On splitting Knebusch-Milnor exact sequence

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Abstract

We show here that the Witt ring of the ring of regular functions is a direct summand of the Witt ring of a formally real algebraic function field over a real closed field. Next, we use this result to show that the Knebusch-Milnor exact sequence of these Witt rings fulfils a certain analogy of splitting.

Key words: Witt ring, Knebusch-Milnor exact sequence, real algebraic curves
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1 Introduction

The notion of a Witt ring of symmetric bilinear forms is central to the algebraic theory of quadratic forms. It is well known that the Witt ring of a Dedekind domain can be embeded into the Witt ring of its fraction field (see [7, Corollary IV.3.3]). Unfortunatelly, in general, such an injection does not split. In only very few cases we can, in fact, show that the image of the Witt ring of a Dedekind domain under above-mentioned embeding is a direct summand of the Witt group (i.e. additive group of the Witt ring) of its field of fractions. A remarkable paper here is [8], where the author finds a criterion for such a splitting for rings of algebraic integers.

In this paper we discuss this question in a more geometric context. Namely, we show that the Witt ring of the ring of regular functions on a real algebraic curve is a direct summand of the Witt group of its fraction field, i.e. the algebraic function field (see Theorem 3.9). As a consequence of this theorem

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we show that, in our case, the Knebusch-Milnor exact sequence slices into and is patched by two split exact sequences (see Corollary 5.2). This in fact is the closest approximation of splitting an exact sequence that we may obtain for a sequence of more than three non-zero terms.

This paper is organised as follows: in section 2 we introduce all the terminology and notation used in this paper. Next, in section 3 we present some preliminary results that eventually lead us to the formulation of our first main result (Theorem 3.9). The following section (i.e. Section 3) is completely devoted to the proof of this theorem. Finally in the last section we summarize our results. In particular we develop an analogon for splitting the Knebusch-Milnor exact sequence (see Theorem 5.1).

2 Notation

Throuhout this paper, letter $\Bbbk$ denotes a real closed field of constants of an algebraic function field K. The field $\Bbbk$ is fixed once and for all. Moreover, we are interested here only in the case when K is formally real, hence in order to simplify wording, the phrase *function field* shall actually mean 'formally real algebraic function field of one variable with the real closed field of constants $\Bbbk$'. The symbol $\Omega(K)$ denotes the set of all points of K trivial on its field of constants. For a given point $\mathfrak{p} \in \Omega(K)$ its residue field is denoted by $K(\mathfrak{p})$ while the completion of K with respect to $\mathfrak{p}$ by $K_{\mathfrak{p}}$.

Among all the points of $\Omega(K)$, we consider those that have the residue field equal to $\Bbbk$, following [3] we call a point of this kind *real*. The set of all real points is denoted by $\gamma = \gamma(K)$. This set may be viewed as a real algebraic curve. It is well known that such a curve is a disjoint union of finitely many semi-algebraically connected components (see [3] for the proof and [1] for a general notion of semi-algebraic connectedness). In what follows N is always the number of the components while the components themselves are denoted by $\gamma_1, \dots, \gamma_N$.

Consider the subring of K consisting of regular functions on γ, we denote it $R = R(K)$. It is given by

$$R = \{f \in K : \operatorname{ord}_{\mathfrak{p}} f \geq 0 \text{ for every } \mathfrak{p} \in \gamma\}.$$

Here, as usually, $\operatorname{ord}_{\mathfrak{p}}$ is the valuation of K associated with the point $\mathfrak{p}$. Observe that R is a holomorphy ring, hence the strong approximation theorem implies that R is a Dedekind domain (see [9, §III.2]).

In addition we use the standard notation and terminology for quadratic forms as they exist in [6]. We also use the notion of: a topology on γ, an orientation and open intervals of a component as they were introduced in [3,4]. We assume that an orientation of γ is arbitrarily chosen and fixed throughout this paper.

Moreover, to simplify the notation, we use the same symbol f to denote both an element of a field and a square class containing this element. Analogously, $\langle f_1, \dots, f_n\rangle$ is (depending on a context) either a quadratic form or its class in the Witt ring.

3 Preliminary results

Lemma 3.1 *The following sequence is exact*

$$0 \to WR \xrightarrow{i} WK \xrightarrow{\partial} \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \xrightarrow{\lambda} \mathbb{Z}^N \to 0. \tag{3.2}$$

PROOF The head of the sequence (3.2):

$$0 \to WR \xrightarrow{i} WK \xrightarrow{\partial} \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p})$$

is exact by [7, Corollary IV.3.3]. On the other hand, the tail

$$WK \xrightarrow{\partial} \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \xrightarrow{\lambda} \mathbb{Z}^N \to 0$$

is exact by [4, Theorem 11.2]. □

Observe that for any real point $\mathfrak{p}$ of K, we have $K(\mathfrak{p}) = \Bbbk$, hence $WK(\mathfrak{p}) \cong \mathbb{Z}$. Therefore, there is a natural isomorphism $\bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \cong \mathbb{Z}^{(\gamma)}$, where $\mathbb{Z}^{(\gamma)}$ denotes a coproduct of $\operatorname{card}(\gamma)$ copies of $\mathbb{Z}$. In what follows we identify these two groups. Similarly, we identify the ring WR with its image $i(WR) = \ker\partial$ under the canonical injection $i\colon WR \hookrightarrow WK$.

Let $\mathbb{E} < \dot K/\dot K^2$ be the subgroup of the group of square classes of K given by

$$\mathbb{E} := \Big\{ f \in \dot K/\dot K^2 : \operatorname{ord}_{\mathfrak{p}} f \equiv 0 \pmod 2 \text{ for all } \mathfrak{p}\in\gamma \Big\}.$$

We have the following consequence of the exactness of (3.2) (see also [2, Proposition 3.1]):

Corollary 3.3 *Let $f \in \dot K/\dot K^2$ be a square class, then $f \in \mathbb{E}$ if and only if a unary form $\langle f\rangle$ belongs to WR, i.e.*

$$f \in \mathbb{E} \iff \langle f\rangle \in WR.$$

Let $H < \mathbb{Z}^{(\gamma)} \cong \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p})$ be the kernel of λ, i.e.

$$H = \Big\{ (n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma} \in \mathbb{Z}^{(\gamma)} : \sum_{\mathfrak{p}\in\gamma_i} n_{\mathfrak{p}} = 0 \text{ for all } 1 \le i \le N \Big\}. \tag{3.4}$$

The exactness of (3.2) implies that $H = \operatorname{im} \partial$.

Proposition 3.5 *Let points $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_N \in \gamma$ be chosen in such a way that in every component γ_i of γ lies exactly one point $\mathfrak{p}_i \in \gamma_i$. Further, let $\hat{H} < \mathbb{Z}^{(\gamma)}$ denote the subgroup*

$$\hat{H} = \Big\{ (n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma} \in \mathbb{Z}^{(\gamma)} : n_{\mathfrak{p}} = 0 \textit{ for } \mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_N\} \Big\}. \tag{3.6}$$

Then $\mathbb{Z}^{(\gamma)}$ is the direct sum $H \oplus \hat{H} = \mathbb{Z}^{(\gamma)}$.

Proof Take an element $(n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma} \in H \cap \hat{H}$. For every $1 \le i \le N$ we have

$$0 = \sum_{\mathfrak{p}\in\gamma_i} n_{\mathfrak{p}} = n_{\mathfrak{p}_i},$$

here the first equality is due to (3.4), while the latter one follows from (3.6). Hence $(n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma} = 0$ and so $H \cap \hat{H} = \{0\}$.

Now, we show that $\mathbb{Z}^{(\gamma)} = H + \hat{H}$. Take an element $(n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma} \in \mathbb{Z}^{(\gamma)}$ and let $m_i := \sum_{\mathfrak{p}\in\gamma_i} n_{\mathfrak{p}_i}$. We define two elements $\hat{h} \in \hat{H}$ and $h \in H$. The former being given by the condition

$$\hat{h} := (m_{\mathfrak{p}})_{\mathfrak{p}\in\gamma} \in \hat{H}, \qquad m_{\mathfrak{p}} = \begin{cases} m_i, & \text{for } \mathfrak{p} = \mathfrak{p}_i \\ 0, & \text{for } \mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_N\} \end{cases}$$

while the latter one by

$$h := (s_{\mathfrak{p}})_{\mathfrak{p}\in\gamma}, \qquad s_{\mathfrak{p}} = \begin{cases} n_{\mathfrak{p}} - m_i, & \text{for } \mathfrak{p} = \mathfrak{p}_i \\ n_{\mathfrak{p}}, & \text{for } \mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_N\}. \end{cases}$$

Observe that $\sum_{\mathfrak{p}\in\gamma_i} s_{\mathfrak{p}} = \left(\sum_{\mathfrak{p}\in\gamma_i} n_{\mathfrak{p}}\right) - m_i = 0$, hence indeed $h \in H$. Clearly $h + \hat{h} = (n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma}$. Consequently $\mathbb{Z}^{(\gamma)} = H \oplus \hat{H}$. $\square$

From now on we assume that the points $\mathfrak{p}_1 \in \gamma_1, \mathfrak{p}_2 \in \gamma_2, \dots, \mathfrak{p}_N \in \gamma_N$ are chosen and fixed once and for all. We denote $\mathfrak{P} := \{\mathfrak{p}_1, \dots, \mathfrak{p}_N\}$ the set of these points and we take $\hat{H}$ to be defined as in the above proposition. We have a natural injection $l\colon \mathbb{Z}^N \to \mathbb{Z}^{(\gamma)}$ defined

$$l(n_1, \dots, n_N) := (n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma} \quad \text{where } n_{\mathfrak{p}} = \begin{cases} n_i, & \text{for } \mathfrak{p} = \mathfrak{p}_i \\ 0, & \text{for } \mathfrak{p} \notin \mathfrak{P}. \end{cases}$$

The following observation is trivial

Observation 3.7 *$\hat{H} = \operatorname{im} l$ and $\lambda \circ l = \mathrm{id}_{\mathbb{Z}^N}$.*

Recall (c.f. [3, §6]) that for every two distinct points $\mathfrak{p}, \mathfrak{q}$ belonging to the same component γ_i there exists an element $\chi = \chi_{(\mathfrak{p},\mathfrak{q})}$ of K such that χ is definite on $\gamma \setminus \{\mathfrak{p}, \mathfrak{q}\}$, positive definite on $\gamma \setminus \gamma_i$ and fulfils $\partial_{\mathfrak{p}}\chi = -1$, $\partial_{\mathfrak{q}}\chi = 1$. Following [3] we call χ an *interval function* for the pair $(\mathfrak{p}, \mathfrak{q})$. An interval function is unique only upto the multiplication by a totally positive element. Hence, in the following construction (as well as in the rest of this paper), for every $\mathfrak{p} \in \gamma_i \setminus \{\mathfrak{p}_i\}$, we assume that $\chi_{(\mathfrak{p}_i,\mathfrak{p})}$ is an arbitrarily chosen and fixed interval function associated with the pair $(\mathfrak{p}_i, \mathfrak{p})$. Later, in corollary 4.4, we show that this choice is not essential.

For any $\mathfrak{q} \in \gamma$, let $\delta_{\mathfrak{q}}$ denote "Kronecker's delta", that is $\delta_{\mathfrak{q}} = (n_{\mathfrak{p}})_{\mathfrak{p}\in\gamma}$ with $n_{\mathfrak{p}} = 0$ for $\mathfrak{p} \neq \mathfrak{q}$ and $n_{\mathfrak{q}} = 1$. The set $\{\delta_{\mathfrak{q}} : \mathfrak{q} \in \gamma\}$ is a base of the $\mathbb{Z}$-module $\mathbb{Z}^{(\gamma)} \cong \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p})$. We define a group homomorphism $d\colon \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \to WK$ by fixing its values on these generators as follows:

$$d(\delta_{\mathfrak{p}}) := \begin{cases} \langle \chi_{(\mathfrak{p}_i,\mathfrak{p})} \rangle, & \text{for } \mathfrak{p} \in \gamma_i \setminus \{\mathfrak{p}_i\} \\ 0, & \text{for } \mathfrak{p} \in \mathfrak{P}. \end{cases} \tag{3.8}$$

Denote the image of this homomorphism by $D := \operatorname{im} d$. We are now ready to state our first main result.

Theorem 3.9 *The Witt group WK of the formally real algebraic function field K over the real closed field $\Bbbk$ is a direct sum of the Witt ring WR of the ring R of regular functions on γ and the group D.*

The proof of this theorem occupies the entirety of the next section.

4 Proof of the theorem 3.9

Lemma 4.1 *Let $f, g \in \dot{K}/\dot{K}^2$ be two square classes of K. If a binary form $\langle f, g\rangle$ represents 1 over K and both forms $\langle f\rangle$, $\langle g\rangle$ belong to the sum $WR + D$, then also $\langle fg\rangle \in WR + D$.*

Proof Since the form $\langle f, g\rangle$ represents 1, so forms $\langle f, g\rangle$ and $\langle 1, fg\rangle$ are isometric. Therefore, in the Witt ring WK we have:

$$\langle fg\rangle = \langle f\rangle + \langle g\rangle + \langle -1\rangle \in (WR + D) + (WR + D) + WR = WR + D.$$

□

For a given point $\mathfrak{p} \in \gamma$, let $\theta_\mathfrak{p} \colon \dot{K}/\dot{K}^2 \to \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^{\,2}$ denote the group homomorphism induced by the canonical injection $K \hookrightarrow K_\mathfrak{p}$ of K into its completion $K_\mathfrak{p}$.

Lemma 4.2 *Let $f, g \in \dot{K}/\dot{K}^2$ be two square classes of K such that for almost every point $\mathfrak{p} \in \gamma$ either $\theta_\mathfrak{p} f$ or $\theta_\mathfrak{p} g$ is the unit element of $\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^{\,2}$. Then the binary form $\langle f, g\rangle$ represents 1 over K.*

Proof This follows immediately form Witt theorem (see [4, Theorem 9.4] or [5, Lemma 2.7]). □

As simple as the above two lemmas are, they have some consequences. Recall (cf. [3, § 2]), that for a given point $\mathfrak{p} \in \gamma$ and a quadratic form $\varphi = \langle f_1, \dots, f_n\rangle$ over K we denote $\tau_\mathfrak{p}\varphi := \operatorname{sgn} f_1(\mathfrak{p}) + \cdots + \operatorname{sgn} f_n(\mathfrak{p})$ the local signature of φ at $\mathfrak{p}$.

Corollary 4.3 *Let $f, g \in \dot{K}/\dot{K}^2$ be two square classes of K such that for almost every point $\mathfrak{p} \in \gamma$ either $\tau_\mathfrak{p} f = 1$ or $\tau_\mathfrak{p} g = 1$. If $\langle f\rangle, \langle g\rangle \in WR + D$, then also $\langle fg\rangle \in WR + D$.*

Observe that the above corollary implies that the square class of a totally positive element belongs to WR. Thus, we have:

Corollary 4.4 *Let $f, g \in \dot{K}/\dot{K}^2$ be two square classes of K such that the form $\langle f\rangle$ belongs to $WR+D$ and g is totally positive. Then $\langle fg\rangle \in WR+D$.*

Corollary 4.5 *Let $f, g \in \dot{K}/\dot{K}^2$ be two square classes of K such that for every point $\mathfrak{p} \in \gamma$ one has $\tau_\mathfrak{p} f = \tau_\mathfrak{p} g$. Then*

$$\langle f\rangle \in WR + D \iff \langle g\rangle \in WR + D.$$

Lemma 4.6 *Let $f, g \in \dot{K}/\dot{K}^2$ be two square classes of K such that $f \in \mathbb{E}$ and $\langle g\rangle \in WR + D$. Then $\langle fg\rangle \in WR + D$.*

Proof The Witt class of the form $\langle g\rangle$ may be expressed as $\langle g\rangle = \varphi + \psi$ for some $\varphi \in WR$ and $\psi \in D$. Therefore, in the Witt ring WK one has:

$$\langle fg\rangle = \langle f\rangle(\varphi + \psi) = \langle f\rangle \cdot \varphi + \langle f\rangle \cdot \psi.$$

The Witt ring WR of R is a subring of WK (recall that we identify here WR with its image under the canonical injection $WR \hookrightarrow WK$) and so the corollary 3.3 implies that $\langle f\rangle \cdot \varphi \in WR \subset WR + D$. Hence, all we need to show is that also $\langle f\rangle \cdot \psi \in WR + D$.

Recall that $\mathbb{Z}^{(\gamma)}$ is generated by the set $\{\delta_\mathfrak{p} : \mathfrak{p} \in \gamma\}$. Consequently, $D = d(\mathbb{Z}^{(\gamma)})$ is spanned by $d\delta_\mathfrak{p}$ for $\mathfrak{p} \in \gamma$. In addition, $d\delta_{\mathfrak{p}_i} = 0$ for all $1 \le i \le N$ and so D is spanned by $\{d\delta_\mathfrak{p} : \mathfrak{p} \in \gamma \setminus \mathfrak{P}\}$. Thus, all we need to show is that

$\langle f\rangle \cdot d\delta_{\mathfrak{p}} = \langle f\cdot \chi_{(\mathfrak{p}_i,\mathfrak{p})}\rangle \in WR + D$ for every $\mathfrak{p}\in\gamma\setminus\mathfrak{P}$. Fix $\mathfrak{p}\in\gamma$ and let $1\le i\le N$ be such that $\mathfrak{p}\in\gamma_i$. Since $f\in\mathbb{E}$, so we see that $\tau_{(\cdot)}f$ does not vanish anywhere on γ. Consequently it is constant on every component of γ. We must consider two cases. First, suppose that f is positive definite on γ_i (i.e. $\tau|_{\gamma_i}(f)\equiv 1$). Take any point $\mathfrak{q}\in\gamma$. If $\mathfrak{q}\in\gamma_i$ then $\tau_{\mathfrak{q}}f=1$; otherwise if $\mathfrak{q}\in\gamma\setminus\gamma_i$ then $\tau_{\mathfrak{q}}\chi_{(\mathfrak{p}_i,\mathfrak{p})}=1$. Hence, the corollary 4.3 shows that $\langle f\cdot\chi_{(\mathfrak{p}_i,\mathfrak{p})}\rangle\in WR+D$. Now consider the second case, when f is negative definite on γ_i. Then $(-f)$ is positive definite on γ_i and the former case implies that $\langle -f\cdot\chi_{(\mathfrak{p}_i,\mathfrak{p})}\rangle\in WR+D$. Consequently also $\langle f\cdot\chi_{(\mathfrak{p}_i,\mathfrak{p})}\rangle\in WR+D$. □

Take $\mathfrak{p}\in\gamma_i$, by the very definition of d, the Witt class $\langle\chi_{(\mathfrak{p}_i,\mathfrak{p})}\rangle$ of the interval function $\chi_{(\mathfrak{p}_i,\mathfrak{p})}$ of the pair $(\mathfrak{p}_i,\mathfrak{p})$ belongs to $D\subset WR+D$. Hence, the corollary 4.5 implies that the Witt class of any interval function associated with the pair $(\mathfrak{p}_i,\mathfrak{p})$ belongs to $WR+D$. Now, let$\chi_{(\mathfrak{p},\mathfrak{p}_i)}$ be an interval function associated with the pair $(\mathfrak{p},\mathfrak{p}_i)$. There exists (see [3, Theorem 2.10]) $f\in\dot K/\dot K^2$ such that f is negative definite on γ_i and positive definite on $\gamma\setminus\gamma_i$. Clearly, $f\in\mathbb{E}$ and so in the group $\dot K/\dot K^2$ we can write

$$\chi_{(\mathfrak{p},\mathfrak{p}_i)} = \chi_{(\mathfrak{p}_i,\mathfrak{p})}\cdot f\cdot g$$

for some totally positive g. The previous lemma, together with the corollary 4.4 imply that the form $\langle\chi_{(\mathfrak{p},\mathfrak{p}_i)}\rangle$ belongs to $WR+D$. Thus we have just proved:

Corollary 4.7 *For any $\mathfrak{p}\in\gamma_i$ $(1\le i\le N)$ the Witt class of the form $\langle\chi_{(\mathfrak{p},\mathfrak{p}_i)}\rangle$ of an interval function of the pair $(\mathfrak{p},\mathfrak{p}_i)$ belongs to $WR+D$.*

So far we have dealt with just single intervals. Now we turn our attention to the case when we have a few intervals, every one in a different component.

Lemma 4.8 *Let $\emptyset\ne I\subseteq\{1,\ldots,N\}$ be some non-empty set of indices. For every $i\in I$ let $\mathfrak{q}_i\in\gamma_i\setminus\{\mathfrak{p}_i\}$ be a fixed point and let g_i denote the square class of either $\chi_{(\mathfrak{q}_i,\mathfrak{p}_i)}$ or $\chi_{(\mathfrak{p}_i,\mathfrak{q}_i)}$. Then the form $\langle g\rangle$ of the square class $g:=\prod_{i\in I}g_i$ belongs to $WR+D$.*

Proof We proceed by induction on the number of elements in I. The previous corollary proves the case $\operatorname{card} I=1$. Hence, assume that we have proved the assertion for all $n\le\operatorname{card} I$ and let $k:=\min I$. Factor g into

$$g=g_k\cdot\hat g,\qquad \text{with } \hat g:=\prod_{\substack{i\in I\\ i\ne k}} g_i.$$

Now, both g_k and $\hat g$ belong to $WR+D$ by the induction, hence the corollary 4.3 implies that also $\langle g\rangle\in WR+D$. □

Recall that for every point $\mathfrak{p}\in\gamma$ there are exactly two orderings $P_+(\mathfrak{p})$ and $P_-(\mathfrak{p})$ of the field K compatible with $\mathfrak{p}$. They may be characterised (see [5,

§ 2]), by formulas:

$$f \in P_+(\mathfrak{p}) \iff \bigvee_{\mathfrak{q}\in\gamma} \bigwedge_{\mathfrak{r}\in\,]\mathfrak{p},\mathfrak{q}[} \tau_{\mathfrak{r}} f = 1$$
$$f \in P_-(\mathfrak{p}) \iff \bigvee_{\mathfrak{q}\in\gamma} \bigwedge_{\mathfrak{r}\in\,]\mathfrak{q},\mathfrak{p}[} \tau_{\mathfrak{r}} f = 1$$

Lemma 4.9 *Let $f \in \dot{K}/\dot{K}^2$ be a square class of K, then $\langle f\rangle \in WR + D$.*

Proof We proceed by induction on $s = \operatorname{card}\{\mathfrak{p} \in \gamma : \tau_{\mathfrak{p}} f = 0\}$. It is well known (see [3, Theorem 3.4]) that s is even. If $s = 0$ then $f \in \mathbb{E}$ and so the corollary 3.3 implies that $\langle f\rangle \in WR \subset WR+D$. Assume that we have proved the result for all even integers less than s.

By the means of [3, Theorem 2.10] we can find square classes $e_1, \dots, e_N \in \mathbb{E}$ such that e_i is positive definite on $\gamma \setminus \gamma_i$ and negative definite on γ_i, i.e.

$$\tau|_{\gamma_i} e_j \equiv \begin{cases} -1, & \text{for } i = j \\ 1, & \text{for } i \neq j. \end{cases}$$

Take $J \subseteq \{1, \dots, N\}$ to be the set of all those indices $1 \le i \le N$ such that f is negative with respect to $P_-(\mathfrak{p}_i)$. In other words

$$J := \{1 \le i \le N : f \notin P_-(\mathfrak{p}_i)\}.$$

Take $f' := f \cdot \prod_{i\in J} e_i$. Lemma 4.6 implies that $\langle f\rangle \in WR + D$ if and only if $\langle f'\rangle \in WR + D$. On the other hand, one has

$$\tau_{\mathfrak{p}} f = 0 \iff \tau_{\mathfrak{p}} f' = 0.$$

Moreover f' is positive with respect to every $P_-(\mathfrak{p}_i)$ for $1 \le i \le N$. Therefore, for the rest of this proof, we can substitute f by f'. Thus, from now on, we assume that $f \in P_-(\mathfrak{p}_i)$ for all $1 \le i \le N$.

A priori, it is possible that $\tau_{\mathfrak{p}_i} f = 0$ for some $\mathfrak{p}_i \in \mathfrak{P}$. Let I be the set of all such i's, i.e. $I := \{1 \le i \le N : \tau_{\mathfrak{p}_i} f = 0\}$. For every $i \in I$ let $\mathfrak{q}_i \in \gamma_i \setminus \{\mathfrak{p}_i\}$ be such a point that $\tau_{\mathfrak{q}_i} f = 0$ and $\tau_{\mathfrak{r}} f \neq 0$ for all $\mathfrak{r} \in\,]\mathfrak{p}_i, \mathfrak{q}_i[$ (in a common language: $\mathfrak{q}_i$ is the first zero of $\tau_{(\cdot)} f$ right to $\mathfrak{p}_i$). Let f_i be either $\chi_{(\mathfrak{p}_i,\mathfrak{q}_i)}$ if $i \in I$ or $f_i := 1$ if $i \notin I$ $(1 \le i \le N)$. Again, $\chi_{(\mathfrak{p}_i,\mathfrak{q}_i)}$ is an interval function for the pair $(\mathfrak{p}_i, \mathfrak{q}_i)$. Take $\hat{f}, \tilde{f} \in \dot{K}/\dot{K}^2$ given by $\hat{f} := f_1 \cdots f_N$ and $\tilde{f} := f \cdot \hat{f}$. Obviously $\tau_{\mathfrak{p}_i} \tilde{f} = 1$ for all $1 \le i \le N$. Moreover

$$\{\mathfrak{p} \in \gamma : \tau_{\mathfrak{p}} \tilde{f} = 0\} = \{\mathfrak{p} \in \gamma : \tau_{\mathfrak{p}} f = 0\} \setminus \bigcup_{i\in I} \{\mathfrak{p}_i, \mathfrak{q}_i\}.$$

Thus, if $I \neq \emptyset$, then $\langle \tilde{f}\rangle \in WR + D$ by the induction hypothesis. Lemma 4.8 implies that also $\langle \hat{f}\rangle \in WR + D$. Finally, for a given point $\mathfrak{p} \in \gamma$ either

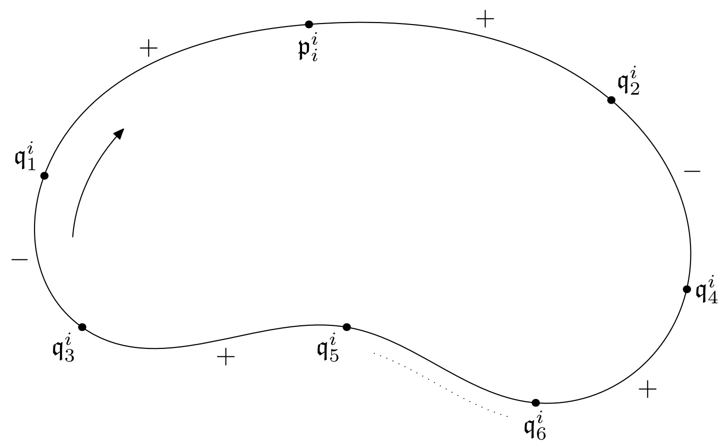

Fig. 1. Numbering of $\mathfrak{q}_j^i$ on γ_i.

$\mathfrak{p} \in \bigcup_{i\in I}]\mathfrak{p}_i, \mathfrak{q}_i[$, then $\tau_{\mathfrak{p}}\tilde{f} = 1$ or $\mathfrak{p} \notin \bigcup_{i\in I}[\mathfrak{p}_i, \mathfrak{q}_i]$, and then $\tau_{\mathfrak{p}}\hat{f} = 1$. All in all, the corollary 4.3 shows that $\langle f \rangle \in WR + D$.

Therefore, assume $I = \emptyset$, this means that $\tau_{\mathfrak{p}_i} f = 1$ for all $1 \le i \le N$. Let

$$\mathfrak{q}_1^1, \ldots, \mathfrak{q}_{s_1}^1, \mathfrak{q}_1^2, \ldots, \mathfrak{q}_{s_2}^2, \ldots, \mathfrak{q}_1^N, \ldots, \mathfrak{q}_{s_N}^N$$

be all points of γ where $\tau_{(\cdot)} f$ is null. We assume here that $\mathfrak{q}_j^i \in \gamma_i$ and so $s_1 + \cdots + s_N = s$, $s_i \in 2\mathbb{Z}$, $s_i \ge 0$. For every $1 \le i \le N$ we renumber points $\mathfrak{q}_j^i$ $(1 \le j \le s_i)$ in such a way (see fig. 1), that

$$\mathfrak{p}_i \in]\mathfrak{q}_1^i, \mathfrak{q}_2^i[\subsetneq]\mathfrak{q}_3^i, \mathfrak{q}_4^i[\subsetneq \cdots \subsetneq]\mathfrak{q}_{s_i-1}^i, \mathfrak{q}_{s_i}^i[\,.$$

Since $\tau_{\mathfrak{p}_i} f = 1$, so we have

$$\partial_{\mathfrak{q}_1^i} f = 1, \partial_{\mathfrak{q}_2^i} f = -1, \partial_{\mathfrak{q}_3^i} f = 1, \ldots, \partial_{\mathfrak{q}_j^i} f = (-1)^{\lfloor \frac{j}{2} \rfloor}, \ldots$$

for every $1 \le i \le N$ and $1 \le j \le s_i$. Let χ_j^i be the interval function for either $(\mathfrak{q}_j^i, \mathfrak{p}_i)$ if $j \notin 2\mathbb{Z}$ or $(\mathfrak{p}_i, \mathfrak{q}_j^i)$ if $j \in 2\mathbb{Z}$. Define square classes g_i, h_i for $1 \le i \le N$ by

$$g_i := \begin{cases} \prod\limits_{\substack{1\le j\le s_i-1 \\ j\notin 2\mathbb{Z}}} \chi_j^i & \text{for } s_i \ne 0 \\ 1 & \text{for } s_i = 0 \end{cases} \qquad h_i := \begin{cases} \prod\limits_{\substack{2\le j\le s_i \\ j\in 2\mathbb{Z}}} \chi_j^i & \text{for } s_i \ne 0 \\ 1 & \text{for } s_i = 0. \end{cases}$$

Take $g := g_1 \cdots g_N$ and $h := h_1 \cdots h_N$. In the square class group $\dot{K}/\dot{K}^2$ we have the equality $f = g \cdot h \cdot e$, for some totally positive $e \in \dot{K}/\dot{K}^2$. By the means of the corollary 4.4 it is enough to show that $\langle g \cdot h \rangle \in WR + D$.

If $s_i \le 2$ for all $1 \le i \le N$, then the lemma 4.8 implies that $\langle g \rangle, \langle h \rangle \in WR+D$. Otherwise, if $s_i > 2$ for some i, then

$$\operatorname{card}\{\mathfrak{p} \in \gamma : \tau_{\mathfrak{p}} g = 0\} = \operatorname{card}\{\mathfrak{p} \in \gamma : \tau_{\mathfrak{p}} h = 0\} = \sum_{1\le i\le N} 2 \cdot \left\lceil \frac{s_i}{4} \right\rceil < \sum_{1\le i\le N} s_i = s,$$

and so $\langle g \rangle, \langle h \rangle \in WR + D$ by our inductive hypothesis. As we see, in both cases forms $\langle g \rangle, \langle h \rangle$ belong to $WR + D$, hence the corollary 4.3 proves that also $\langle g \cdot h \rangle \in WR + D$. $\square$

The Witt group WK of the field K is generated by classes of forms of dimension one. Hence, we conclude:

Corollary 4.10 *$WK = WR + D$.*

All we need to finish our proof of the theorem 3.9 is to show that $WR \cap D = \{0\}$. To accomplish this we need the following lemma.

Lemma 4.11 *The composition $(\partial \circ d)|_H$ is an identity on H.*

Proof Notice that H is spanned by sequences of the form $\delta_{\mathfrak{p}} - \delta_{\mathfrak{p}_i}$ for $\mathfrak{p} \in \gamma_i \setminus \{\mathfrak{p}_i\}$ and $1 \leq i \leq N$. Hence, we restrict ourselves to such elements. Now, however, we have

$$\delta_{\mathfrak{p}} - \delta_{\mathfrak{p}_i} \xmapsto{d} \langle \chi_{(\mathfrak{p}_i, \mathfrak{p})} \rangle \xmapsto{\partial} \delta_{\mathfrak{p}} - \delta_{\mathfrak{p}_i}.$$

$\square$

Recall that the group $\hat{H}$ was defined in (3.6).

Corollary 4.12 *The kernel $\ker d$ of d equals $\hat{H}$.*

Proof The inclusion $\hat{H} \subseteq \ker d$ is trivial. We show the other one. Take any $\varphi \in \ker d$. The proposition 3.5 states that $\bigoplus_{\mathfrak{p} \in \gamma} WK(\mathfrak{p}) = H \oplus \hat{H}$, so we can (uniquely) express φ as a sum $\varphi = h + \hat{h}$, with $h \in H$ and $\hat{h} \in \hat{H}$. As we have already noticed, $\hat{H} \subseteq \ker d$, consequently $d\varphi = dh + d\hat{h} = dh$. The previous lemma implies that $(\partial \circ d)(h) = h$. Now, φ was taken from the kernel of d, thus

$$0 = \partial(0) = (\partial \circ d)(\varphi) = (\partial \circ d)(h) = h.$$

All in all, $\varphi = \hat{h} \in \hat{H}$ and so $\ker d = \hat{H}$, as desired. $\square$

We are now ready to finalise the proof of the theorem 3.9.

Corollary 4.13 *$WR \cap D = \{0\}$.*

Proof Take any $\psi \in WR \cap D$. Since $\psi \in D$, so there is $\varphi \in \bigoplus_{\mathfrak{p} \in \gamma} WK(\mathfrak{p})$ such that $\psi = d\varphi$. By the means of the above corollary we may assume that $\varphi \in H$. Now lemma 4.11 implies that $\varphi = (\partial \circ d)(\varphi) = \partial\psi$. But $WR = \ker \partial$ (see lemma 3.1) and so $\varphi = 0$. Thus also $\psi = d\varphi = 0$. $\square$

Now the theorem 3.9 follows immediately from corollaries 4.10 and 4.13.

5 Main result

Let $\pi\colon WK = WR \oplus D \to WR$ be the projection of WK onto WR.

Theorem 5.1 *The following two sequences are exact*

$$0 \rightleftarrows WR \underset{\pi}{\overset{i}{\rightleftarrows}} WK \underset{d}{\overset{\partial}{\rightleftarrows}} \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \underset{l}{\overset{\lambda}{\rightleftarrows}} \mathbb{Z}^N \rightleftarrows 0.$$

Moreover:

(a) $WK = \operatorname{im} i \oplus \operatorname{im} d = \ker \partial \oplus \ker \pi$*;*
(b) $\bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) = \operatorname{im} \partial \oplus \operatorname{im} l = \ker \lambda \oplus \ker d$*;*
(c) $\lambda \circ l = \mathrm{id}_{\mathbb{Z}^N}$*;*
(d) $\pi \circ i = \mathrm{id}_{WR}$*;*
(e) $(\partial \circ d)|_H = \mathrm{id}_H$*;*
(f) $(d \circ \partial)|_D = \mathrm{id}_D$*.*

Proof The exactness of the upper sequence was shown in the lemma 3.1. The exactness of the lower sequence at $\bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p})$ follows from the observation 3.7 together with the corollary 4.12. Next, the theorem 3.9 implies the exactness at WK. As to the "moreover" part: point (a) follows from the theorem 3.9; point (b) from the proposition 3.5; point (c) from the observation 3.7; (d) is trivial; (e) was proved in lemma 4.11. All in all, we need only to show (f). Take $\psi \in D$, there is $\varphi \in H$ such that $d\varphi = \psi$. Hence, $\partial\psi = (\partial \circ d)(\varphi) = \varphi$ and so $(d \circ \partial)(\psi) = d\varphi = \psi$. □

Notice that the above theorem provides the full analogy for splitting of the Knebusch-Milnor exact sequence, that is normally defined for sequences having only three non-zero terms. We can rephrase the above result, in a more homological favor, using slicing and patching technique, as well.

Corollary 5.2 *The exact sequence*

$$0 \to WR \xrightarrow{i} WK \xrightarrow{\partial} \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \xrightarrow{\lambda} \mathbb{Z}^N \to 0.$$

slices into and is patched by the following two split exact sequences:

$$0 \to WR \to WK \to H \to 0 \qquad \text{and} \qquad 0 \to D \to \bigoplus_{\mathfrak{p}\in\gamma} WK(\mathfrak{p}) \to \mathbb{Z}^N \to 0.$$

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