RELATIVE DETERMINANT OF A BILINEAR MODULE

PRZEMYSŁAW KOPROWSKI

Abstract. This note proposes a new definition of the determinant of a symmetric bilinear form on an finitely generated projective module (over a domain) which is not necessarily free.

The notion of the determinant is fundamental to (linear) algebra. Its most basic variant is the determinant of matrices and endomorphisms of vector spaces. This generalizes naturally to the determinant of endomorphisms on free modules, since free modules (like vector spaces) have bases and associated coordinate systems. In 1960s, Goldman showed that it is possible to generalize the notion of the determinant to endomorphisms of projective modules, that in general lack coordinates. Another standard application of the determinant is in the theory of bilinear forms over fields. It is well known that the determinant of a bilinear form is non-zero if and only if the form is non-degenerate. This can be even taken as a definition of non-degeneracy over fields. Next fundamental property is that the determinant factors over orthogonal sums. Like in the case of endomorphisms, the notion of the determinant generalizes naturally to bilinear forms on free modules (see e.g. [Mar85]). However, to the best of our knowledge, the notion has not been generalized to forms on arbitrary projective modules. Some efforts in this direction may be found in [Pet08], but under quite a strong assumption, that the determinant bundle of the module in question is free. The notion of the determinant in [Pet08] is also implicitly relative to an isomorphism from the determinant bundle to the base ring. Such a relativity with respect to some reference object seems to be intrinsic to the notion of the determinant of forms on projective modules and is present also in our approach.

In this paper, we propose another definition of the determinant of a bilinear form on a finitely generated projective module. Our strategy consists of three steps. First, we define a *relative determinant* of one form with respect to another (necessarily non-degenerate) "reference form" on the same module. In the second step, we show that, for non-degenerate forms, instead of having separate "reference forms" on each module, it suffices to have "reference isomorphism" only for line bundles (with rank ≤ 2 in the Picard group of R). In addition, if R is a domain, then all one needs is one global "reference object" in form of a certain semi-group homomorphism. Finally, in the last step, using the notion of the relative determinant, we extend the definition of the determinant to all forms, including degenerate ones. We also show that with such a definition, the determinant still has the two basic properties: the form is non-degenerate if and only if its determinant is invertible (see Corollary 16) and the determinant factors over orthogonal sums (see Proposition 17).

The notation utilized throughout this note is conventional. For definitions of used terms, we refer the reader to standard textbooks like e.g. [Mar85, MH73] (for the theory of bilinear forms) and [Wei13] (for the terms from K-theory). In particular, all rings here are always commutative, associative and has 1. If M

2010 *Mathematics Subject Classification.* 11E39, 15A63, 13C10.
Key words and phrases. determinant, bilinear forms, projective modules.

is a finitely generated projective R-module, by $M^\vee$ we denote its dual module $M^\vee = \operatorname{Hom}_R(M, R)$. For a symmetric bilinear form $\xi : M \times M \to R$ on M, $\hat{\xi} : M \to M^\vee$ denotes the adjoin homomorphism (i.e. $\big(\hat{\xi}(u)\big)(v) = \xi(u, v)$). The form is said to be *non-degenerate* if $\hat{\xi}$ is an isomorphism.

1. Relative determinant

We begin by defining a relative determinant of a general form with respect to a a non-degenerate "reference form". In this section R is an arbitrary commutative ring and M is a finitely generated projective module of a constant rank. Let $\zeta : M \times M \to R$ be a fixed non-degenerate symmetric bilinear form on M. Take another form $\xi : M \times M \to R$ on the same module M (we do not make any assumptions about non-degeneracy of ξ) and consider an endomorphism $\Delta_{\zeta,\xi} := \hat{\zeta}^{-1} \circ \hat{\xi}$ of M.

Observation 1. *The endomorphism of $\Delta_{\zeta,\xi}$ associated to the pair (ζ, ξ) satisfies the following identity*

$$\xi(u, v) = \zeta\big(u, \Delta_{\zeta,\xi}(v)\big), \qquad \text{for every } u, v \in M.$$

Lemma 2. *The form ξ is non-degenerate if and only if $\Delta_{\zeta,\xi}$ is an automorphism.*

Proof. If ξ is non-degenerate, then $\hat{\xi} : M \to M^\vee$ is an isomorphism. Consequently, $\Delta_{\zeta,\xi}$ is an automorphism as a composition of two isomorphisms. Conversely, if $\Delta_{\zeta,\xi}$ is an automorphism, then $\hat{\xi} = \hat{\zeta} \circ \Delta_{\zeta,\xi}$ is an isomorphism and so ξ is non-degenerate. □

Recall (see e.g. [Wei13, Chapter I, § 3]) that the determinant bundle of a finitely generated projective module M of a constant rank is defined as the highest exterior power of M, namely

$$\det M := \underbrace{M \wedge \cdots \wedge M}_{n \text{ times}}, \qquad \text{where } n = \operatorname{rank} M.$$

By functoriality, any endomorphism $\varphi \in \operatorname{End} M$ induces an endomorphism $\wedge^n\varphi$ of $\det M$ of the form $(\wedge^n\varphi)(x) = d \cdot x$ for a unique element $d \in R$, depending only on φ (because $\det M$ is a line bundle). This element is called the *determinant* of the endomorphism φ and denoted $\det \varphi$ (see e.g. [Wei13]).

Definition 3. Let $\zeta, \xi : M \times M \to R$ be two symmetric bilinear forms on a projective R-module M with ζ non-degenerate. We define the *relative determinant* of ξ with respect to ζ by the formula:

$$\det_\zeta(\xi) := \det \Delta_{\zeta,\xi}.$$

One of the fundamental facts in the theory of quadratic form over fields, is that a form is non-degenerate if and only if its determinant is non-zero. Combining Lemma 2 with [Gol61, Proposition 1.3], we get an analog of this property for the relative determinant.

Proposition 4. *The form $\xi : M \times M \to R$ is non-degenerate if and only if its determinant $\det_\zeta(\xi)$ is invertible in R.*

As the theory of bilinear forms over fields has already been called upon, it is worth to make also another observation.

Observation 5. *If M is a free module (e.g. when R is a field), then the classical determinant of ξ is the relative determinant (in sense of the above definition) of ξ with respect to the dot product.*

Another basic property of the determinant is that it factors over orthogonal sums.

Proposition 6. *Let M, N be two finitely generated projective R-modules. Take symmetric bilinear forms $\zeta, \xi : M \times M \to R$ and $\varsigma, \rho : N \times N \to R$ and assume that ζ, ς are non-degenerate. Then*

$$\det_{\zeta \perp \varsigma}(\xi \perp \rho) = \det_{\zeta}(\xi) \cdot \det_{\varsigma}(\rho).$$

Proof. Fix any $u \oplus v \in M \oplus N$. We claim that $\Delta_{\zeta\perp\varsigma}(\xi \perp \rho)(u \oplus v) = \big(\Delta_\zeta(\xi) \oplus \Delta_\varsigma(\rho)\big)(u \oplus v)$. Indeed, the left-hand-side reads as

$$\Delta_{\zeta\perp\varsigma}(\xi \perp \rho)(u \oplus v) = \Big(\widehat{\zeta \perp \varsigma}^{-1} \circ \widehat{\xi \perp \rho}\Big)(u \oplus v) = \Big(\widehat{\zeta \perp \varsigma}^{-1}\Big)\big(\xi(u, \cdot) + \rho(v, \cdot)\big),$$

while the right-hand-side evolves into

$$\Big(\Delta_\zeta(\xi) \oplus \Delta_\varsigma(\rho)\Big)(u \oplus v) = \Delta_\zeta(\xi)(u) \oplus \Delta_\varsigma(\rho)(v) = \hat{\zeta}^{-1}\big(\xi(u, \cdot)\big) \oplus \hat{\varsigma}^{-1}\big(\rho(v, \cdot)\big).$$

Apply $\widehat{\zeta \perp \varsigma}$ to both side to get $\xi(u, \cdot) + \rho(v, \cdot)$ in both cases. Now, $\widehat{\zeta \perp \varsigma}$ is an isomorphism, hence our claim is proved. Consequently $\Delta_{\zeta\perp\varsigma}(\xi \perp \rho) = \big(\Delta_\zeta(\xi) \oplus \Delta_\varsigma(\rho)\big)$ and the assertion follows from [Wei13, Proposition II.2.6]. □

The relative determinant satisfies also the following "chain-rule", that has no direct analog for a classical determinant:

Proposition 7. *Let ζ, ς be two non-degenerate forms and ξ be any bilinear form, all three defined on the same R-module M. Then*

$$\det_\zeta(\xi) = \det_\zeta(\varsigma) \cdot \det_\varsigma(\xi).$$

Proof. We have $\Delta_{\zeta,\xi} = \hat{\zeta}^{-1} \circ \hat{\xi} = \hat{\zeta}^{-1} \circ \hat{\varsigma} \circ \hat{\varsigma}^{-1} \circ \hat{\xi} = \Delta_{\zeta,\varsigma} \circ \Delta_{\varsigma,\xi}$. Therefore $\det_\zeta(\xi) = \det \Delta_{\zeta,\xi} = \det(\Delta_{\zeta,\varsigma} \circ \Delta_{\varsigma,\xi}) = \det \Delta_{\zeta,\varsigma} \cdot \det \Delta_{\varsigma,\xi} = \det_\zeta(\varsigma) \cdot \det_\varsigma(\xi)$. □

2. Determinant of non-degenerate forms

In this section we show that, when dealing with non-degenerate forms, the relativity in the definition of the determinant may be restricted entirely to modules of constant rank 1 (i.e line bundles). We begin with a proposition that is essentially due to M. Ciemała and K. Szymiczek (c.f. [CS05, Theorem 2.5]).

Proposition 8. *Let L be a line bundle. If L admits any non-degenerate form, then L has rank ≤ 2 in the Picard group of R.*

Proof. Let $\lambda : L \times L \to R$ be a non-degenerate form, then $\hat{\lambda} : L \to L^\vee$ is an isomorphism and so we have $L \otimes L \cong L \otimes L^\vee \cong R$. □

For line bundles, there is an alternative formula for computing the relative determinant (the proof is immediate):

Observation 9. *If $\lambda, \zeta : L \times L \to R$ are two non-degenerate forms on the same line bundle L and $\Lambda, \Xi : L \otimes L \xrightarrow{\sim} R$ be the associated isomorphisms: $\Lambda(x \otimes y) = \lambda(x, y)$, $\Xi(x \otimes y) = \xi(x, y)$, then the following identity holds:*

$$\det_\lambda(\xi) = \det(\Lambda^{-1} \circ \Xi).$$

Now, let $\xi : M \times M \to R$ be a non-degenerate symmetric bilinear form on an arbitrary finitely generated projective R-module M. In particular $\hat{\xi} : M \to M^\vee$, $\big(\hat{\xi}(u)\big)(v) := \xi(u, v)$ is an isomorphism. By functoriality of the (fixed) exterior power, $\wedge^n\hat{\xi} : \wedge^n M \to \wedge^n M^\vee$ is again an isomorphism. Recall that $\operatorname{rank} M = \operatorname{rank} M^\vee$ (see e.g. [Wei13, p. 17]), hence for $n = \operatorname{rank} M$ we get an isomorphism $\wedge^n\hat{\xi} : \det M \to \det M^\vee$. Define $\delta : \det M \times \det M \to R$ by the formula

$$\delta(x, y) := \big(\wedge^n\hat{\xi}(x)\big)(y).$$

It is straightforward to check that δ symmetric bilinear. Notice that $\hat{\delta} = \wedge^n \hat{\xi}$ is an isomorphism and so we have:

Observation 10. *The form $\delta : \det M \times \det M \to R$ is non-degenerate.*

This let us define a relative determinant of a (non-degenerate) form ξ with respect to a (fixed) isomorphism $\Lambda : (\det M)^{\otimes 2} \xrightarrow{\sim} R$ or equivalently with respect to a (fixed) non-degenerate form $\lambda : \det M \times \det M \to R$.

Definition 11. The *relative determinant* of a non-degenerate form $\xi : M \times M \to R$ with respect to an isomorphism $\Lambda : (\det M)^{\otimes 2} \to R$ is

$$\det_\Lambda(\xi) := \det(\Lambda^{-1} \circ \Delta) = \det_\lambda(\delta),$$

where $\delta : \det M \times \det M \to R$ is the form constructed above and $\Delta : (\det M)^{\otimes 2} \xrightarrow{\sim} R$ is the associated isomorphism of line bundles.

This definition of the determinant is still relative, but contrary to Definition 3, instead of having a separate reference form for each module, we have a separate reference isomorphism only for each line bundle isomorphic to R. In order to have just one global reference object, we need to assume that R is a domain. It is well known (see e.g. [Wei13, Proposition I.3.5]), that over a domain, every line bundle can be identified with an invertible ideal. We assume that such an identification is fixed once and for all (the definition that follows will still depend on the identification).

The global reference object, with respect to which we may now define an "absolute" determinant, is a (fixed) semi-group homomorphism $\mathscr{F}$ from $\{I \lhd R : I^2 \text{ is principal}\}$ to the multiplicative semi-group $(R, \cdot)$ such that $\mathscr{F}(I)$ is a generator for I^2. Having fixed $\mathscr{F}$, we can coherently define all the Λ's, namely $\Lambda : I^2 \to R$ is $\Lambda(x) := \frac{x}{\mathscr{F}(I)}$. This way the determinants of forms on ideals are uniformly defined by the means of Observation 9 and consequently, they are uniformly defined on all finitely generated projective modules using Definition 11, as well.

It is known (see [CS05, Theorem 3.1]) that every non-degenerate bilinear form on an ideal I is given by the formula $\delta(x, y) = \frac{u}{\mathscr{F}(I)}\, xy$ for some invertible u. Clearly, the determinant of δ with respect to the above defined Λ (i.e with respect to $\mathscr{F}$) equals now $\det_\Lambda(\delta) = u$. To emphasize the uniformity, in what follows, we shall write $\det_{\mathscr{F}}(\delta)$ or even drop the subscript all together, when $\mathscr{F}$ is known.

Proposition 12. *Let $\xi : M \times M \to R$ and $\zeta : N \times N \to R$ be two non-degenerate symmetric bilinear forms on some finitely generated projective R-modules M and N, respectively. Then*

$$\det(\xi \perp \zeta) = \det \xi \cdot \det \zeta.$$

Proof. Let I, J be two invertible ideals of R such that $\det M \cong I$ and $\det N \cong J$. Further, let $d := \mathscr{F}(I)$, $e := \mathscr{F}(J)$ and

$$\begin{aligned} \delta &: I \times I \to R, & \delta(x, y) &= \frac{u}{d} xy, \\ \epsilon &: J \times J \to R, & \epsilon(x, y) &= \frac{v}{e} xy \end{aligned}$$

be the associated bilinear forms, as explained in the above construction of the determinant. Now, $\det(M \oplus N) \cong \det M \otimes \det N$ by [Wei13, Proposition II.2.6]. Denote by $\partial : (\det M \otimes \det N) \times (\det M \otimes \det N) \to R$ the bilinear form associated to $\xi \perp \zeta$. With the earlier convention, we use the same letter to denote the isometric form $\partial : IJ \times IJ \to R$. Now, $\mathscr{F}$ is a semi-group homomorphism and so $\mathscr{F}(IJ) = d \cdot e$.

Then, for some $xx', yy' \in IJ$ we have

$$\begin{aligned}\partial(xx', yy') &= \\ &= \Big(\big(\wedge^{m+n}\widehat{\xi \perp \zeta}\big)(x_1 \otimes x_1' \wedge \ldots \wedge x_m \otimes x_n')\Big)(y_1 \otimes y_1' \wedge \ldots \wedge y_m \otimes y_n') = \\ = \Big(\big(\wedge^m\hat{\xi}\big)(x_1 \wedge \ldots \wedge x_m)\Big)(y_1 \wedge \ldots \wedge y_m) &\cdot \Big(\big(\wedge^m\hat{\zeta}\big)(x_1' \wedge \ldots \wedge x_n')\Big)(y_1' \wedge \ldots \wedge y_n') = \\ &= \delta(x,y)\cdot\epsilon(x,y) = \frac{uv}{de}xx'\,yy'.\end{aligned}$$

It follows that $\det(\xi \perp \zeta) = uv = \det\xi \cdot \det\zeta$, as desired. □

3. Determinant of general forms

The above construction of the determinant relies on the fact that for a non-degenerate bilinear module the square $(\det M)^2$ of the determinant line bundle is principal. It is not so for degenerate bilinear forms. Hence this construction does not admit a direct generalization to such forms. We may, however, omit this obstacle using our earlier definition of the relative determinant.

Observe that combining Observation 5 with the chain-rule (i.e Proposition 7) one gets the following identity for the determinant of a bilinear form over a field (or more generally for a form on a free module):

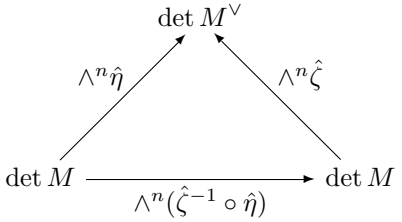
$$\det\xi = \det\zeta \cdot \det_\zeta(\xi),$$

where ξ is an arbitrary form and ζ is non-degenerate. We shall use it to define the determinant of a general form. First, however, we need to check that this formula agrees with our definition of the determinant of a non-degenerate form on a projective module, ensuring the correctness of the definition that follows. As before, we assume that an isomorphism of a given line bundle with an invertible ideal of R remains fixed and $\mathscr{F}$ is a fixed reference isomorphism relative to which we define all determinants.

Lemma 13. *If $\zeta, \eta : M \times M \to R$ are two non-degenerate forms, then*

$$\det\eta = \det\zeta \cdot \det_\zeta(\eta).$$

Proof. Assume that $\det M = \wedge^n M \cong I$ for some invertible ideal I of R and $\mathscr{F}(I) = d$. Let $u = \det\zeta$, $v = \det\eta$ and $w = \det_\zeta(\eta)$ with $u, v, w \in UR$. Thus $\big((\wedge^n\hat{\zeta})(x)\big)(y) = \frac{u}{d}xy$, $\big((\wedge^n\hat{\eta})(x)\big)(y) = \frac{v}{d}xy$ and $\big(\wedge^n(\hat{\zeta}^{-1}\circ\hat{\eta})\big)(x) = wx$. By functoriality of the exterior power, the following diagram commutes



Hence, $\wedge^n\hat{\eta} = \big(\wedge^n\hat{\zeta}\big)\circ\big(\wedge^n(\hat{\zeta}^{-1}\circ\hat{\eta})\big)$ and so $v = u\cdot w$. □

Definition 14. Let $\xi : M \times M \to R$ be a symmetric bilinear form on some finitely generated projective module M over a domain R. If there exists any non-degenerate form ζ on M, the (absolute) *determinant* of ξ is defined as

$$\det\xi := \det\zeta \cdot \det_\zeta(\xi).$$

If, on the other hand, M does not admit any non-degenerate form, take $\det\xi := 0$.

Proposition 15. *The definition of $\det\xi$ does not depend on the choice of ζ.*

Proof. Let ζ, η be two non-degenerate forms on M, then

$$\det\zeta \cdot \det_\zeta(\xi) = \det\zeta \cdot \det_\zeta(\eta) \cdot \det_\eta(\xi) = \det\eta \cdot \det_\eta(\xi),$$

here the first equality follows from the chain-rule (Proposition 7) and the second is due to Lemma 13. □

In the view of the above definition, we have the following immediate consequence of Proposition 4:

Corollary 16. *The form $\xi : M \times M \to R$ is non-degenerate if and only if $\det\xi$ is invertible in R.*

Now, having completed the definition of the determinant we should generalize Proposition 12.

Proposition 17. *Let M, N be two finitely generated projective modules over a common domain R, both admitting some non-degenerate forms. Then, for any two forms $\xi : M \times M \to R$ and $\rho : N \times N \to R$ the following holds:*

$$\det(\xi \perp \rho) = \det\xi \cdot \det\rho.$$

Proof. Let $\zeta : M \times M \to R$ and $\varsigma : N \times N \to R$ be two non-degenerate forms. Write

$$\det\xi = \det\zeta \cdot \det_\zeta(\xi) \qquad \text{and} \qquad \det\rho = \det\varsigma \cdot \det_\varsigma(\rho).$$

The assertion follows now from Proposition 12 and Proposition 6. Indeed:

$$\begin{aligned}\det(\xi \perp \rho) &= \det(\zeta \perp \varsigma) \cdot \det_{\zeta\perp\varsigma}(\xi \perp \rho) = \\ &= \det\zeta \cdot \det\varsigma \cdot \det_\zeta(\xi) \cdot \det_\varsigma(\rho) = \\ &= \det\xi \cdot \det\rho.\end{aligned}$$

□

Example. As an example application of the above theory, we shall reprove [Rot13, Theorem 2.9] using the introduced notion of the determinant. Take a domain R and assume that the reference semi-group homomorphism $\mathscr{F}$ is fixed. Let $M = I_1 \oplus \cdots \oplus I_n$ be a direct sum of invertible ideals of R. A symmetric bilinear form $\xi : M \times M \to R$ is given by a formula (see [Rot13, Proposition 2.8]):

$$\xi\big(x_1 \oplus \cdots \oplus x_n, y_1, \oplus \cdots \oplus y_n\big) := \frac{1}{d} \cdot \big(x_1, \ldots, x_n\big) \cdot A \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

where $d = \mathscr{F}(I_1 \cdots I_n)$ and $A = \big(a_{ij}\big)_{1\leq i,j\leq n}$ is a symmetric square matrix with entries $a_{ij} \in I_1^{\varepsilon_1} \cdots I_n^{\varepsilon_n}$ with $\varepsilon_k = 2$ for $k \notin \{i, j\}$, $\varepsilon_k = 1$ for $k = i \neq j$ or $k = j \neq i$ and $\varepsilon_k = 0$ when $k = i = j$.

Keep the notation used in previous sections. The determinant bundle of M is $\det M = \det(I_1 \oplus \cdots \oplus I_n) \cong \det(I_1) \otimes \cdots \otimes \det(I_n) \cong I_1 \cdots I_n \lhd R$. The isomorphism $\Lambda : (I_1 \cdots I_n)^2 \xrightarrow{\sim} R$ is given by $\Lambda(x) = {}^{x}\!/_{d}$. Therefore, one can express $\delta : \det M \times \det M \to R$ in the form

$$\delta\big(x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots y_n\big) = \det\Big(\xi(x_i, y_j)\Big)_{1\leq i,j\leq n}.$$

Consequently, the determinant of ξ is

$$\det\xi = \det\big(\Lambda^{-1} \circ \Delta\big) = d \cdot \det\Big(\frac{a_{ij}}{d}\Big)_{1\leq i,j\leq n} = \frac{1}{d^{n-1}} \det A.$$

It follows from Corollary 16 that ξ is non-degenerate if and only of $\det A = u \cdot d^{n-1}$ for some unit $u \in UR$ of R. This (re)proves [Rot13, Theorem 2.9].

References

[CS05] Marzena Ciemała and Kazimierz Szymiczek. On the existence of nonsingular bilinear forms on projective modules. *Tatra Mt. Math. Publ.*, 32:1–13, 2005.

[Gol61] Oscar Goldman. Determinants in projective modules. *Nagoya Math. J.*, 18:27–36, 1961.

[Mar85] Murray A. Marshall. *Bilinear forms and orderings on commutative rings*, volume 71 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University, Kingston, ON, 1985.

[MH73] John Milnor and Dale Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York, 1973.

[Pet08] Holger P. Petersson. Polar decompositions of quaternion algebras over arbitrary rings. preprint, 2008.

[Rot13] Beata Rothkegel. Nonsingular bilinear forms on direct sums of ideals. *Math. Slovaca*, 63(4):707–724, 2013.

[Wei13] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.

E-mail address: pkoprowski@member.ams.org

Faculty of Mathematics, Univeristy of Silesia, ul. Bankowa 14, 40-007 Katowice, Poland