Project 4 - Peano axioms. Mathematical induction.

The following axioms, known as Peano axioms, define the second order theory of natural numbers. We denote by $\mathbb{N}$ the set of all natural numbers and by $S(n)$ the successor function (that is $S(n) = n+1$, although we don't use the "+" sign in formulation of the theory):

(1) $\forall n \in \mathbb{N}(n = n)$
(2) $\forall n, m \in \mathbb{N}(n = m \Rightarrow m = n)$
(3) $\forall n, m, k \in \mathbb{N}(n = m \wedge m = k \Rightarrow n = k)$
(4) $\forall n \in \mathbb{N} \forall m(n = m \Rightarrow n \in \mathbb{N})$
(5) $0 \in \mathbb{N}$
(6) $\forall n \in \mathbb{N}(S(n) \in \mathbb{N})$
(7) $\forall n \in \mathbb{N}(S(n) \neq 0)$
(8) $\forall n, m \in \mathbb{N}(S(n) = S(m) \Rightarrow n = m)$
(9) $\forall K \subset \mathbb{N}(0 \in K \wedge \forall n \in \mathbb{N}(n \in K \Rightarrow S(n) \in K) \Rightarrow K = \mathbb{N})$

This last axiom shows why we are dealing with a second order theory here (we quantify over sets instead of just individuals). It's, in many ways, most fascinating axiom, that we shall call the **axiom of mathematical induction.**

Prove the following theorems:

Sums of powers of consecutive integers.

(1) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = s_1$
(2) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = s_2$ (see above)
(3) $\sum_{k=1}^{n} k^3 = s_1^2$
(4) $\sum_{k=1}^{n} k^4 = \frac{1}{5}s_2(3n^2 + 3n - 1)$
(5) $\sum_{k=1}^{n} k^5 = \frac{1}{3}s_1^2(2n^2 + 2n - 1)$
(6) $\sum_{k=1}^{n} k^6 = \frac{1}{7}s_2(3n^4 + 6n^3 - 3n + 1)$
(7) $\sum_{k=1}^{n} k^7 = \frac{1}{6}s_1^2(3n^4 + 6n^3 - n^2 - 4n + 2)$
(8) $\sum_{k=1}^{n} k^8 = \frac{1}{15}s_2(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3)$

Sums of powers of consecutive odd integers.

(1) $\sum_{k=1}^{n}(2k-1) = n^2 = \sigma_1$
(2) $\sum_{k=1}^{n}(2k-1)^2 = \frac{1}{3}n(4n^2 - 1) = \sigma_2$
(3) $\sum_{k=1}^{n}(2k-1)^3 = \sigma_1(2n^2 - 1)$
(4) $\sum_{k=1}^{n}(2k-1)^4 = \frac{1}{5}\sigma_2(12n^2 - 7)$
(5) $\sum_{k=1}^{n}(2k-1)^5 = \frac{1}{3}\sigma_1(16n^4 - 20n^2 + 7)$
(6) $\sum_{k=1}^{n}(2k-1)^6 = \frac{1}{7}\sigma_2(48n^4 - 72n^2 + 31)$

Divisibility.

(1) $2|n^2 - n$
(2) $6|n^3 - n$
(3) $30|n^5 - n$
(4) $42|n^7 - n$
(5) $546|n^13 - n$
(6) $9|10^n - 1$
(7) $12|10^n - 4$
(8) $11|10^n - (-1)^n$
(9) $101|10^{2n} - (-1)^n$
(10) $1001|10^{3n} - (-1)^n$
(11) $7|10^{3n+1} - 3(-1)^n$
(12) $13|10^{3n+1} + 3(-1)^n$
(13) $14|10^{3n+2} - 2(-1)^n$
(14) $52|10^{3n+2} + 4(-1)^n$
(15) $11|2^{6n+1} + 3^{2n+2}$
(16) $10|2^{2^n} - 6$
(17) $41|5 \cdot 7^{2(n+1)} + 2^{3n}$
(18) $25|2^{n+2}3^n + 5n - 4$
(19) $169|3^{3n} - 26n - 1$
(20) $11|5^{5n+1} + 4^{5n+2} + 3^{5n}$

Bernoulli inequality and some of its generalizations.

(1) $(1+a)^n \geq 1 + na$, $a > -1$ (Bernoulli, 1689)
(2) $(1+a)^n \geq 1 + na + \frac{n(n-1)}{2}a^2$, $a \geq 0$
(3) $(1+a)^n \geq 1 + na + \frac{n(n-1)}{2}a^2 + \frac{n(n-1)(n-2)}{6}a^3$, $a > -1$

(4) $(1+a)^{1/n} \leq 1 + \frac{a}{n}$, $a > -1$
(5) $(1+a)^{1/n} \geq 1 + \frac{a}{n(1+a)}$, $a > -1$
(6) $(1+a)^{1+1/n} \geq 1 + (1+\frac{1}{n})a$, $a > -1$
(7) $(1+a)^{1+m/n} \geq 1 + (1+\frac{m}{n})a$, $a > -1$
(8) $(1+a)^{p/q} \geq 1 + \frac{p}{q}a$, $a > -1$, $p \geq q \geq 1$
(9) $(1+a)^{p/q} \leq 1 + \frac{p}{q}a$, $a > -1$, $1 \leq p \leq q$

Fibonacci sequence. Set

$$u_0 = 0, \qquad u_1 = 1,$$
$$u_{n+2} = u_n + u_{n+1}$$

(1) $\sum_{k=0}^{n} u_k = u_{n+2} - 1$
(2) $\sum_{k=0}^{n} u_{2k+1} = u_{2n+2}$
(3) $\sum_{k=0}^{n} u_{2k} = u_{2n+1} - 1$
(4) $\sum_{k=0}^{2n} (-1)^k u_k = u_{2n-1} - 1$
(5) $\sum_{k=0}^{2n+1} (-1)^{k+1} u_k = u_{2n} + 1$
(6) $\sum_{k=0}^{n} u_k^2 = u_n u_{n+1}$
(7) $\sum_{k=0}^{2n-1} u_k u_{k+1} = u_{2n}^2$
(8) $u_{n-1}u_{n+1} - u_n^2 = (-1)^n$
(9) $u_{n+1} = u_m u_{n-m} + u_{m+1}u_{n-m+1}$, $n \geq m \geq 0$
(10) $u_{2n+1} = u_n^2 + u_{n+1}^2$
(11) $u_{2n} = u_{n+1}^2 - u_{n-1}^2$
(12) $u_{3n} = u_n^3 + u_{n+1}^3 - u_{n-1}^3$
(13) $u_n^4 = 1 + u_{n-2}u_{n-1}u_{n+1}u_{n+2}$
(14) $\sum_{k=0}^{n} \frac{u_{k+2}}{u_{k+1}u_{k+3}} = \frac{u_3}{u_1 u_2} - \frac{u_{n+4}}{u_{n+2}u_{n+3}}$
(15) $u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where α i β are two distinct solutions of the equation

$$x^2 = x + 1.$$