5 Applications of Hilbert Nullstellensatz.

1. Give an example of an ideal $\mathfrak{a} \lhd k[x_1, ..., x_n]$ such that $\mathcal{Z}(\mathfrak{a})$ is a singleton, but $\mathfrak{a}$ is not maximal.

2. Show that the ideal $(f) \lhd k[x_1, ..., x_n]$ is radical if and only if the polynomial f is not divisible by a square of a non-constant polynomial.

3. Let $\operatorname{Spec} A$ denote the set of all prime ideals of a ring A. For $\mathfrak{a} \lhd A$ define

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

   a) Show that for $\mathfrak{a}, \mathfrak{b} \lhd A$:

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b}).$$

   b) Show that for $\mathfrak{a}_i \lhd A$, $i \in I$:

$$\bigcap_{i \in I} V(\mathfrak{a}_i) = V\left(\left\langle \bigcup_{i \in I} \mathfrak{a}_i \right\rangle\right).$$

   c) Deduce that there is a topology on $\operatorname{Spec} A$ whose closed sets are $V(\mathfrak{a})$, $\mathfrak{a} \lhd A$. We shall call it the **Zariski topology** on $\operatorname{Spec} A$.

4. Show that $\operatorname{Spec} A$ with the Zariski topology is a compact space.

5. Show that if $\operatorname{Spec} A$ is a union of two disjoint closed sets in the Zariski topology, then there exists an idempotent element in A different from 0 and 1.

Homework: Problems 3, 4 and 5.