This is an early version of a paper "Graph of even points on an arithmetic curve". I leave it online since some of the proofs presented here are completely different from the ones included in the final article.

GRAPH OF EVEN POINTS IN A GLOBAL FUNCTION FIELD

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Abstract. Let K be a global function field of characteristic $\neq 2$ and X be an associated smooth curve. A point $\mathfrak{p} \in X$ is called even if its class in the Picard group of X is 2-divisible. The even points admit a certain symmetric relation, discovered while investigating automorphisms of the Witt ring of K. The set of even points of X equipped with this relation is an (undirected infinite) graph. In this note we investigate properties of this graph. In particular we prove that it is connected and has diameter 2.

Let $\mathbb{F}_q$ be a finite field of odd characteristic. Further let $K = \mathbb{F}_q(X)$ be a global function field, with a full field of constants $\mathbb{F}_q$, and X be the associated smooth projective curve. For a (closed) point $\mathfrak{p} \in X$, denote by $[\mathfrak{p}]$ its class in the Picard group of X. We say that $\mathfrak{p} \in X$ is *even*, if its class in the Picard group is 2-divisible, i.e. $[\mathfrak{p}] \in 2 \operatorname{Pic} X$. Even points emerge naturally when one studies theory of quadratic forms over global function fields. It was proved in [2] that a point is even if and only if it is a unique wild point of some self-equivalence of K. The set of even points admit a symmetric relation $\smile$ (defined below) that controls the formation of bigger wild sets of self-equivalences of K (see [2, Proposition 4.5] and [1, Proposition 4.7]). The aim of this short note is to show that the set of even points equipped with this relation is a connected graph and its diameter equals 2.

We use the following notation. Let $\mathfrak{p} \in X$ be a (closed) point, then $\operatorname{ord}_\mathfrak{p}$ denotes the associated valuation, $\mathcal{O}_\mathfrak{p} = \{\lambda \in K \mid \operatorname{ord}_\mathfrak{p} \lambda \geq 0\}$ is the valuation ring, $K_\mathfrak{p}$ is the completion of K at $\mathfrak{p}$ and $K(\mathfrak{p})$ is the residue field. Throughout this paper we make free use of basic facts from class field theory and density of primes (points) in global function fields. Standard references for all these facts are [3, 9].

It is well known that the exact sequence

$$0 \to \operatorname{Pic}^0 X \to \operatorname{Pic} X \xrightarrow{\deg} \mathbb{Z} \to 0$$

splits (because $\mathbb{Z}$ is a projective $\mathbb{Z}$-module). Equivalently

$$\operatorname{Pic} X \cong \operatorname{Pic}^0 X \oplus \mathbb{Z} \tag{1}$$

and the projection onto the second coordinate is the degree homomorphism. Therefore the necessary condition for a point $\mathfrak{p} \in X$ to be even is $\deg \mathfrak{p} \in 2\mathbb{Z}$. This condition is not sufficient, though, unless $|\operatorname{Pic}^0 X|$ is odd. A convenient condition of evenness of $\mathfrak{p}$ makes use of a certain subgroup of the square class group of K. Let $Y \subseteq X$ be an open nonempty set. The group morphism $\operatorname{div}_Y : K^\times \to \operatorname{Div} Y$, that assigns to a nonzero element of K its principal divisor, induces a morphism of the quotient groups $K^\times/K^{\times 2} \to \operatorname{Div} Y/2 \operatorname{Div} Y$. Harmlessly abusing the notation, we denote the latter morphism by div_Y, too. Define the subgroup $\mathbb{E}_Y < K^\times/K^{\times 2}$ to be the kernel of this map:

$$\mathbb{E}_Y := \ker\bigl(\operatorname{div}_Y : K^\times/K^{\times 2} \to \operatorname{Div} Y/2 \operatorname{Div} Y\bigr).$$

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It is clear that $\mathbb{E}_Y$ consists of the square classes of elements of K having even valuations everywhere on Y:

$$\mathbb{E}_Y = \bigl\{\lambda \in {}^{K^\times}\!/\!{}_{K^{\times 2}} \mid \operatorname{ord}_{\mathfrak{p}} \lambda \equiv 0 \pmod 2 \text{ for all } \mathfrak{p} \in Y\bigr\}.$$

Proposition 1. $\mathbb{E}_X \cong {}^{\operatorname{Pic} X}\!/\!{}_{2\operatorname{Pic} X}$.

Proof. The group $\operatorname{Pic}^0 X$ is finite and so ${}^{\operatorname{Pic}^0 X}\!/\!{}_{2\operatorname{Pic}^0 X}$ is isomorphic to a group $(\operatorname{Pic}^0 X)_2$ of elements of order not exceeding 2. In view of Eq. (1) we may write

$${}^{\operatorname{Pic} X}\!/\!{}_{2\operatorname{Pic} X} \cong (\operatorname{Pic}^0 X)_2 \oplus {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}.$$

Consider a group epimorphism $\mathbb{E}_X \twoheadrightarrow (\operatorname{Pic}^0 X)_2$ that sends $\lambda \in \mathbb{E}_X$ to $[\frac{1}{2}\operatorname{div}_X \lambda]$. The kernel of this map is isomorphic to ${}^{\mathbb{F}_q^\times}\!/\!{}_{\mathbb{F}_q^{\times 2}}$, the square-class group of of the base field. The characteristic of $\mathbb{F}_q$ is odd, therefore ${}^{\mathbb{F}_q^\times}\!/\!{}_{\mathbb{F}_q^{\times 2}} \cong {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}$. The exact sequence

$$0 \to {}^{\mathbb{F}_q^\times}\!/\!{}_{\mathbb{F}_q^{\times 2}} \to \mathbb{E}_X \to (\operatorname{Pic}^0 X)_2 \to 0$$

splits since $(\operatorname{Pic}^0 X)_2$, being an elementary 2-group, is a projective module over ${}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}$. All in all, we have

$$\mathbb{E}_X \cong {}^{\mathbb{F}_q^\times}\!/\!{}_{\mathbb{F}_q^{\times 2}} \oplus (\operatorname{Pic}^0 X)_2 \cong {}^{\operatorname{Pic} X}\!/\!{}_{2\operatorname{Pic} X}. \qquad \square$$

Proposition 2. *Let $\mathfrak{p} \in X$ be a point. Set $Y := X \setminus \{\mathfrak{p}\}$. Then $\mathfrak{p}$ is even if and only if* $[\mathbb{E}_Y : \mathbb{E}_X] = 2$.

Proof. We have $\mathbb{E}_X \cong {}^{\operatorname{Pic} X}\!/\!{}_{2\operatorname{Pic} X}$. Moreover [2, Proposition 2.3.(1)] implies that

$$\mathbb{E}_Y \cong {}^{\operatorname{Pic} Y}\!/\!{}_{2\operatorname{Pic} Y} \oplus {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}.$$

It now follows from [2, Proposition 2.7] that $\mathfrak{p}$ is even if and only if the right hand side is isomorphic to

$$\bigl({}^{\operatorname{Pic}^0 X}\!/\!{}_{2\operatorname{Pic}^0 X} \oplus {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}\bigr) \oplus {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}.$$

Using again Eq. (1), we may replace the direct sum in the parenthesis by $\mathbb{E}_X$. Thus $[\mathfrak{p}] \in 2\operatorname{Pic} X$ if and only if

$$\mathbb{E}_Y \cong \mathbb{E}_X \oplus {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}.$$

This proves the proposition. $\square$

Let $\mathfrak{p}$ be an even point. It follows from the above proposition that there is a (non-unique) square class $\lambda_{\mathfrak{p}} \in {}^{K^\times}\!/\!{}_{K^{\times 2}}$ such that $\mathbb{E}_Y$ is the disjoint union of $\mathbb{E}_X$ and its coset $\lambda_{\mathfrak{p}} \cdot \mathbb{E}_X$. In particular, $\mathfrak{p}$ is the unique point at which $\lambda_{\mathfrak{p}}$ has an odd valuation. It turns out that this property characterizes even points. A couple of results in this note rely on the following criterion of evenness, obtained in [2]. For the convenience of the reader and to make the exposition more complete we restate it here explicitly.

Proposition 3 ([2, Proposition 3.2])**.** *A point $\mathfrak{p} \in X$ is even if and only if there is an element $\lambda \in \mathbb{E}_{X\setminus\{\mathfrak{p}\}}$ such that* $\operatorname{ord}_{\mathfrak{p}} \lambda \equiv 1 \pmod 2$.

Proof. We have:

$$\begin{aligned}
[\mathfrak{p}] \in 2\operatorname{Pic} X &\iff \exists_{\lambda\in K^\times} \exists_{\mathscr{D}\in\operatorname{Div} X}\, \mathfrak{p} + 2\mathscr{D} = \operatorname{div}_X \lambda \\
&\iff \exists_{\lambda\in K^\times} \exists_{\mathscr{D}\in\operatorname{Div} X} \Bigl(\operatorname{ord}_{\mathfrak{p}} \lambda \equiv 1 \pmod 2 \wedge \forall_{\mathfrak{q}\neq\mathfrak{p}} \operatorname{ord}_{\mathfrak{q}} \lambda = \operatorname{ord}_{\mathfrak{q}} 2\mathscr{D}\Bigr) \\
&\iff \exists_{\lambda\in K^\times} \Bigl(\operatorname{ord}_{\mathfrak{p}} \lambda \equiv 1 \pmod 2 \wedge \forall_{\mathfrak{q}\neq\mathfrak{p}} \operatorname{ord}_{\mathfrak{q}} \lambda \equiv 0 \pmod 2\Bigr). \qquad \square
\end{aligned}$$

The next result emphasizes the link between $\mathbb{E}_X$ and even points.

Theorem 4. *For every $\mu \in K^\times$ the following two conditions are equivalent:*

(1) $\mu \in \mathbb{E}_X$,
(2) *μ is a local square at every even point.*

Proof. The implication (1) $\implies$ (2) follows from [2, Proposition 3.4]. We need to prove the other implication. Assume that μ is a local square at every even point and suppose that there is a point $\mathfrak{q}_1 \in X$ such that $\operatorname{ord}_{\mathfrak{q}_1} \mu \equiv 1 \pmod 2$. The divisor of μ has a form

$$\operatorname{div}_X \mu = \sum_{i=1}^{k} \mathfrak{q}_i + 2\mathscr{D},$$

for some divisor $\mathscr{D} \in \operatorname{Div} X$. The points $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_k$ are all the points where μ has odd valuations. Consequently $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_k$ cannot be even, because μ is a local square at all even points. Moreover k must be greater than 1, since otherwise $\mathfrak{q}_1$ would be an even point by Proposition 3. The local square class group ${}^{K_{\mathfrak{q}_1}^{\times}}/{}_{K_{\mathfrak{q}_1}^{\times 2}}$ consists of four classes:

$$K_{\mathfrak{q}_1}^{\times}/K_{\mathfrak{q}_1}^{\times 2} = \{1, u, \pi, u\pi\},$$

where $\operatorname{ord}_{\mathfrak{q}_1} u \equiv 0 \pmod 2$ and $\operatorname{ord}_{\mathfrak{q}_1} \pi \equiv 1 \pmod 2$. It follows from [6, Lemma 2.1] that there is $\lambda \in K^{\times}$ and a point $\mathfrak{p} \in X$ such that:

$$\begin{array}{ll}
\operatorname{ord}_{\mathfrak{p}} \lambda = 1 & \\
\operatorname{ord}_{\mathfrak{q}_1}(\lambda - u) \geq 1 & \\
\operatorname{ord}_{\mathfrak{q}_i}(\lambda - 1) \geq 1 & \text{for } i \in \{2, \ldots, k\} \\
\operatorname{ord}_{\mathfrak{r}} \lambda \equiv 0 \pmod 2 & \text{for every } \mathfrak{r} \in X \setminus \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}.
\end{array}$$

Then $\mathfrak{p}$ is the unique point where λ has an odd valuation. Hence $\mathfrak{p}$ is even by Proposition 3 and $\lambda = \lambda_{\mathfrak{p}}$. We claim that the Hilbert symbol $(\lambda, \mu)_{\mathfrak{r}}$ vanishes for all $\mathfrak{r} \in X \setminus \{\mathfrak{p}, \mathfrak{q}_1\}$. Indeed, if $\mathfrak{r} \notin \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$, then both λ and μ have even valuations at $\mathfrak{r}$ and so $(\lambda, \mu)_{\mathfrak{r}} = 1$. On the other hand, if $\mathfrak{r} = \mathfrak{q}_i$ for some $i \geq 2$, then the condition $\operatorname{ord}_{\mathfrak{r}}(\lambda - 1) \geq 1$ implies that $\lambda \in K_{\mathfrak{r}}^{\times 2}$. Thus again $(\lambda, \mu)_{\mathfrak{q}} = 1$. Therefore, Hilbert reciprocity law asserts that

$$(\lambda, \mu)_{\mathfrak{p}} \cdot (\lambda, \mu)_{\mathfrak{q}_1} = 1.$$

Now $(\lambda, \mu)_{\mathfrak{q}_1} = -1$, because $\lambda \equiv u \pmod{K_{\mathfrak{q}_1}^{\times 2}}$ and $\operatorname{ord}_{\mathfrak{q}_1} \mu \equiv 1 \pmod 2$. This implies that also $(\lambda, \mu)_{\mathfrak{p}} = -1$. This, however, is clearly impossible since $\mathfrak{p}$ is an even point and so λ is a square at $\mathfrak{p}$. □

The following proposition substantially strengthens [2, Proposition 3.11], which was proved using different, more elementary methods.

Proposition 5. *The set of even points has a positive density.*

Proof. A classical theorem by F.K. Schmidt (see e.g. [8, Corollary V.1.11]) asserts that $\min\{\deg \mathscr{D} : \mathscr{D} \in \operatorname{Div} X, \deg \mathscr{D} > 0\} = 1$. It follows that there must exist a point $\mathfrak{o} \in X$ of an odd degree. Denote $Y := X \setminus \{\mathfrak{o}\}$, then $\operatorname{Pic} Y$ may be viewed as the ideal class group of a Dedekind domain $\mathcal{O}_Y = \bigcap_{\mathfrak{q} \in Y} \mathcal{O}_{\mathfrak{q}}$. The group $\operatorname{Pic} Y$ is finite. Moreover, the points whose classes in $\operatorname{Pic} Y$ are 2-divisible are precisely the even points of X (i.e. points whose classes are 2-divisible in $\operatorname{Pic} X$) by [2, Proposition 3.6]. Take now any effective divisor $\mathscr{D} \in \operatorname{Div} Y$, $\mathscr{D} \gneqq 0$ and consider a set

$$E(\mathscr{D}) := \{\mathfrak{p} \in Y \mid [\mathfrak{p}]_Y = [2\mathscr{D}]_Y\}.$$

Here $[\mathfrak{p}]_Y$ stands for the class of $\mathfrak{p}$ in $\operatorname{Pic} Y$. Dirichlet density theorem for function fields says that $E(\mathscr{D})$ has density ${}^1/_{|\operatorname{Pic} Y|} > 0$. On the other hand, it is clear that the set of even points is a finite disjoint union $E(\mathscr{D}_1) \dot\cup \ldots \dot\cup E(\mathscr{D}_k)$, where the classes of $\mathscr{D}_1, \ldots, \mathscr{D}_k$ give rise to all 2-divisible elements in $\operatorname{Pic} Y$. Thus its density is ${}^k/_{|\operatorname{Pic} Y|} > 0$. □

In [2] we defined a relation $\smile$ on the set of even points by the condition:

$$\mathfrak{p} \smile \mathfrak{q} \qquad \text{if and only if} \qquad \mathbb{E}_{X \setminus \{\mathfrak{p}\}} \setminus \mathbb{E}_X \subseteq K_{\mathfrak{q}}^{\times 2}.$$

Let $\lambda_\mathfrak{p}$ and $\lambda_\mathfrak{q}$ be squares classes corresponding to $\mathfrak{p}$ and $\mathfrak{q}$, respectively. Hilbert reciprocity law asserts that

$$\prod_{\mathfrak{r}\in X}(\lambda_\mathfrak{p},\lambda_\mathfrak{q})_\mathfrak{r}=1.$$

Here $(-,-)_\mathfrak{r}$ stands for the Hilbert symbol. If $\mathfrak{r}$ is neither $\mathfrak{p}$ nor $\mathfrak{q}$, then $\lambda_\mathfrak{p}$ and $\lambda_\mathfrak{q}$ have even valuations at $\mathfrak{r}$ and so $(\lambda_\mathfrak{p},\lambda_\mathfrak{q})_\mathfrak{r}=1$. Therefore the above formula simplifies to

$$(\lambda_\mathfrak{p},\lambda_\mathfrak{q})_\mathfrak{p}\cdot(\lambda_\mathfrak{p},\lambda_\mathfrak{q})_\mathfrak{q}=1.$$

If we assume in addition that $\mathfrak{p}\smile\mathfrak{q}$, then $\lambda_\mathfrak{p}\in K_\mathfrak{q}^{\times 2}$ and so $(\lambda_\mathfrak{p},\lambda_\mathfrak{q})_\mathfrak{q}=1$, which implies that also $(\lambda_\mathfrak{p},\lambda_\mathfrak{q})_\mathfrak{p}=1$. Now, $\operatorname{ord}_\mathfrak{p}\lambda_\mathfrak{p}\equiv 1 \pmod 2$, hence $\lambda_\mathfrak{q}$ must be a local square at $\mathfrak{p}$. This shows that $\mathfrak{q}\smile\mathfrak{p}$, proving a symmetry of the relation.

Observation 6 ([2, Lemma 4.3])**.** *The relation $\smile$ is symmetric.*

We will need the following characterization of the relation $\smile$.

Lemma 7. *Let $\mathfrak{p},\mathfrak{q}\in X$ be two even points and $\lambda_\mathfrak{p}\in {}^{K^\times}\!/_{K^{\times 2}}$ be defined as above. Then $\mathfrak{p}\smile\mathfrak{q}$ is and only if $\mathfrak{q}$ splits in $K(\sqrt{\lambda_\mathfrak{p}})$.*

Proof. Take a polynomial $f:=t^2-\lambda_\mathfrak{p}\in K[t]$ and let $\overline{f}\in K(\mathfrak{q})[t]$ be the reduction of f modulo $\mathfrak{q}$. Then $\mathfrak{q}$ splits in $K(\sqrt{\lambda_\mathfrak{p}})\cong {}^{K[t]}\!/_{\langle f\rangle}$ if and only if $\overline{f}$ factors into linear terms, if and only if $\lambda_\mathfrak{p}(\mathfrak{q})$ is a square in the residue field $K(\mathfrak{q})$. Now, $\operatorname{ord}_\mathfrak{q}\lambda_\mathfrak{p}$ is even, so we may choose a representative of the square class $\lambda_\mathfrak{p}$ that has valuation 0 at $\mathfrak{q}$. Abusing the notation slightly, denote it $\lambda_\mathfrak{p}$ again. Then $\lambda_\mathfrak{p}(\mathfrak{q})\in K(\mathfrak{q})^{\times 2}$ if and only if $\lambda_\mathfrak{p}\in K_\mathfrak{q}^{\times 2}$ by the well known correspondence between square classes in a local field and in its residue field (see e.g. [5, Lemma VI.1.1]). The last condition means that $\mathfrak{p}\smile\mathfrak{q}$. □

For the next result, we need the following (rather basic) fact from group theory.

Lemma 8. *Let G be an abelian group and H be its finite subgroup. Assume that H has an odd number of elements. Then for every $g\in G$ we have*

$$g\in 2G \iff (g+H)\in 2({}^{G}\!/_{H}).$$

Proof. Assume that $g+H=2g'+H$ for some $g'\in G$. Then there is $h\in H$ such that $g=2g'+h$. Now, $|H|\equiv 1 \pmod 2$, hence every element of H is 2-divisible. In particular $h=2h'$ for some $h'\in H$. Consequently, $g=(2g'+h')$. This proves one implication. The other one is (even more) trivial. □

Proposition 9. *Take $\lambda\in K^\times$ and let*

$$A=\{\mathfrak{p}\in X \mid [\mathfrak{p}]\in 2\operatorname{Pic}X, \lambda\in K_\mathfrak{p}^{\times 2}\}$$

be the set of the even points at which λ is a local square. Then A has a positive density.

Proof. The proof of this proposition combines ideas from the proofs of Proposition 5 and Lemma 7. Let $\overline{\mathrm{K}}$ be a fixed algebraic closure of K. As in the proof of Proposition 5, fix a point $\mathfrak{o}\in X$ of an odd degree and denote $Y:=X\setminus\{\mathfrak{o}\}$. Consider the ray class group $C_\mathfrak{o}$ with the modulus $\mathfrak{o}$. Take the subgroup $H:=C_\mathfrak{o}$ of $C_\mathfrak{o}$ and let $L_1\subset\overline{\mathrm{K}}$ be the class field for H. Then $\operatorname{Gal}(L_1/K)\cong {}^{C_\mathfrak{o}}\!/_{H}$.

We claim that the points that split completely in L_1 are precisely the even points of X. Indeed, a point $\mathfrak{p}\in X$ splits completely in L_1 if and only if its Artin symbol vanishes, i.e. $\left(\frac{L_1/K}{\mathfrak{p}}\right)=1\in\operatorname{Gal}(L_1/K)$. This happens if and only if the class of $\mathfrak{p}$ in $C_\mathfrak{o}$ sits in H, i.e. the class of $\mathfrak{p}$ is 2-divisible in $C_\mathfrak{o}$. We have an exact sequence (see [4, Section 9], this is a function field analog of [7, Theorem V.1.7]):

$$1\to\mathbb{F}_q^\times\to K(\mathfrak{o})^\times\to C_\mathfrak{o}\to\operatorname{Pic}X\to 0.$$

In particular, if $\deg \mathfrak{o} = 1$, then $C_{\mathfrak{o}} \cong \operatorname{Pic} X$ and the claim follows trivially. If $\deg \mathfrak{o} \neq 1$, then

$$\operatorname{Pic} X \cong {}^{C_{\mathfrak{o}}}\!/\!_{\ker(C_{\mathfrak{o}} \twoheadrightarrow \operatorname{Pic} X)},$$

where

$$\ker(C_{\mathfrak{o}} \twoheadrightarrow \operatorname{Pic} X) \cong {}^{K(\mathfrak{o})^\times}\!/\!_{\mathbb{F}_q^\times}.$$

The cardinality of the latter group is

$$\left|{}^{K(\mathfrak{o})^\times}\!/\!_{\mathbb{F}_q}\right| = \frac{q^{\deg \mathfrak{o}} - 1}{q - 1} = q^{\deg \mathfrak{o} - 1} + q^{\deg \mathfrak{o} - 2} + \cdots + 1.$$

Recall that $\deg \mathfrak{o}$ is odd. Thus, in the above sum we have an odd number of odd integers. Therefore $\left|{}^{K(\mathfrak{o})^\times}\!/\!_{\mathbb{F}_q}\right| \equiv 1 \pmod 2$ and the claim follows from the preceding lemma. In particular, it follows from Chebotarev density theorem that the density of even points is ${}^{1}\!/\!_{[L_1\,:\,K]}$. This provides an alternative proof of Proposition 5.

So far we have not used λ, at all. Hence, now let $L_2 := K(\sqrt{\lambda}) \subset \overline{\mathrm{K}}$. As in the proof of Lemma 7, observe that a point $\mathfrak{p} \in X$ splits in L_2 if and only if λ is a local square at $\mathfrak{p}$.

There are two cases to consider. Either L_2 is contained in L_1 or it is not. The first case happens when λ is a local square at every even point of X. In this case the assertion is trivial. In particular, the relative density of A in set of even points is 1.

The more interesting case is $L_2 \not\subset L_1$. We then take the composite field $L := L_1 L_2$. Then a point $\mathfrak{p} \in X$ splits completely in L if and only if $\mathfrak{p}$ is even (splitting in L_1) and λ is a local square at $\mathfrak{p}$ (splitting in L_2). Again using Chebotarev density theorem, we obtain the density of A equal $\delta(A) = {}^{1}\!/\!_{[L\,:\,K]} > 0$. In this case, the relative density of A in the set of all even points is then ${}^{1}\!/\!_{[L\,:\,L_1]} = {}^{1}\!/\!_{2}$. □

Corollary 10. *For every $\lambda \in K^\times$, the set*

$$B = \left\{\mathfrak{p} \in X \mid [\mathfrak{p}] \in 2 \operatorname{Pic} X, \lambda \notin K_{\mathfrak{p}}^{\times 2}\right\}$$

is either empty or has a positive density.

Proof. If $\lambda \in \mathbb{E}_X$, then the set B is empty by [2, Proposition 3.4]. Otherwise, B is the complement of the set A in the set of all even points. These two sets have densities, hence so does B. It follows from the analysis of relative densities at the end of the proof of Proposition 9 that for $\lambda \notin \mathbb{E}_X$, we have

$$\delta(B) = \delta(\left\{[\mathfrak{p}] \in 2 \operatorname{Pic} X\right\}) - \delta(A) = \frac{1}{2} \cdot \delta(\left\{[\mathfrak{p}] \in 2 \operatorname{Pic} X\right\}) > 0. \qquad \square$$

Remark. The celebrated Global Square Theorem (GST) says that an element is locally a square *almost everywhere* if and only if it is a square globally if and only if it is a local square *everywhere*. Proposition 9 together with Theorem 4 can be considered to be a certain analog of GST for even points. These two results say that an element is a local square at almost very even point if and only if it is a local square at every even point if and only if it lies in $\mathbb{E}_X$.

We may now define an (infinite) graph $\mathscr{E} = (V, E)$, whose vertices are the even points of X and edges are defined by the relation $\smile$, that is:

$$V = \left\{\mathfrak{p} \in X \mid [\mathfrak{p}] \in 2 \operatorname{Pic} X\right\}, \qquad E = \left\{(\mathfrak{p}, \mathfrak{q}) \in V \times V \mid \mathfrak{p} \smile \mathfrak{q}\right\}.$$

Proposition 11. *No vertex of $\mathscr{E}$ is adjacent to all other vertices. In particular, $\mathscr{E}$ is not complete.*

Proof. Take an even point $\mathfrak{p} \in X$ and let $\mu \in K^\times$ be an element such that $\operatorname{ord}_\mathfrak{p} \mu = 0$ and $\mu \notin K_\mathfrak{p}^{\times 2}$. Using [6, Lemma 2.1] we show that there is a point $\mathfrak{q} \in X$ and $\lambda \in \mathbb{E}_{X \setminus \{\mathfrak{q}\}}$ such that

$$\operatorname{ord}_\mathfrak{p}(\lambda - \mu) \geq 1 \qquad \text{and} \qquad \operatorname{ord}_\mathfrak{q} \lambda = 1.$$

Thus we can write $\lambda = \mu + \rho$ for some $\rho \in K_\mathfrak{p}$, $\operatorname{ord}_\mathfrak{p} \rho \geq 1$. It follows that in the residue field we have

$$(\lambda\mu)(\mathfrak{p}) = \mu(\mathfrak{p})^2 \in K(\mathfrak{p})^{\times 2}.$$

The well know correspondence between square class groups of a local field in its residue field ensures that $\lambda \equiv \mu \pmod{K_\mathfrak{p}^{\times 2}}$. In particular λ is not a local square at $\mathfrak{p}$.

Now $\mathfrak{q}$ is the only point where λ has an odd valuation, hence $[\mathfrak{q}] \in 2 \operatorname{Pic} X$ by Proposition 3. Moreover $\lambda = \lambda_\mathfrak{q}$ and the classes of λ and μ coincide in ${}^{K_\mathfrak{p}^\times}/{}_{K_\mathfrak{p}^{\times 2}}$. Hence $\lambda_\mathfrak{q} \notin K_\mathfrak{p}^{\times 2}$, which means that $\mathfrak{q} \not\smile \mathfrak{p}$. □

Theorem 12. *The graph $\mathscr{E}$ is connected and has a diameter* 2.

We present two different proofs of the theorem.

Proof A. The diameter of $\mathscr{E}$ is greater than 1 by Proposition 11. We show that it does not exceed 2. Let $\mathfrak{p}, \mathfrak{q} \in X$, $\mathfrak{p} \neq \mathfrak{q}$ be two even points. Suppose that $\mathfrak{p} \not\smile \mathfrak{q}$, i.e. they are not connected by an edge of $\mathscr{E}$. We will shows that there is an even point $\mathfrak{r} \in X$ adjacent to both of them simultaneously. Fix square classes $\lambda_\mathfrak{p}, \lambda_\mathfrak{q} \in {}^{K^\times}/{}_{K^{\times 2}}$ such that

$$\mathbb{E}_{X \setminus \{\mathfrak{p}\}} = \mathbb{E}_X \mathbin{\dot\cup} \lambda_\mathfrak{p} \cdot \mathbb{E}_X \qquad \text{and} \qquad \mathbb{E}_{X \setminus \{\mathfrak{q}\}} = \mathbb{E}_X \mathbin{\dot\cup} \lambda_\mathfrak{q} \cdot \mathbb{E}_X.$$

Take $L := K(\sqrt{\lambda_\mathfrak{p}}, \sqrt{\lambda_\mathfrak{q}})$, then a point $\mathfrak{r} \in X$ splits completely in L if and only if its Artin symbol is $\left(\frac{L/K}{\mathfrak{r}}\right) = 1 \in \operatorname{Gal}(L/K)$. Thus Chebotarev density theorem implies that the relative density of the even points that split completely in L in the set of all even points is ${}^1/{}_{[L:K]} = {}^1/{}_4$. Take $\mathfrak{r} \in X$ to be any even point that splits completely in L. Then $\mathfrak{r}$ splits simultaneously in $K(\sqrt{\lambda_\mathfrak{p}})$ and $K(\sqrt{\lambda_\mathfrak{q}})$, hence $\mathfrak{r} \smile \mathfrak{p}$ and $\mathfrak{r} \smile \mathfrak{q}$ by Lemma 7. □

Proof B. As in proof A, let $\mathfrak{p}, \mathfrak{q} \in X$, $\mathfrak{p} \neq \mathfrak{q}$ be two non-adjacent even points. It follows from [6, Lemma 2.1] that there is a point $\mathfrak{r} \in X$ and an element $\lambda \in \mathbb{E}_{X \setminus \{\mathfrak{r}\}}$ such that

$$\operatorname{ord}_\mathfrak{p}(\lambda - 1) \geq 1, \qquad \operatorname{ord}_\mathfrak{q}(\lambda - 1) \geq 1 \qquad \text{and} \qquad \operatorname{ord}_\mathfrak{r} \lambda = 1$$

In particular, $\mathfrak{r}$ is the only point where λ has an odd valuation, hence $[\mathfrak{r}] \in 2 \operatorname{Pic} X$ by Proposition 3. On the other hand, the condition $\operatorname{ord}_\mathfrak{p}(\lambda - 1) = 1$ implies that λ must be a local square at $\mathfrak{p}$. The same holds for $\mathfrak{q}$. Consequently, $\mathbb{E}_{X \setminus \{\mathfrak{r}\}} \setminus \mathbb{E}_X = \lambda \cdot \mathbb{E}_X \subset K_\mathfrak{p}^{\times 2} \cap K_\mathfrak{q}^{\times 2}$. This means that $\mathfrak{r} \smile \mathfrak{p}$ and $\mathfrak{r} \smile \mathfrak{q}$. □

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