ALGEBRAIC APPROACH TO GEOMETRIC CHARACTERIZATION OF PARAMETRIC CUBICS

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Abstract. We reprove the result of Stone and DeRose, which gives the geometric classification of the affine type of an untrimmed Bézier curve, using classical algebraic geometry. We show how to derive the characterization of Stone and DeRose from three classical results: Bézout theorem, polynomial parametrization criterion and classification of the singularity type of an algebraic curve given in Weierstrass normal form.

1. Introduction

In 1989 Stone and DeRose presented a geometric criterion to determine the affine type of a parametric cubic curve (see [7]). Having three, out of four, control points of Bézier reprezentation of the curve fixed to specified locations one can decide if the curve has a cusp, a node, or one or two inflection points by examining the position of the fourth point. Stone and DeRose showed that the real plane is divided into regions where the curve has respectively: a node, a cusp, one or two inflection points. Those regions are given by a parabola and its tangent. Their proof of this result is based entirely upon the parametric form of the curve and is purily analitic in nature. In fact, the authors support their analysis on the earlier papers bu Su and Liu [8] and Wang [11], that are also purely analitic. On the other hand, characterization of the affine type of a (singular) cubic curve has a long history in algebraic geometry dating back to 19^{th} century and beyond! The aim of this paper is to show how the result of Stone and DeRose relates to this classical theory. To this end we reprove the above-mentioned theorem using the language of classical algebraic geometry. In particular we show that the result of Stone and DeRose is closely realted to the well known classification of singular cubics given in Weierstrass normal form. Our proof has two advantages. The main is that using the well established basis of algebraic geometry our method not only proves the assertion but, more importantly, offers an insight *explaining* the phenomenon. Secondly, the presented proof is realitively shorter since it is completely based upon the classical theory (and so can be considered self-contained), in contrast Stone and DeRose base their results on two earlier papers: [8] and [11].

Because we only reprove here an already known result, using completely classical tools, this paper is rather of an expository nature. In particular, we give full references to all the results we use, no matter how classical they are. We want to emphasize the fact that the paper of Stone and DeRose presents more nice results than the one reproved here. In particular the authors give also a characterization of trimmed curves and show that similar characterizations can be obtained for other representatations of parametric cubics by appropriate planar slice of a common three-dimensional "characterisation space". The similar result were obtained also in the papers by Su and Liu [8], Wang [11] and Forrest [3]. More recently Vincent (see [9]) presented another algorithm to decide the type of a trimmed Bézier cubic. The idea of using the language of (classical) algebraic geometry to tackle problems from the realms of geometric modelling, that we use here, is not new. For cubic curves it was effectively used for example in the Patterson's paper [5].

2. Stone–DeRose Theorem

Here we reprove the theorem of Stone and DeRose (c.f. [7]) using the language of (classical) algebraic geometry. Let C be an untrimmed polynomial Bézier cubic curve, with control points P_0, P_1, P_2, P_3:

$$C(t) = P_0 \cdot (1-t)^3 + P_1 \cdot 3t(1-t)^2 + P_2 \cdot 3t^2(1-t) + P_3 \cdot t^3. \tag{1}$$

We assume that the control points P_0, P_1, P_2, P_3 are in *general* position, i.e. they are not colinear and no two of them are coincident. Since reversing the order of control points of a Bézier curve reverses only the parametrization but does not affect the shape of the curve, we may assume that P_0, P_1 and P_2 are not colinear. The Bézier reprezentation is affine invariant (see e.g. [2, §4.3]). Choosing an appropriate affine transformation, we may fix the positions of these three control points so that:

$$P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2}$$

Now, the position of $P_3 = \binom{P_x}{P_y}$ determines the class of the curve C with respect to affine equivalence ("the characteristic" of the curve in terms of [7]). Substituting the coordinates of control points into Eq. (1) leads us to:

$$C(t) = \begin{pmatrix} (P_x - 3)t^3 + 3t^2 \\ P_y t^3 - 3t^2 + 3t \end{pmatrix}.$$

If $P_x - 3 = 0 = P_y$ the curve degenerates to a conic. Since a parabola is the only conic with a polynomial parametrization (see e.g. [1, Chapter 1]) we have:

Observation 3. *If $P_3 = \binom{3}{0}$, then the curve C is a parabola.*

Form now on, we assume that $P_3 \neq \binom{3}{0}$. Implicitize C computing the Bézout resultant (see e.g. [6, §3.3]) for $x - (P_x - 3)t^3 - 3t^2$ and $y - P_y t^3 + 3t^2 - 3t$. We get

$$\begin{aligned} &\operatorname{Res}_t\bigl(x - (P_x-3)t^3 - 3t^2, y - P_y t^3 + 3t^2 - 3t\bigr) = \\ &\quad = \det \begin{pmatrix} 3x & -3x-3y & P_y x - (P_x-3)y \\ -3x-3y & 9 + P_y x - (P_x-3)y & 3(P_x-3) \\ P_y x - (P_x-3)y & 3(P_x-3) & -3(P_x-3) - 3P_y \end{pmatrix} = \\ &\qquad\qquad = -(P_y x - (P_x-3)y)^3 + 9\bigl(A_1 x^2 + A_2 xy + A_3 y^2\bigr) - 27 A_3 x, \end{aligned}$$

with $A_1 = 3P_x - 3P_xP_y - 2P_y^2 + 12P_y - 9$, $A_2 = 3P_x^2 + P_xP_y - 12P_x + 3P_y + 9$ and $A_3 = P_x^2 - 3P_x + 3Py$. Let F be the homogenization of f and take $\hat{C} := \{(x : y : w) \in \mathbb{P}^2\mathbb{R} : F(x, y, w) = 0\}$ the Zarisky's closure of C in the projective plane $\mathbb{P}^2\mathbb{R}$. The curve C has a parametrization, hence its genus is zero. Thus $\hat{C}$ must have a singular point. It is well know (see e.g. [4, Chapter 7]) that there are only three types of real singular cubics:

- **A:** cuspidal—it has a single real cusp with the double real tangent; it has a unique real flex;

B: crunodal—it has a single real node with two different real tangents; it has a unique real flex and two complex flexes;

C: acnodal—it has a single real node with two complex conjugate tangents; it has three distinct real flexes.

Now, we know that C has a *polynomial* parametrization, consequently $\hat{C}$ has exactly one place at infinity (see e.g. [1, Chapter 1]). Thus Bézout theorem (see e.g. [4, Theorem 14.7] or [10, Theorem IV.5.4]) implies that there are only two possibilities:

1: $\hat{C}$ has a cusp at infinity and its (double) tangent is a line at infinity;

2: $\hat{C}$ has a flex at infinity and its tangent is again a line at infinity.

In the first case the curve is of type **A**, and so has exactly one affine flex and no affine singularities. In the other case it may be of any type, but since it already has one flex at infinity it may have either two or zero affine inflection points. If it has no affine inflection points it is of type **A** or **B** and so it has a cusp or a crunode. If it has two affine inflection points it is of type **C** so has an acnode, and since the acnode is an isolated point of the real algebraic curve, it does not belong to the parametric curve. Thus, we have the following possibilities:

1A: the Bézier curve C has one inflection point and no singular points;

2A: the Bézier curve C has no inflection points and one cusp;

2B: the Bézier curve C has no inflection points and one crunode;

2C: the Bézier curve C has two inflection points and no singular points.

The following theorem due to Stone and DeRose correlates the position of the control point P_3 to one of the above cases.

Theorem 4 (Stone, DeRose)**.** *If C is an untrimmed Bézier curve with control points P_0, P_1, P_2, P_3 satisfying Eq.* (2) *then:*

0: *it is a parabola if and only if $P_3 = \binom{3}{0}$;*

1A: *it has one inflection point if and only if P_3 belongs to the line $x+y-3=0$ and $P_3 \neq \binom{3}{0}$;*

2A: *it has a cusp if and only if P_3 lies on the parabola $(x-3)(x+1)+4y=0$ and $P_3 \neq \binom{3}{0}$;*

2B: *it has a crunode if and only if P_3 lies below the parabola $(x-3)(x+1)+4y=0$;*

2C: *it has two inflection points if and only if P_3 lies above the parabola $(x-3)(x+1)+4y=0$ but does not belong to the line $x+y-3=0$.*

The five cases mentioned above are illustrated in Figure 1.

Proof. We have already solved the degenerate case in Observation 3. Next, we know that the curve $\hat{C}$ has exactly one place at infinity. Substituting $w=0$ to $F(x,y,w)=0$ we have $((P_x-3)y-P_yx)^3=0$. Hence the point at infinity has coordinates $(P_x-3:P_y:0)$. From our earlier disscussion we know that the curve is of type **1A** if and only if this point is singular (and then it is necesserally a cusp) and the double tangent is the line at infinity. Compute the partial derivative

$$\frac{\partial F}{\partial w}(P_x-3:P_y:0)=27(P_x+P_y-3)^3.$$

Thus $(P_x-3:P_y:0)$ is singular if and only if the coordinates of P_3 satisfy $P_x+P_y-3=0$. This proves **1A**.

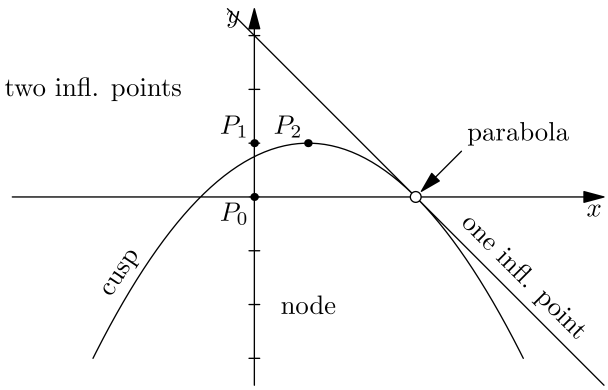

FIGURE 1. The characterization diagram of Stone and DeRose.

On the other hand suppose that the point at infinity is not singular, hence it is a flex and the tangent is the line at infinity. Consider a change of variables $x \mapsto x + (P_x - 3)/P_y y$, $y \mapsto y$, $w \mapsto w$. It transforms $\hat{C}$ into $\hat{C}_1$ given by

$$G_1(x, y, w) := F(x + (P_x - 3)/P_y y, y, w).$$

The curve $\hat{C}_1$ has a flex at $(0 : 1 : 0)$ and its tangent is the line at infinity $\{w = 0\}$. Notice that we are now at the initial position for the classical derivation of Weierstrass normal form (see e.g. [4, §15.2] or [10, §6.4]). Repeating the classical scheme we dehomogenize G_1 to obtain g_1. Now, there is an affine change of variables that transforms the curve into Weierstrass normal form:

$$y^2 = x^3 + \alpha x + \beta.$$

With a direct computation[1] we find out that

$$\alpha = -\frac{3\big((P_x - 3)(P_x + 1) + 4P_y\big)^2}{16(P_x + P_y - 3)^4}, \qquad \beta = \frac{\big((P_x - 3)(P_x + 1) + 4P_y\big)^3}{32(P_x + P_y - 3)^6}.$$

It follows from the already proved part that the denominators are non-zero. Now, the classification of cubics given in Weierstrass normal form is well known (see [4, §15.3]): it is cuspidal (hence of type **2A**) if and only if $\alpha = \beta = 0$ if and only if $(P_x - 3)(P_x + 1) + 4P_y = 0$; it has a crunode (i.e. it is of type **2B**) iff $\beta > 0$; and finally it has an acnode (hence its type is **2C**) when $\beta < 0$. $\square$

It is worth to stress the point that singularities and inflection points are mutually exclusive only for polynomial curves. In rational case it is not hard to show a Bézier curve (in fact even a segment) having a cusp and an inflection point, a node and an inflection point or three inflection points (see Figure 2).

REFERENCES

[1] S. S. Abhyankar. *Algebraic geometry for scientists and engineers*, volume 35 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[2] G. Farin. *Curves and surfaces for CAGD: a practical guide*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2002.

[1] We used a computer algebra system Mathematica 3.01 (c.f. [12]).

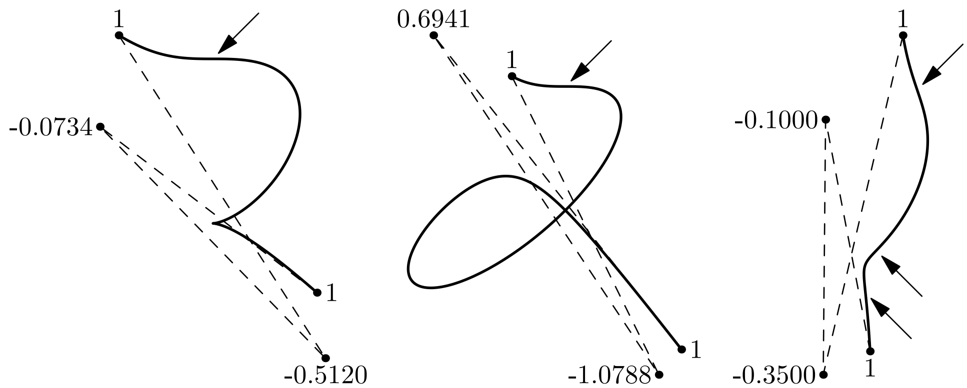

FIGURE 2. Examples of rational cubic segments having: (a) a cusp and an inflection point; (b) a node and an inflection point; (c) three inflection points. The inflection points are marked with arrows. Labels near the control points show their weigths.

[3] A. R. Forrest. The twisted cubic curve: A computer-aided geometric design approach. *Comput. Aided Des.*, 4(12):165–172, July 1980.

[4] C. G. Gibson. *Elementary geometry of algebraic curves: an undergraduate introduction.* Cambridge University Press, Cambridge, 1998.

[5] R. R. Patterson. Parametric cubics as algebraic curves. *Comput. Aided Geom. Des.*, 5(2):139–160, 1988.

[6] T. W. Sederberg. Applications to computer aided geometric design. In *Applications of computational algebraic geometry (San Diego, CA, 1997)*, volume 53 of *Proc. Sympos. Appl. Math.*, pages 67–89. Amer. Math. Soc., Providence, RI, 1998.

[7] M. Stone and T. DeRose. A geometric characterization of parametric cubic curves. *ACM Trans. Graph.*, 8(3):147–163, 1989.

[8] B. Su and D. Liu. An affine invarient and its application in computational geometry. *Scientia Sinica (Series A)*, 3(24):259–267, March 1983.

[9] S. Vincent. Fast detection of the geometric form of two-dimensional cubic Bézier curves. 7(3):43–51, 2002.

[10] R. J. Walker. *Algebraic curves.* Dover Publications Inc., New York, 1962.

[11] C. Wang. Shape classification of the parametric cubic curve and parametric B-spline cubic curve. *Computer-Aided Design*, 13(4):199–296, 1981.

[12] Wolfram Research, Inc. *Mathematica, Version 3.01.* Wolfram Research, Inc., Champaign, Illinois, 1996.