Problem set 19: analysis in $\mathbb{R}^n$: differentiation.

(1) If $f(x,y) = x^3 + x^2y^3 - 5y$, find $\frac{\partial f}{\partial x}(2,1)$ and $\frac{\partial f}{\partial y}(2,1)$.

(2) If $f(x,y) = \cos\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(3) Assume that $z = z(x,y)$ is implicitly given by

$$x^3 + y^2 + z^2 + 12xyz = 2.$$

Determine $\partial z/\partial x$ and $\partial z/\partial y$.

(4) Assume that $f(x,y,z) = e^{x^2z}\log(z)$. Find $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

(5) Find the second partial derivatives of $f(x,y) = x^5 + x^3y^2 + x$.

(6) Calculate $\frac{\partial^4 f}{\partial x^2 \partial y \partial z}$ of $f(x,y,z) = \cos(x^3 + yz)$.

(7) Laplace's equation in the plane is

$$\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0.$$

Show that $f(x,y) = e^x\cos(y)$ is a solution.

(8) A special form of the wave equation looks like

$$\frac{\partial^2 f}{\partial t^2}(x,t) = \frac{\partial^2 f}{\partial x^2}(x,t).$$

Show that $f(x,t) = \cos(x-t)$ is a solution.

(9) Find the first partial derivatives of $f(x,y) = \int_x^y \sin(2t^2)\,dt$.

(10) Let $f(x,y)$ be a twice continuously differentiable function defined on $\mathbb{R}^2$ and $F(x) = \int_0^x f(x,y)\,dy$. Calculate $F'(x)$.

(11) Let $f(x,y) = x^2e^{xy}$ be defined on the plane. Show that it is everywhere differentiable.

(12) Let $f(x,y) = 3x^2 - ay^2 + y$ be defined on the plane and a be a real parameter. Find a such that $Df(0,1) = 0$.

(13) Is $f(x,y) = \sqrt{xy}\sin\left(\frac{1}{x^2+y^2}\right)$ for $(x,y) \neq 0$ and $f(0,0) = 0$ differentiable at 0?

(14) Find the local minima and maxima of $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sin(x)\cos(y)$.

(15) Determine the derivative of $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = \begin{bmatrix} x^2 - y^2 \\ x^2 + y^2 \end{bmatrix}.$$

(16) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^2 + 3(y-1)^2 - 4$. Show that there is some $\varepsilon > 0$ and a C^1 function $g\colon (1-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ such that

$$f(g(y), y) = 0$$

for all $y \in (1-\varepsilon, 1+\varepsilon)$.

(17) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^2 + (y-1)^2 - 4$. Is there is some $\varepsilon > 0$ and a C^1 function $g\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$f(g(y), y) = 0$$

for all $y \in (3-\varepsilon, 3+\varepsilon)$?

(18) Assume that $f\colon [a,b] \to \mathbb{R}$ is strictly monotone and continuous. Show that it has an inverse $f^{-1}\colon [f(a), f(b)] \to \mathbb{R}$.

(19) Let us have a look at the system

$$x - y^3 = a,$$
$$x^2 + y + y^2 = b.$$

If $(a, b) = 0$, then we see that $(x, y) = 0$ is a solution. Assume that we are not completely sure that a and b are really 0. Is the system also solvable for a and b near 0?

(20) Determine the Taylor series up to order 2 of $F(x, y) = \sin(xy)$ at 0.