Bonus problem set: problems to help you study for the exam

(1) Solve the following equations:
(a) $5x^2+5x+1=0$ in $\mathbb{Z}_{11}$, (b) $x^2+x+3=0$ in $\mathbb{Z}_5$,
(c) $2x^2+2x+2=0$ in $\mathbb{Z}_{13}$, (d) $2x^3+3x^2+x-4=0$ in $\mathbb{Z}_7$.

(2) Solve the following systems of equations

(a) $\begin{cases} 3x+5y=2 \\ 4x+9y=4 \end{cases}$ in $\mathbb{Z}_{13}$ and $\mathbb{Z}_7$ (b) $\begin{cases} 5x+4y=a \\ 4x+3y=b \end{cases}$ in $\mathbb{Z}_{11}$ and $\mathbb{Z}_5$.

(3) Solve the following systems of equations:

(a) $\begin{cases} (1+i)z+(2-i)w=2-2i \\ (1-i)z-(3+i)w=-3+3i \end{cases}$; (b) $\begin{cases} (3-i)z+(4+2i)w=2+6i \\ (4+2i)z-(2+3i)w=5+4i \end{cases}$;

(c) $\begin{cases} \dfrac{z}{2-i}+\dfrac{w}{1+i}=2 \\ \dfrac{5z}{(2-i)^2}+\dfrac{2w}{(1+i)^2}=3 \end{cases}$.

(4) Find the trigonometric form of the following complex numbers:
1, -1, i, $-i$,
$1+i$, $1-i$, $-1+i$, $1+i\sqrt{3}$,
$-1-i\sqrt{3}$, $\sqrt{3}-i$, $\sqrt{6}+\sqrt{2}+i(\sqrt{6}-\sqrt{2})$, $\cos\frac{\pi}{3}+i\sin\frac{\pi}{6}$,
$\cos\frac{\pi}{2}+i\sin\frac{\pi}{3}$.

(5) Evaluate:
(a) $\dfrac{(1+i\sqrt{3})^{76}+1}{(1-i)^{37}}$, (b) $\dfrac{(1-i\sqrt{3})^{32}+5}{(1+i)^{17}}$.

(6) Solve the following equations:
(a) $(1+i)z^2-(3+7i)z+10i=0$;
(b) $(1+2i)z^2-(-1+8i)z+(-5+5i)=0$;
(c) $(1+2i)z^2-(1+7i)z+(-2+6i)=0$;
(d) $(1+i)z^2-(1+5i)z+(-2+6i)=0$;
(e) $(1-i)z^2-(7+3i)z+10i=0$;
(f) $(1-2i)z^2-(4+7i)z+(7+i)=0$;
(g) $(1+i)z^2-(3+3i)z+(4+2i)=0$;

(7) Solve the following systems of linear equations over $\mathbb{R}$:

(a) $\begin{cases} 2x-3y+5z+7t=1 \\ 4x-6y+2z+3t=2 \\ 2x-3y-11z-15t=1 \end{cases}$; (b) $\begin{cases} 2x+5y-8z=8 \\ 4x+3y-9z=9 \\ 2x+3y-5z=7 \\ x+8y-7z=12 \end{cases}$;

(c) $\begin{cases} 3x+4y+z+2t=3 \\ 6x+8y+2z+5t=7 \\ 9x+12y+3z+10t=13 \end{cases}$; (d) $\begin{cases} 3x-5y+2z+4t=2 \\ 7x-4y+z+3t=5 \\ 5x+7y-4z-6t=3 \end{cases}$;

(e) $\begin{cases} 3x-2y+5z+4t=2 \\ 6x-4y+4z+3t=3 \\ 9x-6y+3z+2t=4 \end{cases}$; (f) $\begin{cases} 8x+6y+5z+2t=21 \\ 3x+3y+2z+t=10 \\ 4x+2y+3z+t=8 \\ 3x+5y+z+t=15 \\ 7x+4y+5z+2t=18 \end{cases}$;

(g) $\begin{cases} x+y+3z-2t+3w=1 \\ 2x+2y+4z-t+3w=2 \\ 3x+3y+5z-2t+3w=1 \\ 2x+2y+8z-3t+9w=2 \end{cases}$; (h) $\begin{cases} 2x-y+z+2t+3w=2 \\ 6x-3y+2z+4t+5w=3 \\ 6x-3y+2z+8t+13w=9 \\ 4x-2y+z+t+2w=1 \end{cases}$;

(i) $\begin{cases} 6x+4y+5z+2t+3w=1 \\ 3x+2y+4z+t+2w=3 \\ 3x+2y-2z+t \qquad =-7 \\ 9x+6y+z+3t+2w=2 \end{cases}$.

(8) Solve the following systems of equations over $\mathbb{Q}$ and $\mathbb{Z}_p$:

(a) $\begin{cases} 2x+7y+3z+t=6 \\ 3x+5y+2z+2t=4 \\ 9x+4y+z+7t=2 \end{cases}$, $p=11$; (b) $\begin{cases} 9x-3y+5z+6t=4 \\ 9x-3y+5z+6t=4 \\ 3x-y+3z+14t=-8 \end{cases}$, $p=13$;

(c) $\begin{cases} 6x+3y+2z+3t+4w=5 \\ 4x+2y+z+2t+w=4 \\ 4x+2y+3z+2t+w=0 \\ 2x+y+7z+3t+2w=1 \end{cases}$, $p=11$; (d) $\begin{cases} 2x-y+3z-7t=5 \\ 6x-3y+z-4t=7 \\ 4x-2y+14z-31t=18 \end{cases}$, $p=37$;

(e) $\begin{cases} x+2y+3z-2t+w=4 \\ 3x+6y+5z-4t+3w=5 \\ x+2y+7z-4t+w=11 \\ 2x+4y+2z-3t+3w=6 \end{cases}$, $p=13$; (f) $\begin{cases} 3x+2y+2z+2t=2 \\ 2x+3y+2z+5t=3 \\ 9x+y+4z-5t=1 \\ 2x+2y+3z+4t=5 \\ 7x+y+6z-t=7 \end{cases}$, $p=7$;

(g) $\begin{cases} 2x+3y+z+2t=4 \\ 4x+3y+z+t=5 \\ 5x+11y+3z+2t=2 \\ 2x+5y+z+t=1 \\ x-7y-z+2t=7 \end{cases}$, $p=17$.

(9) Solve the following systems of equations over $\mathbb{Z}_5$, $\mathbb{Z}_7$, and $\mathbb{Z}_{11}$:

(a) $\begin{cases} x+4y+3z=2 \\ 3x+2y+4z=3 \\ 4x+y+z=0 \end{cases}$, (b) $\begin{cases} 2x+3y+z=1 \\ x+4y+3z=3 \\ 4x \qquad +3z=2 \end{cases}$.

(10) Show that the system of equations $\begin{cases} x+y+z=1 \\ 2x+y-z=2 \\ x-y+3z=0 \end{cases}$ has no solutions over $\mathbb{Z}_p$ if and only if $p=2$.

(11) Solve the following system of equations over $\mathbb{C}$:

$$\begin{cases} 6ix+(-3+6i)y+(4+2i)z+(1+2i)t=0 \\ (5+5i)x+(3+5i)y+(7-3i)z+(4+2i)t=0 \\ (-3+3i)x+(-6+3i)y+(-1+3i)z \qquad -t=0 \\ (1+11i)x+(1+12i)y+(11+7i)z+7it=0 \end{cases}$$

assuming that:

(a) $x=0$, (b) $y=0$, (c) $z=0$, (d) $t=0$, (e) $x+y=0$.

(12) Solve the following systems of equations over $\mathbb{C}$:

(a) $\begin{cases} (1+i)x+2iy-z=3+2i \\ (3+i)x+(1-i)y+4z=6+i \\ 5x+y-iz=2 \end{cases}$, (b) $\begin{cases} (1+i)x+2y-iz=2-3i \\ 3x+iy+(2-i)z=6+4i \\ (4+i)x+y+3z=6+6i \end{cases}$.

(13) Solve the following systems of linear equations over $\mathbb{R}$:

(a) $\begin{cases} 2x-3y+5z+7t=1 \\ 4x-6y+2z+3t=2 \\ 2x-3y-11z-15t=1 \end{cases}$; (b) $\begin{cases} 2x+5y-8z=8 \\ 4x+3y-9z=9 \\ 2x+3y-5z=7 \\ x+8y-7z=12 \end{cases}$;

(c) $\begin{cases} 3x+4y+z+2t=3 \\ 6x+8y+2z+5t=7 \\ 9x+12y+3z+10t=13 \end{cases}$; (d) $\begin{cases} 3x-5y+2z+4t=2 \\ 7x-4y+z+3t=5 \\ 5x+7y-4z-6t=3 \end{cases}$;

(e) $\begin{cases} 3x-2y+5z+4t=2 \\ 6x-4y+4z+3t=3 \\ 9x-6y+3z+2t=4 \end{cases}$; (f) $\begin{cases} 8x+6y+5z+2t=21 \\ 3x+3y+2z+t=10 \\ 4x+2y+3z+t=8 \\ 3x+5y+z+t=15 \\ 7x+4y+5z+2t=18 \end{cases}$;

(g) $\begin{cases} x+y+3z-2t+3w=1 \\ 2x+2y+4z-t+3w=2 \\ 3x+3y+5z-2t+3w=1 \\ 2x+2y+8z-3t+9w=2 \end{cases}$; (h) $\begin{cases} 2x-y+z+2t+3w=2 \\ 6x-3y+2z+4t+5w=3 \\ 6x-3y+2z+8t+13w=9 \\ 4x-2y+z+t+2w=1 \end{cases}$;

(i) $\begin{cases} 6x+4y+5z+2t+3w=1 \\ 3x+2y+4z+t+2w=3 \\ 3x+2y-2z+t\qquad=-7 \\ 9x+6y+z+3t+2w=2 \end{cases}$.

(14) Solve the following systems of equations over $\mathbb{Q}$ and $\mathbb{Z}_p$:

(a) $\begin{cases} 2x+7y+3z+t=6 \\ 3x+5y+2z+2t=4 \\ 9x+4y+z+7t=2 \end{cases}$, $p=11$; (b) $\begin{cases} 9x-3y+5z+6t=4 \\ 9x-3y+5z+6t=4 \\ 3x-y+3z+14t=-8 \end{cases}$, $p=13$;

(c) $\begin{cases} 6x+3y+2z+3t+4w=5 \\ 4x+2y+z+2t+w=4 \\ 4x+2y+3z+2t+w=0 \\ 2x+y+7z+3t+2w=1 \end{cases}$, $p=11$; (d) $\begin{cases} 2x-y+3z-7t=5 \\ 6x-3y+z-4t=7 \\ 4x-2y+14z-31t=18 \end{cases}$, $p=37$;

(e) $\begin{cases} x+2y+3z-2t+w=4 \\ 3x+6y+5z-4t+3w=5 \\ x+2y+7z-4t+w=11 \\ 2x+4y+2z-3t+3w=6 \end{cases}$, $p=13$; (f) $\begin{cases} 3x+2y+2z+2t=2 \\ 2x+3y+2z+5t=3 \\ 9x+y+4z-5t=1 \\ 2x+2y+3z+4t=5 \\ 7x+y+6z-t=7 \end{cases}$, $p=7$;

(g) $\begin{cases} 2x+3y+z+2t=4 \\ 4x+3y+z+t=5 \\ 5x+11y+3z+2t=2 \\ 2x+5y+z+t=1 \\ x-7y-z+2t=7 \end{cases}$, $p=17$.

(15) Solve the following systems of equations over $\mathbb{Z}_5$, $\mathbb{Z}_7$, and $\mathbb{Z}_{11}$:

$$\text{(a)} \begin{cases} x+4y+3z=2 \\ 3x+2y+4z=3 \\ 4x+y+z=0 \end{cases}, \quad \text{(b)} \begin{cases} 2x+3y+z=1 \\ x+4y+3z=3 \\ 4x \qquad +3z=2 \end{cases}.$$

(16) Find the products of the following matrices:

$$\text{(a)} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 0 \\ -1 & 5 \end{bmatrix}, \text{(b)} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ 7 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}, \text{(c)} \begin{bmatrix} -3 & 4 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & -1 \end{bmatrix}^2,$$

$$\text{(d)} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^3, \text{(e)} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix},$$

$$\text{(f)} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T, \text{(g)} \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 3 & 2 \end{bmatrix}.$$

(17) Find the following determinants:

$$\text{(a)} \begin{vmatrix} 1 & 2 & 3 & 4 \\ -3 & 2 & -5 & 13 \\ 1 & -2 & 10 & 4 \\ -2 & 9 & -8 & 25 \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} 1 & -1 & 1 & -2 \\ 1 & 3 & -1 & 3 \\ -1 & -1 & 4 & 3 \\ -3 & 0 & -8 & -13 \end{vmatrix}, \quad \text{(c)} \begin{vmatrix} 7 & 6 & 9 & 4 & -4 \\ 1 & 0 & -2 & 6 & 6 \\ 1 & -1 & -2 & 4 & 5 \\ 1 & -1 & -2 & 4 & 4 \\ -7 & 0 & -9 & 2 & -2 \end{vmatrix},$$

$$\text{(d)} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad \text{(e)} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad \text{(f)} \begin{vmatrix} 4 & 4 & -1 & 0 & -1 & 8 \\ 2 & 3 & 7 & 5 & 2 & 3 \\ 3 & 2 & 5 & 7 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 & 2 \\ 1 & 7 & 6 & 6 & 5 & 7 \\ 2 & 1 & 1 & 2 & 2 & 1 \end{vmatrix},$$

$$\text{(g)} \begin{vmatrix} 1001 & 1002 & 1003 & 1004 \\ 1002 & 1003 & 1001 & 1002 \\ 1001 & 1001 & 1001 & 999 \\ 1001 & 1000 & 998 & 999 \end{vmatrix}, \quad \text{(h)} \begin{vmatrix} 30 & 20 & 15 & 12 \\ 20 & 15 & 12 & 15 \\ 15 & 12 & 15 & 20 \\ 12 & 15 & 20 & 30 \end{vmatrix}, \quad \text{(i)} \begin{vmatrix} 5 & -4 & 4 & 0 & 0 & 0 \\ 9 & -7 & 6 & 0 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \end{vmatrix},$$

$$\text{(j)} \begin{vmatrix} 1 & 6 & 20 & 50 & 140 & 140 \\ 0 & -16 & -70 & -195 & -560 & -560 \\ 0 & 26 & 125 & 366 & 1064 & 1064 \\ 0 & -31 & -154 & -460 & -1344 & -1344 \\ 0 & 4 & 20 & 60 & 176 & 175 \\ 0 & 4 & 20 & 60 & 175 & 176 \end{vmatrix}, \quad \text{(k)} \begin{vmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ -1 & 3 & 1 & 1 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 \\ -1 & -1 & -1 & 3 & 1 & 1 \\ -1 & -1 & -1 & -1 & 3 & 1 \\ -1 & -1 & -1 & -1 & -1 & 3 \end{vmatrix}.$$

(18) Evaluate:

$$\text{(a)} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 5 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 2 & 1 & 4 \end{vmatrix} \text{ over } \mathbb{Z}_7, \quad \text{(b)} \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ 1 & 1 & 4 & 3 \\ 3 & 0 & 8 & 10 \end{vmatrix} \text{ over } \mathbb{Z}_{11}, \quad \text{(c)} \begin{vmatrix} 7 & 6 & 11 & 4 & 4 \\ 1 & 0 & 2 & 6 & 6 \\ 7 & 8 & 9 & 1 & 6 \\ 1 & 10 & 2 & 4 & 5 \\ 7 & 0 & 9 & 2 & 2 \end{vmatrix} \text{ over } \mathbb{Z}_{13}.$$

(19) Determine which of the following matrices are invertible and find their inverses where possible:

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 3 & -5 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$,

(e) $\begin{bmatrix} 2 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

(20) Solve the following matrix equations:

(a) $X \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} X = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}$,

(c) $X \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{bmatrix}$,

(d) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} X \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$.

(21) Solve the following systems of matrix equations:

(a) $\begin{cases} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} Y = \begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} \\ \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} Y = \begin{bmatrix} 4 & 9 \\ -1 & -4 \end{bmatrix} \end{cases}$,

(b) $\begin{cases} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} Y = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} Y = \begin{bmatrix} 1 & 1 \\ 5 & 3 \end{bmatrix} \end{cases}$.

(22) Check which of the following subsets of the space K^4 are subspaces, where K is an arbitrary field.

(a) $U = \{[t, t+1, 0, 1] : t \in K\}$;
(b) $U = \{[t, u, t+u, t-u] : t, u \in K\}$;
(c) $U = \{[tu, u, t, 0] : t, u \in K\}$;
(d) $U = \{[x, y, z, t] : x + y - z = 0\}$;
(e) $U = \{[x, y, z, t] : xy = 0\}$;
(f) $U = \{t[1, 0, 1, 0] + u[0, -1, 0, 1] : t, u \in K\}$.

(23) Check which of the following subsets of the space $\mathbb{R}^4$ are subspaces:

(a) $U = \{[t, u, t+u, t-u] : t \leq u\}$;
(b) $U = \{[t, u, t, 0] : tu \geq 0\}$;
(c) $U = \{[x, y, z, t] : x, y, z, t \in \mathbb{Q}\}$.

(24) Let $\mathbb{R}^\infty$ be the space of sequences of elements of the field $\mathbb{R}$. Check which of the following subsets are subspaces:

(a) $U_1 = \{[a_1, a_2, \ldots] : a_{i+1} = a_i + a_{i-1}$ for every $i = 2, 3, \ldots\}$;
(b) $U_2 = \{[a_1, a_2, \ldots] : a_i = \frac{1}{2}(a_{i-1} + a_{i+1})$ for every $i = 2, 3, \ldots\}$;
(c) the set of all sequences $[a_1, a_2, \ldots]$, whose entries are almost all zero;

(d) the set of all bounded sequences.

(25) Show that $\mathbb{R}^4 = U_1 \oplus U_2$, if

(a) U_1 is the set of solutions of $x_1 + x_2 + x_3 + x_4 = 0$, and $U_2 = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$;

(b) U_1 is the set of solutions of $\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 0 \\ -x_1 + x_2 + x_3 = 0 \end{cases}$, and $U_2 = \operatorname{lin}\left(\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

(26) Show that $\mathbb{R}^4 = U_1 + U_2$, but $\mathbb{R}^4 \neq U_1 \oplus U_2$, if U_1 is the set of solutions of $3x_1 - 2x_2 + x_3 + 4x_4 = 0$, and $U_2 = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}\right)$.

(27) Show that

$$\begin{aligned} \mathbb{R}^3 &= \operatorname{lin}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \oplus \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = lin(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\Bigg) \oplus \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \\ &= \operatorname{lin}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \oplus \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right). \end{aligned}$$

(28) Check if the vectors α and β are linear combinations of the system $\mathcal{A}$ of vectors of the space $\mathbb{R}^4$, if

(a) $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \end{bmatrix}\right)$, $\alpha = \begin{bmatrix} 9 \\ 6 \\ 5 \\ -1 \end{bmatrix}$, $\beta = \begin{bmatrix} 9 \\ 6 \\ 5 \\ 0 \end{bmatrix}$;

(b) $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}\right)$, $\alpha = \begin{bmatrix} 9 \\ 6 \\ 5 \\ -1 \end{bmatrix}$, $\beta = \begin{bmatrix} 9 \\ 6 \\ 5 \\ 0 \end{bmatrix}$.

(29) Check if the system $\left(\begin{bmatrix} i \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ i \end{bmatrix}\right)$ of vectors of the space $\mathbb{C}^3$ is linearly independent.

Express the vector $\begin{bmatrix} 2 \\ 3 \\ 1+2i \end{bmatrix}$ as the linear combination of the above vectors.

(30) Check if the system of vectors $(\alpha_1, \dots, a_n)$ of the space K^4 is linearly independent, if

(a) $K = \mathbb{Z}_7$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}$;

(b) $K = \mathbb{R}$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 6 \\ 3 \\ 10 \\ 5 \end{bmatrix}$;

(c) $K = \mathbb{C}$, $\alpha_1 = \begin{bmatrix} 1 \\ i \\ 3 \\ -i \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 4+i \\ 0 \\ 5+3i \\ 5 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 2i \\ i \\ 2 \end{bmatrix}$;

(d) $K = \mathbb{Z}_5$, $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}$.

(31) Show that the vectors $\alpha_1, \ldots, \alpha_n$ form a basis of the space $\mathbb{Q}^n$ and find the coordinates of the vector β in such a basis, if

(a) $n = 3$; $\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\beta = \begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}$;

(b) $n = 3$; $\alpha_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\beta = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$;

(c) $n = 4$; $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$, $\beta = \begin{bmatrix} 7 \\ 14 \\ -1 \\ 2 \end{bmatrix}$.

(32) Find bases of the subspaces of solutions of the following systems of linear equations (over $\mathbb{R}$):

(a) $\begin{cases} x_1 + 3x_2 + 2x_3 = 0 \\ 2x_1 - x_2 + 3x_3 = 0 \\ 3x_1 - 5x_2 + 4x_3 = 0 \end{cases}$; (b) $\begin{cases} x_1 + x_2 \qquad - 3x_4 = 0 \\ x_1 - x_2 + 2x_3 - x_4 = 0 \\ 4x_1 - 2x_2 + 6x_3 + 3x_4 = 0 \end{cases}$.

(33) Find a basis and the dimension of a given subspace $\operatorname{lin}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the space $\mathbb{Q}^4$ if:

(a) $\alpha_1 = \begin{bmatrix} 5 \\ 2 \\ -3 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 3 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}$;

(b) $\alpha_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 3 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 4 \\ -1 \\ 15 \\ 17 \end{bmatrix}$, $\alpha_5 = \begin{bmatrix} 7 \\ -6 \\ -7 \\ 0 \end{bmatrix}$;

(c) $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ -5 \\ 8 \\ -3 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 5 \\ 26 \\ -9 \\ -12 \end{bmatrix}$, $\alpha_5 = \begin{bmatrix} 3 \\ -4 \\ 1 \\ 2 \end{bmatrix}$.

(34) Find a basis for each of the subspaces of $\mathbb{R}^4$ listed below as well as a basis of the sum $U_i + U_j$ and the intersection $U_i \cap U_j$, if:

(a) $U_1 = \operatorname{lin}\left(\begin{bmatrix}1\\1\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\2\\3\\-1\end{bmatrix}\right)$, $U_2 = \left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \in \mathbb{R}^4 : x_1 + x_2 - 2x_3 + x_4 = 0\right\}$;

(b) $U_1 = \operatorname{lin}\left(\begin{bmatrix}2\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}4\\-1\\1\\-3\end{bmatrix}\right)$, $U_2 = \operatorname{lin}\left(\begin{bmatrix}1\\-1\\2\\-2\end{bmatrix}, \begin{bmatrix}4\\0\\0\\-3\end{bmatrix}\right)$,

$U_3 = \left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \in \mathbb{R}^4 : x_1 - x_2 + x_3 + x_4 = 0\right\}$;

(c) $U_1 = \left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \in \mathbb{R}^4 : 2x_1 - x_2 + x_3 - 2x_4 = 0\right\}$,

$U_2 = \operatorname{lin}\left(\begin{bmatrix}2\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}4\\3\\1\\5\end{bmatrix}\right)$;

(d) $U_1 = \operatorname{lin}\left(\begin{bmatrix}1\\2\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\3\\5\\4\end{bmatrix}, \begin{bmatrix}3\\0\\1\\5\end{bmatrix}\right)$, $U_2 = \operatorname{lin}\left(\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$.

(35) Which of the following maps $\varphi : K^n \to K^m$ are linear, if:

(a) $n = m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+z\\2x+z\\3x-y+z\end{bmatrix}$; (b) $n = m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x\\y+1\\z+2\end{bmatrix}$;

(c) $n = m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x+y\\x+z\\z\end{bmatrix}$; (d) $n = m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-y+z\\z\\y\end{bmatrix}$;

(e) $n = 4$, $m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right) = \begin{bmatrix}x-y+2t\\2x+3y+5z-t\\x+z-t\end{bmatrix}$;

(f) $n = 4$, $m = 3$, $\varphi\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right) = \begin{bmatrix}x-y+2t\\2x-3y+5z-t\\x-z-t\end{bmatrix}$;

(g) $n = m = 4$, $\varphi\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right) = \begin{bmatrix}x+3y-2t\\x+y+z\\2y+t\\y+z\end{bmatrix}$;

(h) $n = m = 4$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) = \begin{bmatrix} x+3y-2t \\ x+y+z \\ 2y-3t \\ 2x+4y+z-2t \end{bmatrix}$;

(i) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ 2xz \\ 3x-y+z \end{bmatrix}$.

If φ is a linear map, check if it is a monomorphism, or an epimorphism.

(36) Find kernels and images of linear maps from Problem (35).

(37) Find kernels and images of the symmetry (projection) of V_1 (onto V_1) along V_2 (see Problem (**??**)).

(38) A linear map $\varphi : K^2 \to K^3$ is given by $\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ 3y \end{bmatrix}$. Find:

(a) images of the following subspaces: K^2, $\operatorname{lin}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $\operatorname{lin}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, $\operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in K^2 : 2x+3y=0\right\}$;

(b) counterimages of the following subspaces: K^3, $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$, $\operatorname{lin}\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right)$, $\operatorname{lin}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right)$, $\operatorname{lin}\left(\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$, $\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in K^3 : x+y+z=0\right\}$.

(39) Find a linear map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\operatorname{Ker}\psi = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ and $\operatorname{Im}\psi = \operatorname{lin}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

How many solutions are there?

(40) In the vector space K^3 consider the bases $\mathcal{A}_3 = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$ and $\mathcal{B}_3 = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$, and in the vector space K^4 consider the bases $\mathcal{A}_4 = \left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$ and $\mathcal{B}_4 = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$. Find the matrix of a linear map $\varphi : K^n \to K^m$ in the bases $\mathcal{A}_n$ and $\mathcal{B}_m$ ($\mathcal{A}_n$ and $\mathcal{A}_m$; $\mathcal{B}_n$ and $\mathcal{B}_m$; $\mathcal{B}_n$ and $\mathcal{A}_m$), if:

(a) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ 2x+z \\ 3x-y+z \end{bmatrix}$; (b) $n = m = 3$, $\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y+z \\ y \\ z \end{bmatrix}$;

(c) $n=4, m=3,\ \varphi\left(\begin{bmatrix} x\\ y\\ z\\ t\end{bmatrix}\right)=\begin{bmatrix} x-y+2t\\ 2x+3y+5z-t\\ x+z-t\end{bmatrix}$;

(d) $n=4, m=3,\ \varphi\left(\begin{bmatrix} x\\ y\\ z\\ t\end{bmatrix}\right)=\begin{bmatrix} x-y+2t\\ 2x-3y+5z-t\\ x-z-t\end{bmatrix}$;

(e) $n=3, m=4,\ \varphi\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=\begin{bmatrix} x+3y-2z\\ x+y+z\\ 2y\\ y+z\end{bmatrix}$; (f) $n=3, m=4,\ \varphi\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=\begin{bmatrix} x+3y-2z\\ x+y+z\\ 2y-3z\\ 2x+4y+z\end{bmatrix}$.

(41) Consider the vector space $\mathbb{R}^n$ and its bases $\mathcal{A}$ and $\mathcal{B}$. Denote by $\mathcal{E}$ the canonical basis $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Find the transition matrices from $\mathcal{E}$ to $\mathcal{A}$, from $\mathcal{E}$ to $\mathcal{B}$, from $\mathcal{A}$ to $\mathcal{E}$ and from $\mathcal{A}$ to $\mathcal{B}$, if:

(a) $n=2,\ \mathcal{A}=\left(\begin{bmatrix} 1\\ 2\end{bmatrix}, \begin{bmatrix} -3\\ 5\end{bmatrix}\right),\ \mathcal{B}=\left(\begin{bmatrix} -1\\ 6\end{bmatrix}, \begin{bmatrix} 0\\ 4\end{bmatrix}\right)$;

(b) $n=3,\ \mathcal{A}=\left(\begin{bmatrix} 8\\ -6\\ 7\end{bmatrix}, \begin{bmatrix} -16\\ 7\\ -13\end{bmatrix}, \begin{bmatrix} 9\\ -3\\ 7\end{bmatrix}\right),\ \mathcal{B}=\left(\begin{bmatrix} 1\\ -2\\ 1\end{bmatrix}, \begin{bmatrix} 3\\ -1\\ 2\end{bmatrix}, \begin{bmatrix} 2\\ 1\\ 2\end{bmatrix}\right)$;

(c) $n=4,\ \mathcal{A}=\left(\begin{bmatrix} 1\\ 0\\ 1\\ 1\end{bmatrix}, \begin{bmatrix} -1\\ 1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix} 2\\ 0\\ 1\\ 0\end{bmatrix}, \begin{bmatrix} 0\\ 0\\ 0\\ 1\end{bmatrix}\right),\ \mathcal{B}=\left(\begin{bmatrix} 1\\ 2\\ 0\\ 0\end{bmatrix}, \begin{bmatrix} -1\\ 0\\ 2\\ 1\end{bmatrix}, \begin{bmatrix} 1\\ 1\\ 1\\ 1\end{bmatrix}, \begin{bmatrix} 1\\ 0\\ 0\\ 0\end{bmatrix}\right)$.

(42) Find the matrix of $\varphi : K^3 \to K^3$ in the basis $(\varepsilon_1, \varepsilon_2+\varepsilon_3, \varepsilon_1+\varepsilon_2)$ if the matrix of φ in the basis

(a) $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, (b) $(\varepsilon_1+\varepsilon_2, \varepsilon_2, \varepsilon_3)$

is equal to $\begin{bmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$.

(43) An endomorphism $\varphi \in \mathrm{End}(\mathbb{C}^2)$ has the following matrix in the basis $\mathcal{A}=\left(\begin{bmatrix} 1\\ 1\end{bmatrix}, \begin{bmatrix} 0\\ i\end{bmatrix}\right)$:

(a) $\begin{bmatrix} 3 & 4\\ 5 & 2\end{bmatrix}$; (b) $\begin{bmatrix} 2 & 1\\ -1 & 5\end{bmatrix}$.

Find eigenvalues and eigenvectors of φ. What will be the solution if we assume that $\mathcal{A}$ is the canonical basis? And if we assume that $\varphi \in \mathrm{End}(\mathbb{R}^2)$?

(44) A is the matrix of an endomorphism $\varphi \in \mathrm{End}(\mathbb{C}^n)$ in the canonical basis. Find eigenvalues and eigenvectors of φ. If possible, find a basis of $\mathbb{C}^n$ consisting of eigenvectors of φ, as well as a matrix $C \in GL(n, \mathbb{C})$ such that the matrix $C^{-1}AC$ is diagonal.

$n=2:$ (a) $A=\begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$; (b) $A=\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$; (c) $A=\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$; (d) $A=\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$;

$n=3:$ (e) $A=\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$; (f) $A=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; (g) $A=\begin{bmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{bmatrix}$;

$n=4:$ (h) $A=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$; (i) $A=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$; (j) $A=\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$;

(k) $A=\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$; (l) $A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 7 & -1 \end{bmatrix}$; (m) $A=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix}$.

(45) Find the characteristic polynomial of an endomorphism, which in a certain basis has the following matrix:

(a) $\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$.

(46) Find eigenvalues and corresponding eigenvectors of endomorphisms of real vector spaces whose matrices in the canonical bases are equal to:

(a) $\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$; (d) $\begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$;

(e) $\begin{bmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$; (f) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; (g) $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$.

(47) Find eigenvalues and corresponding eigenvectors of endomorphisms of complex vector spaces whose matrices in the canonical bases are equal to:

(a) $\begin{bmatrix} -1 & 2i \\ -2i & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ for $a \in \mathbb{R}$;

(c) $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}$.

(48) Find a formula for A^n, if A equals to

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$; (d) $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

(49) Find a formula for a_n, if

(a) $a_0 = 0$, $a_1 = 1$, $a_{n+2} = a_{n+1} + a_n$ (Fibonacci sequence);

(b) $\mathrm{a}_0 = 1$, $a_1 = 2$, $a_{n+2} = 3a_n - 2a_{n+1}$.

(50) Calculate the following limit in case it exists:

$$\lim_{n\to\infty} \sqrt[n]{5^n + 6^n + 7^n}.$$

(51) Does $\lim_{n\to\infty}(-1)^n$ exist?
(52) Calculate

$$\lim_{n\to\infty} \frac{\log_3 n^8}{\log_9 n}.$$

(53) Does the following series converge

$$\sum_{n=1}^{\infty} \frac{\cos(4^n)}{3^n}?$$

(54) Does the following series converge

$$\sum_{n=1}^{\infty} \frac{1}{5n-2}?$$

(55) Does $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n-1}\right)$ exist? If so, calculate it.
(56) Does the following limit exist? If so, what is its value?

$$\lim_{x\to\infty} \frac{\sin(2x)}{x^2}.$$

(57) Compute the following limit if it exists

$$\lim_{x\to 7} \frac{\sqrt{x}-7}{x^2-49}.$$

(58) Compute the following limit in case it exists

$$\lim_{x\to 0} \frac{5x}{6\sin(3x)}.$$

(59) Is $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right), & \text{otherwise.} \end{cases}$$

continuous?
(60) Determine the points where $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x] + [-x]$ is continuous.
(61) Does $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{e^{\frac{1}{x}-2}}{e^{\frac{1}{x}+3}}$$

have a continuous extension to $\mathbb{R}$?
(62) Does $f\colon [-1,1] \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{\sqrt{\sin(x)}}$$

have a continuous extension to $[-1,1]$?

(63) What is the derivative of

$$\sqrt[3]{x^4\sqrt{x}}$$

on $(0,\infty)$?

(64) Compute the derivative of the function

$$x \mapsto \frac{1-x^2}{7x^2+9}.$$

(65) Calculate the derivative of

$$\frac{x^3 - 7x\sqrt[5]{x^2}}{2\sqrt{x}}$$

defined on $(0,\infty)$.

(66) Compute the derivative of $\operatorname{tg}^4(3x)$ where the function is defined.

(67) Compute $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(x)}$.

(68) Find all local minima of $f\colon [0,\infty) \to \mathbb{R}$ defined as $f(x) = (x-3)^3 + 5$.

(69) $\lim_{x\to 1+} \frac{\log x}{\sqrt{x^2-1}}$.

(70) Compute $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin(x)+\cos(x)+x-\frac{\pi}{2}}{\sin(2x)-\cos(x)}$.

(71) Give the maximal open intervals where $f\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^3$ is convex and concave, respectively.

(72) Find all local maxima and minima of $f(x) = \sin(x^2)$ defined on $\mathbb{R}$.

(73) Compute $\int \frac{1}{x^2+2x+2}\,dx$.

(74) Compute $\int \frac{1}{\sqrt{2-x^2}}\,dx$.

(75) Compute $\int x^2\sqrt{3x^3-3}\,dx$.

(76) Give $\int 3x^2 e^{x^3}\,dx$.

(77) Compute $\int \operatorname{tg}(x)\,dx$.

(78) Calculate $\int \frac{x}{1-\sin^2 x}\,dx$.

(79) Compute $\int x\operatorname{tg}^2(x)\,dx$.

(80) Determine $\int \frac{4x^3+x^2+2x-1}{x^4-1}\,dx$.

(81) Calculate $\int \sqrt{a+bx}\,dx$.

(82) Calculate $\int \sin(x)\cos(x)\,dx$.

(83) Determine the primitive of $\cos^4(x)\sin(x)$.

(84) Give $\int \sqrt{1-x^2}\,dx$ for $x \in (-1,1)$.

(85) Compute $\int \sqrt{1+x^2}\,dx$.

(86) Determine $\int \frac{1}{2+4\cos^2(x)}\,dx$.

(87) What is $\int \frac{x}{(x^2+a^4)^n}\,dx$, where $n \in \mathbb{N}$?

(88) Compute the area between the x-axis and e^{-x} given on the positive real numbers.

(89) Calculate the area of

$$\{(x,y) : x,y \in [0,1], x^2 \le y \le x\}.$$

(90) A bike has an initial velocity of $4\frac{\mathrm{m}}{\mathrm{s}}$ and accelerates at the rate of $a(t) = 2.4t\frac{\mathrm{m}}{\mathrm{s}^2}$. How fast is the bike going after 4 seconds and how far has it traveled?

(91) What is the length of the curve $f(x) = \cos(x)$ from $x = 0$ to $x = 2\pi$?

(92) The region in the first quadrant enclosed by the y-axis, the line through $x = 2\pi$, and the graphs of $y = \sin(x)$ and $y = \frac{1}{2}\sin(x)$ is revolved around the x-axis. What is the volume of the generated solid?

(93) Find the area of the region enclosed by the parabola $y = 3 - x^2$ and $y = -x$.