Omega(K)$ a 2-equivalence (t, T) (i.e. a Hilbert-symbol equivalence) induces an isomorphism $\operatorname{Br}_2 K_{\mathfrak{p}} \to \operatorname{Br}_2 L_{T\mathfrak{p}}$ sending $\left(\frac{f,g}{K_{\mathfrak{p}}}\right)$ to $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$.

Lemma 3.1. *If (t, T) is a 2-equivalence, then it is an N-equivalence for every $N \geq 2$.*

Proof. We treat separately even and odd N's. Assume that N is even. By [5, V.3.13] we can write

$$c_{\mathfrak{p}}\langle f_1, \ldots, f_{N-2}, f_{N-1}, f_N\rangle = \\ = c_{\mathfrak{p}}\langle f_1, \ldots, f_{N-2}\rangle \cdot c_{\mathfrak{p}}\langle f_{N-1}, f_N\rangle \cdot c_{\mathfrak{p}}\big\langle \operatorname{disc}\langle f_1, \ldots, f_{N-2}\rangle, \operatorname{disc}\langle f_{N-1}, f_N\rangle\big\rangle.$$

By a simple induction we can decompose the above formula into a product of Witt invariants of binary forms. The 2-equivalence induces an isomorphism $\operatorname{Br}_2 K_{\mathfrak{p}} \to \operatorname{Br}_2 L_{T\mathfrak{p}}$, hence the product is preserved when passing to $L_{T\mathfrak{p}}$. Recomposing it back into $c_{T\mathfrak{p}}\langle tf_1, \ldots, tf_N\rangle$ we get the assertion.

For N odd first decompose $c_{\mathfrak{p}}\langle f_1, \ldots, f_N\rangle$ by [5, V.3.13]:

$$c_{\mathfrak{p}}\langle f_1, \ldots, f_{N-1}, f_N\rangle = \\ = c_{\mathfrak{p}}\langle f_1, \ldots, f_{N-1}\rangle \cdot c_{\mathfrak{p}}\langle f_N\rangle \cdot c_{\mathfrak{p}}\langle \operatorname{disc}\langle f_1, \ldots, f_{N-1}\rangle, -\operatorname{disc}\langle f_N\rangle\rangle = \\ = c_{\mathfrak{p}}\langle f_1, \ldots, f_{N-1}\rangle \cdot c_{\mathfrak{p}}\langle \operatorname{disc}\langle f_1, \ldots, f_{N-1}\rangle, -f_N\rangle$$

and then proceed as in the even case. □

Alternatively, one can show the above lemma using the fact that a Hilbert-symbol equivalence induces local Witt equivalence (see [7, Proposition 1.4]).

Combining the above lemma with the observation 2.3, we see that any even equivalence of global function fields implies all other equivalences. Hence, the next obvious step is to examine a 3-equivalence.

Lemma 3.2. *If (t, T) is a 3-equivalence, then it is a 2-equivalence (hence also an N-equivalence for every $N \geq 2$).*

Proof. Fix a point $\mathfrak{p} \in \Omega(K)$ and consider two cases. First assume that $-1 \notin K_{\mathfrak{p}}^2$. Take a form $\langle p, -1, -1\rangle$ with p a fixed uniformizer. By [5, Proposition V.3.22] we know that a ternary form has a trivial Witt invariant iff it is isotropic. Hence $c_{\mathfrak{p}}\langle p, -1, -1\rangle \neq 1$ and so $c_{T\mathfrak{p}}\langle tp, t(-1), t(-1)\rangle = c_{T\mathfrak{p}}\langle tp, -1, -1\rangle \neq 1$. From this we see two things: first the form $\langle tp, -1, -1\rangle$ is anisotropic, hence $-1 \notin L_{T\mathfrak{p}}^2$, which means that a 3-equivalence preserves local levels. Second, we have $\operatorname{ord}_{T\mathfrak{p}} tp \equiv 1 \pmod 2$, so it preserves parity of valuation. Using a terminology established for a Hilbert-symbol equivalence we would say that it is 'tame'.

In turn assume that $-1 \in K_{\mathfrak{p}}^2$. From the previous part we see that also $-1 \in L_{T\mathfrak{p}}^2$. We claim that a form $\langle f, g, fg\rangle$ is anisotropic over $K_{\mathfrak{p}}$ if and only if $\langle f, g, fg\rangle = \langle u, p, up\rangle$, here as usually $\operatorname{ord}_{\mathfrak{p}} u \equiv 0 \pmod 2$ and $\operatorname{ord}_{\mathfrak{p}} p \equiv 1 \pmod 2$. Indeed, by [5, Corollary VI.2.5(3)] we obtain that an anisotropic ternary form φ over $K_{\mathfrak{p}}$ represents all square classes of $K_{\mathfrak{p}}$ except $-\det\varphi = \det\varphi$. Now there are exactly four anisotropic ternary forms over $K_{\mathfrak{p}}$ (since $-1 \in K_{\mathfrak{p}}^2$). These are:

$$\langle 1, u, p\rangle, \quad \langle 1, u, up\rangle, \quad \langle 1, p, up\rangle, \quad \langle u, p, up\rangle.$$

All but the last represents $1 = \det\langle f, g, fg\rangle$. This proves our claim.

Take now two square classes $f, g \in \dot{K}/\dot{K}^2$ and assume that $\left(\frac{f,g}{K_{\mathfrak{p}}}\right) \neq 1$. The only non-split quaternion algebra over $K_{\mathfrak{p}}$ is $\left(\frac{u,p}{K_{\mathfrak{p}}}\right)$. Hence $\langle 1, f, g, fg\rangle = \langle 1, u, p, up\rangle$. Cancelling $\langle 1\rangle$ we have $\langle f, g, fg\rangle = \langle u, p, up\rangle$ over $K_{\mathfrak{p}}$. Now [5, Propositon V.3.22] implies $c_{\mathfrak{p}}\langle f, g, fg\rangle \neq 1$, hence $c_{T\mathfrak{p}}\langle tf, tg, tftg\rangle \neq 1$. Thus $\langle tf, tg, tftg\rangle$ is anisotropic over $L_{T\mathfrak{p}}$ and so by our claim $\langle tf, tg, tftg\rangle = \langle v, q, vq\rangle$ where $\operatorname{ord}_{T\mathfrak{p}} v \equiv 0 \pmod 2$ and q is a class of a $T\mathfrak{p}$-adic uniformizer. Hence $\langle 1, tf, tg, tftg\rangle = \langle 1, v, q, vq\rangle$ and so $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right) \neq 1$. □

All in all we have just proved:

Proposition 3.3. *Let K, L be global function fields, for any $N, M \geq 2$ the pair (t, T) is an N-equivalence if and only if it is an M-equivalence.*

4. Real function fields

In this section K and L are two formally real algebraic function fields over a fixed real closed field $\Bbbk$. Recall that in this case we default an N-equivalence to be taken with respect to (γ^K, γ^L).

Fix $N \geq 2$ and let (t, T) be an N-equivalence, for a given square class $f \in \dot{K}/\dot{K}^2$ consider a quadratic form $\varphi_f := \langle f, -1\rangle$ if N is even and $\varphi_f := \langle f, -1, -1\rangle$ if N is odd. Padding φ_f with hyperbolic planes we arrive at a form of dimension N laying in the same Witt class. Now for any point $\mathfrak{p}$ we have $\left(\frac{f,-1}{K_\mathfrak{p}}\right) = c_\mathfrak{p}\langle f, -1\rangle = 1$ iff f is a square in $K_\mathfrak{p}$. Similarly, using [5, Proposition V.3.22] we have $c_\mathfrak{p}\langle f, -1, -1\rangle = 1$ iff $\langle f, -1, -1\rangle$ is isotropic over $K_\mathfrak{p}$ iff f is a square in $K_\mathfrak{p}$. Hence,

$$f \in K_\mathfrak{p}^2 \iff c_\mathfrak{p}\varphi_f = 1 \iff c_{T\mathfrak{p}}\varphi_{tf} = 1 \iff tf \in L_{T\mathfrak{p}}^2,$$

where the middle equivalence follows from the very definition of an N-equivalence. Thus we have just proved:

Observation 4.1. *Any N-equivalence preserves local squares.*

The above observation implies that the mapping $\left(\frac{f,g}{K_\mathfrak{p}}\right) \mapsto \left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ is well defined; denote it by Υ. Observe that by Springer theorem (see [5, Proposition VI.1.9]) a quaternion algebra $\left(\frac{f,g}{K_\mathfrak{p}}\right)$ splits if and only if at least one of the following holds:

$$f \in K_\mathfrak{p}^2 \quad \text{or} \quad g \in K_\mathfrak{p}^2 \quad \text{or} \quad -fg \in K_\mathfrak{p}^2.$$

The previous observation implies that this is possible if and only if at least one f the following holds:

$$tf \in L_{T\mathfrak{p}}^2 \quad \text{or} \quad tg \in L_{T\mathfrak{p}}^2 \quad \text{or} \quad -tftg \in L_{T\mathfrak{p}}^2,$$

which means that $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ splits. Hence Υ maps a unit element of $\operatorname{Br}_2 K_\mathfrak{p}$ onto a unit element of $\operatorname{Br}_2 L_{T\mathfrak{p}}$. Now since $\operatorname{Br}_2 K_\mathfrak{p} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \operatorname{Br}_2 L_{T\mathfrak{p}}$, the map Υ is in fact an isomorphism.

Take now $N, M \geq 2$. Let (t, T) be an N-equivalence and $\varphi = \langle f_1, \ldots, f_M\rangle$ any M-dimensional quadratic form. By the means of [5, V.3.13], we can decompose the Witt invariant $c_\mathfrak{p}\varphi$ of φ into a product of quaternion algebras

$$c_\mathfrak{p}\varphi = \prod_{i,j} \left(\frac{\pm f_i, \pm f_j}{K_\mathfrak{p}}\right).$$

The isomorphism Υ induced by (t, T) carries this product to $\operatorname{Br}_2 L_{T\mathfrak{p}}$:

$$\begin{aligned}\Upsilon\big(c_\mathfrak{p}\langle f_1, \ldots, f_M\rangle\big) &= \\ = \Upsilon\Bigg(\prod_{i,j}\left(\frac{\pm f_i, \pm f_j}{K_\mathfrak{p}}\right)\Bigg) &= \prod_{i,j}\Upsilon\left(\frac{\pm f_i, \pm f_j}{K_\mathfrak{p}}\right) = \prod_{i,j}\left(\frac{\pm tf_i, \pm tf_j}{L_{T\mathfrak{p}}}\right) = \\ &= c_{T\mathfrak{p}}\langle tf_1, \ldots, tf_M\rangle\end{aligned}$$

Hence we have just proved the real analogue of 3.3

Proposition 4.2. *Let K, L be two formally real algebraic function fields over a real closed field $\Bbbk$, for any $N, M \geq 2$ the pair (t, T) is an N-equivalence if and only if it is an M-equivalence.*

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