

# Functional limit theorems for linear processes in the domain of attraction of stable laws

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## Abstract

We study functional limit theorems for linear type processes with short memory under the assumption that the innovations are dependent identically distributed random variables with infinite variance and in the domain of attraction of stable laws.

*Key words:* Lévy stable processes, functional limit theorem, Skorohod topologies,  $\rho$ -mixing  
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## 1. Introduction

We consider the linear process  $\{Z_j: j \in \mathbb{Z}\}$  defined by

$$Z_j = \sum_{k=-\infty}^{\infty} a_k \xi_{j-k}, \quad (1)$$

where the innovations  $\{\xi_j: j \in \mathbb{Z}\}$  are identically distributed random variables with infinite variance and the sequence of constants  $\{a_k: k \in \mathbb{Z}\}$  is such that  $\sum_{k \in \mathbb{Z}} |a_k| < \infty$ . This case is referred to as short memory, or as short range dependence. The functional central limit theorem (FCLT) for the partial sums of the linear process, properly normalized, merely follows from the corresponding FCLT for the innovations being in the domain of attraction of the normal law, see Peligrad and Utev (2006). Then the limiting process has continuous sample paths and choosing the right topology in the Skorohod space  $\mathbb{D}[0, 1]$  is not problematic. However, as shown by Avram and Taqqu (1992), the weak convergence of the partial sums of the linear process with independent innovations (i.i.d. case) in the domain of attraction of non-normal laws is impossible in the Skorohod  $J_1$  topology on  $\mathbb{D}[0, 1]$ , but the functional limit theorem might still hold, under additional assumptions, in the weaker Skorohod  $M_1$  topology (see Skorohod (1956)). Avram and Taqqu (1992) use the standard approach through tightness plus convergence of finite dimensional distributions. Here, we use approximation techniques and study weak convergence in  $\mathbb{D}[0, \infty)$ , i.e. the space of functions on  $[0, \infty)$  that have finite left-hand limits and are continuous from the right. Given processes  $X_n, X$  with sample paths in  $\mathbb{D}[0, \infty)$ , we will denote by  $X_n(t) \Rightarrow X(t)$  the weak convergence in  $\mathbb{D}[0, \infty)$  with one of the Skorohod topologies  $J_1$  or  $M_1$ , and write  $\xrightarrow{J_1}$  or  $\xrightarrow{M_1}$ , if the indicated topology is used. Note that if the limiting process  $X$  has continuous sample paths then  $\Rightarrow$  in  $\mathbb{D}[0, \infty)$  with one of the Skorohod topologies is equivalent to weak convergence in  $\mathbb{D}[0, \infty)$  with the local uniform topology. For definitions and properties of the topologies we refer to Jacod and Shiryaev (2003) and Whitt (2002).

To motivate our approach we first consider the linear process  $\{Z_j: j \in \mathbb{Z}\}$  as in (1), where  $\{\xi_j: j \in \mathbb{Z}\}$  is a sequence of i.i.d. random variables. There exist sequences  $b_n > 0$  and  $c_n$  such that the partial sum

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processes of the i.i.d. sequence  $\{\xi_j : j \in \mathbb{Z}\}$  converge weakly to an  $\alpha$ -stable Lévy process  $X$  with  $0 < \alpha < 2$  (see, e.g. Resnick, 1986, Proposition 3.4)

$$\frac{1}{b_n} \sum_{j=1}^{\lfloor nt \rfloor} (\xi_j - c_n) \xrightarrow{J_1} X(t) \quad (2)$$

if and only if there is convergence in distribution

$$\frac{1}{b_n} \sum_{j=1}^n (\xi_j - c_n) \xrightarrow{d} X(1) \quad \text{in } \mathbb{R},$$

or, equivalently, there exist  $p \in [0, 1]$  and a slowly varying function  $L$ , i.e.,  $L(sx)/L(x) \rightarrow 1$  as  $x \rightarrow \infty$  for every  $s > 0$ , such that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(\xi_1 > x)}{\mathbb{P}(|\xi_1| > x)} = p \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(|\xi_1| > x)}{x^{-\alpha} L(x)} = 1; \quad (3)$$

in that case the sequences  $\{b_n, c_n : n \in \mathbb{N}\}$  in (2) can be chosen as

$$b_n = \inf \left\{ x : \mathbb{P}(|\xi_1| \leq x) \geq 1 - \frac{1}{n} \right\} \quad \text{and} \quad c_n = \mathbb{E}(\xi_1 I(|\xi_1| \leq b_n)). \quad (4)$$

We have  $b_n \rightarrow \infty$ ,  $n\mathbb{P}(|\xi_1| > b_n) \rightarrow 1$ , and  $nb_n^{-\alpha} L(b_n) \rightarrow 1$ , as  $n \rightarrow \infty$ , by (3). We refer to Feller (1971) for  $\alpha$ -stable random variables and their domains of attraction. If  $\alpha = 2$ , condition (2) holds if and only if the function  $x \mapsto \mathbb{E}(\xi_1^2 I(|\xi_1| \leq x))$  is slowly varying; in that case the sequence  $b_n$  can be chosen as satisfying  $nb_n^{-2} \mathbb{E}(\xi_1^2 I(|\xi_1| \leq b_n)) \rightarrow 1$  as  $n \rightarrow \infty$ , the  $c_n$  as in (4), and  $X$  is a Brownian motion. In any case, (2) implies that the function  $x \mapsto \mathbb{E}(\xi_1^2 I(|\xi_1| \leq x))$  is regularly varying with index  $2 - \alpha$ , that is, there exists a slowly varying function  $\ell$  such that (if  $\alpha < 2$  then  $\ell(x) = \alpha L(x)/(2 - \alpha)$  where  $L$  is as in (3))

$$\lim_{x \rightarrow \infty} \frac{\mathbb{E}(\xi_1^2 I(|\xi_1| \leq x))}{x^{2-\alpha} \ell(x)} = 1. \quad (5)$$

Astrauskas (1983) and Davis and Resnick (1985) show that if the coefficients  $\{a_k : k \in \mathbb{Z}\}$  are such that

$$\sum_{k=-\infty}^{\infty} |a_k|^r < \infty \quad \text{for some } r < \alpha, 0 < r \leq 1, \quad (6)$$

and if (2) holds then the linear process  $\{Z_j : j \in \mathbb{Z}\}$  defined by (1) satisfies

$$\frac{1}{b_n} \sum_{j=1}^n (Z_j - Ac_n) \xrightarrow{d} AX(1) \quad \text{in } \mathbb{R}, \quad \text{where } A = \sum_{k \in \mathbb{Z}} a_k.$$

For the case  $\alpha \in (0, 2)$ , Avram and Taqqu (1992) show that if  $a_k \geq 0$ ,  $k \in \mathbb{Z}$ , satisfy (6) and if additional constraints are imposed for  $\alpha \geq 1$  (see Avram and Taqqu, 1992, Theorem 2), then (2) implies

$$\frac{1}{b_n} \sum_{j=1}^{\lfloor nt \rfloor} (Z_j - Ac_n) \xrightarrow{M_1} AX(t) \quad (7)$$

and that the convergence in (7) is impossible in the  $J_1$ -topology. We show in Corollary 1 that no additional assumptions are needed for  $\alpha \geq 1$ . It is still not known to what extent one can relax the condition that all  $a_k$  have the same sign to get convergence in (7) with any topology weaker than  $J_1$ . With our approach we reduce this problem to continuity properties of addition in a given topology (see Section 3). When  $\alpha = 2$  then (7) holds with any real constants  $a_k$  satisfying (6) (see, e.g. Peligrad and Utev, 2006; Moon, 2008, and the references therein).

We now consider identically distributed, possibly dependent, random variables  $\{\xi_j : j \in \mathbb{Z}\}$  with  $\mathbb{E}\xi_1^2 = \infty$ . Note that if (5) holds with  $\alpha \in (0, 2]$  then  $\mathbb{E}|\xi_1|^\beta < \infty$  for every  $\beta \in (0, \alpha)$ , thus condition (6) ensures that each  $Z_j$  in (1) is a.s. converging series, since

$$\mathbb{E}|Z_j|^r \leq \sum_{k \in \mathbb{Z}} |a_k|^r \mathbb{E}|\xi_{j-k}|^r = \mathbb{E}|\xi_1|^r \sum_{k \in \mathbb{Z}} |a_k|^r < \infty.$$

Our main result is the following.

**Theorem 1.** *Let a linear process  $\{Z_j : j \in \mathbb{Z}\}$  be defined by (1), where  $\{\xi_j : j \in \mathbb{Z}\}$  and  $\{a_k : k \in \mathbb{Z}\}$  satisfy (5) and (6) with  $\alpha \in (0, 2]$ . Assume that  $\{b_n, c_n : n \in \mathbb{N}\}$  are sequences satisfying the following conditions:*

$$b_n \rightarrow \infty, \quad \frac{c_n}{b_n} \rightarrow 0, \quad \text{and} \quad \limsup_{n \rightarrow \infty} n b_n^{-2} E(\xi_1^2 I(|\xi_1| \leq b_n)) < \infty, \quad (8)$$

there exists  $s \geq 1$  such that

$$\limsup_{n \rightarrow \infty} \sup_k \mathbb{E} \left( \max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l (\xi_{j-k} I(|\xi_{j-k}| \leq b_n) - c_n) \right|^s \right) < \infty \quad (9)$$

for all  $T > 0$ , and there exists a process  $X$  such that

$$\frac{1}{b_n} \sum_{j=1}^{[nt]} (\xi_j - c_n) \Longrightarrow X(t) \quad (10)$$

in  $\mathbb{D}[0, \infty)$  with the topology  $J_1$  or  $M_1$ . If the constants  $a_k$  are nonnegative or the process  $X$  has continuous sample paths, then the linear process  $\{Z_j : j \in \mathbb{Z}\}$  satisfies (7) with the same process  $X$ . Moreover, assumption (9) can be omitted, if  $\alpha < 1$  and  $\limsup_{n \rightarrow \infty} n b_n^{-1} |c_n| < \infty$ .

*Remark 1.* Note that if  $b_n \rightarrow \infty$  then

$$\frac{1}{b_n} \mathbb{E}(|\xi_1| I(|\xi_1| \leq b_n)) = \int_0^1 \mathbb{P}(y b_n < |\xi_1| \leq b_n) dy \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

by Lebesgue's dominated convergence theorem, since  $\mathbb{P}(y b_n < |\xi_1| \leq b_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Observe also that if (3) holds with  $\alpha \in (0, 1) \cup (1, 2)$  and the sequences  $\{b_n, c_n : n \in \mathbb{N}\}$  are as in (4) then

$$\frac{1}{b_n} \sum_{j=1}^{[nt]} (\xi_j - c_n) \Longrightarrow X(t) \quad \text{if and only if} \quad \frac{1}{b_n} \sum_{j=1}^{[nt]} (\xi_j - c) \Longrightarrow X(t) + \tilde{c}t,$$

where  $c = 0$  for  $\alpha < 1$ ,  $c = \mathbb{E}\xi_1$  for  $\alpha > 1$ , and  $\tilde{c}$  is the limit of  $n b_n^{-1} (c_n - c)$ , which exists and is finite. The equivalence is also valid when (5) holds with  $\alpha = 2$ ,  $b_n$  is such that  $n b_n^{-2} \ell(b_n) \rightarrow 1$ , and  $c_n$  as above.

The proof of Theorem 1 is given in Section 2. We now comment on condition (9). The choice of  $c_n = \mathbb{E}(\xi_1 I(|\xi_1| \leq b_n))$  might allow us to use known moment maximal inequalities for partial sums of random variables with mean zero such as Doob's maximal inequality for martingales or maximal inequalities for strongly mixing sequences. In the i.i.d. case Theorem 1 implies the following.

**Corollary 1.** *Let  $\{\xi_j : j \in \mathbb{Z}\}$  be a sequence of i.i.d. random variables such that condition (2) holds, where  $X$  is an  $\alpha$ -stable process with  $\alpha \in (0, 2)$ . If the nonnegative coefficients  $\{a_k : k \in \mathbb{Z}\}$  satisfy (6), then the linear process  $\{Z_j : j \in \mathbb{Z}\}$  satisfies (7) with the same process  $X$ .*

*Proof.* We can choose the sequences  $\{b_n, c_n : n \in \mathbb{N}\}$  as in (4). Then (8) holds. For each  $n$  and  $k \in \mathbb{Z}$ , define

$$\zeta_{n,k,j} := \xi_{j-k} I(|\xi_{j-k}| \leq b_n) - \mathbb{E}(\xi_1 I(|\xi_1| \leq b_n)), \quad j \in \mathbb{Z}.$$

From Doob's maximal inequality for martingales it follows that

$$\mathbb{E}\left(\max_{1 \leq l \leq [nT]} \left| \sum_{j=1}^l \zeta_{n,k,j} \right|^2\right) \leq 2\mathbb{E}\left(\sum_{j=1}^{[nT]} \zeta_{n,k,j}\right)^2 = 2[nT]\mathbb{E}(\zeta_{n,k,j})^2 \leq 2nT\mathbb{E}(\xi_1^2 I(|\xi_1| \leq b_n)).$$

Since  $nb_n^{-\alpha}L(b_n) \rightarrow 1$  as  $n \rightarrow \infty$ , we have  $nb_n^{-2}\mathbb{E}(\xi_1^2 I(|\xi_1| \leq b_n)) \rightarrow \alpha/(2-\alpha)$ . Therefore, (9) holds.  $\square$

*Remark 2.* In Corollary 1, the convergence in (7) can not be strengthened to the  $J_1$  topology, by Theorem 1 of Avram and Taqu (1992). This can also be derived from Theorem 2.4 of Davis and Resnick (1985) and Theorem 3.1 of Tyran-Kamińska (2009) (see Tyran-Kamińska, 2009, Remark 3.3, for more details).

Recall that a sequence  $\{\xi_j : j \in \mathbb{Z}\}$  is said to be  $\rho$ -mixing, if  $\rho(n) \rightarrow \infty$ , where

$$\rho(n) = \sup\{|\text{corr}(f, g)| : f \in L^2(\mathcal{F}_k), g \in L^2(\mathcal{F}^{n+k}), k \in \mathbb{Z}\}$$

and  $\mathcal{F}_l$  ( $\mathcal{F}^l$ ) denotes the  $\sigma$ -algebra generated by  $\xi_j$  with indices  $j \leq l$  ( $j \geq l$ ).

**Corollary 2.** *Let the sequence  $\{\xi_j : j \in \mathbb{Z}\}$  be strictly stationary and  $\rho$ -mixing with  $\sum_{i \geq 1} \rho(2^i) < \infty$ . Suppose that condition (5) holds with  $\alpha \in [1, 2]$ , and that the sequence  $\{b_n : n \in \mathbb{N}\}$  satisfies (8), where  $c_n = \mathbb{P}(\xi_1 I(|\xi_1| \leq b_n))$ ,  $n \geq 1$ . Then condition (9) holds for all  $T > 0$  with  $s = 2$ .*

*Moreover, if the coefficients  $\{a_k : k \in \mathbb{Z}\}$  satisfy (6), then (2) implies (7) with the same process  $X$  provided that either the  $\{a_k : k \in \mathbb{Z}\}$  are nonnegative or  $X$  has continuous sample paths.*

*Proof.* Note that for each  $n$  and  $k$  the sequence  $\{\zeta_{n,k,j} : j \in \mathbb{Z}\}$  defined in the proof of Corollary 1 is a stationary sequence of square integrable random variables with mean zero and is  $\rho$ -mixing with at least the same mixing rate as  $\{\xi_j : j \in \mathbb{Z}\}$ . By Theorem 1.1 of Shao (1995), there exists a constant  $C > 0$ , depending only on  $\rho(\cdot)$ , such that

$$\mathbb{E}\left(\max_{1 \leq l \leq [nT]} \left(\sum_{j=1}^l \zeta_{n,k,j}\right)^2\right) \leq CnT\mathbb{E}\zeta_{n,k,1}^2.$$

Since  $\mathbb{E}\zeta_{n,k,1}^2 \leq \mathbb{E}(\xi_1^2 I(|\xi_1| \leq b_n))$ , condition (8) implies (9) and the result follows from Theorem 1.  $\square$

In the setting of Corollary 2, if we assume that  $\rho(1) < 1$ , then (5) with  $\alpha = 2$  implies that (2) holds with a sequence  $\{b_n : n \in \mathbb{N}\}$  satisfying (8) and with  $X$  being a standard Brownian motion (see Shao, 1993). Hence, we recover the result of Moon (2008). If  $\alpha \in (0, 2)$  then (3) in general does not imply (2) as the example of linear processes shows; see Tyran-Kamińska (2009) for sufficient conditions when (3) implies (2). In particular, if in Corollary 2 we take  $\alpha \in [1, 2)$ , replace (5) with (3), and choose the sequences  $\{b_n, c_n : n \in \mathbb{N}\}$  as in (4), then Theorem 1.1 of Tyran-Kamińska (2009) shows that condition (2) holds with  $X$  being an  $\alpha$ -stable Lévy process if and only if for any  $\varepsilon > 0$  there exist sequences of integers  $r_n = r_n(\varepsilon), l_n = l_n(\varepsilon) \rightarrow \infty$  such that

$$r_n = o(n), \quad l_n = o(r_n), \quad n\rho(l_n) = o(r_n), \quad \text{as } n \rightarrow \infty,$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{2 \leq j \leq r_n} |\xi_j| > \varepsilon b_n \mid |\xi_1| > \varepsilon b_n\right) = 0.$$

## 2. Proof of Theorem 1

We need the following maximal inequality which follows from Theorem 1 of Kounias and Weng (1969).

**Lemma 1.** *Let  $\tau \in (0, 1]$ . If  $\zeta_1, \dots, \zeta_N$  are random variables with  $E|\zeta_j|^\tau < \infty$  for  $j = 1, \dots, N$ , then, for any  $\delta > 0$ ,*

$$\mathbb{P}\left(\max_{1 \leq l \leq N} |\zeta_1 + \dots + \zeta_l| > \delta\right) \leq \frac{\sum_{j=1}^N \mathbb{E}(|\zeta_j|^\tau)}{\delta^\tau}.$$

The next lemma will allow us to use Theorem 4.2 of Billingsley (1968). For  $m, n \in \mathbb{N}$ , define

$$X_n^{(m)} := \frac{1}{b_n} \sum_{j=1}^{[nt]} \sum_{|k| \leq m} a_k (\xi_{j-k} - c_n), \quad X_n(t) := \frac{1}{b_n} \sum_{j=1}^{[nt]} (Z_j - Ac_n), \quad t \geq 0.$$

**Lemma 2.** *Assume (5), (6), and (8). If condition (9) holds with  $s \geq 1$  and  $T > 0$  then*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\sup_{0 \leq t \leq T} |X_n(t) - X_n^{(m)}(t)| > \delta) = 0 \quad \text{for all } \delta > 0. \quad (11)$$

If  $\alpha < 1$  and  $\limsup_{n \rightarrow \infty} nb_n^{-1} |c_n| < \infty$ , then (11) holds for all  $T > 0$ .

*Proof.* Define  $\xi_{n,j} = \xi_j I(|\xi_j| \leq b_n) - c_n$ ,  $j \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ . First note that

$$X_n(t) - X_n^{(m)}(t) = \frac{1}{b_n} \sum_{j=1}^{[nt]} \sum_{|k| > m} a_k \xi_{n,j-k} + \frac{1}{b_n} \sum_{j=1}^{[nt]} \sum_{|k| > m} a_k \xi_{j-k} I(|\xi_{j-k}| > b_n).$$

Therefore, the probability in (11) is less than

$$\mathbb{P}\left(\max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k| > m} a_k \xi_{n,j-k} \right| > \frac{\delta}{2}\right) + \mathbb{P}\left(\max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k| > m} a_k \xi_{j-k} I(|\xi_{j-k}| > b_n) \right| > \frac{\delta}{2}\right). \quad (12)$$

We now find an upper bound for the first term in (12). Let  $s \geq 1$  be such that condition (9) holds. By Hölder's inequality, we have

$$\left( \sum_{|k| > m} |a_k| \left| \sum_{j=1}^l \xi_{n,j-k} \right| \right)^s \leq \left( \sum_{|k| > m} |a_k| \right)^{s-1} \sum_{|k| > m} |a_k| \left| \sum_{j=1}^l \xi_{n,j-k} \right|^s,$$

and therefore

$$\mathbb{E}\left(\max_{1 \leq l \leq [nT]} \left| \sum_{j=1}^l \sum_{|k| > m} a_k \xi_{n,j-k} \right|^s\right) \leq \left( \sum_{|k| > m} |a_k| \right)^s \sup_{|k| > m} \mathbb{E}\left(\max_{1 \leq l \leq [nT]} \left| \sum_{j=1}^l \xi_{n,j-k} \right|^s\right),$$

which, by Markov's inequality, leads to

$$\mathbb{P}\left(\max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k| > m} a_k \xi_{n,j-k} \right| > \frac{\delta}{2}\right) \leq \frac{2^s}{\delta^s b_n^s} \left( \sum_{|k| > m} |a_k| \right)^s \sup_{|k| > m} \mathbb{E}\left(\max_{1 \leq l \leq [nT]} \left| \sum_{j=1}^l \xi_{n,j-k} \right|^s\right).$$

From assumption (9) we conclude that there exists a constant  $C_1$  such that, for any  $m \geq 1$ , we have

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k| > m} a_k \xi_{n,j-k} \right| > \frac{\delta}{2}\right) \leq C_1 \left( \sum_{|k| > m} |a_k| \right)^s. \quad (13)$$

To estimate the second term in (12), we consider separately the case of  $\alpha \in (1, 2]$  and  $\alpha \in (0, 1]$ . Let us note that (5) and Karamata's theorem imply (see Feller, 1971, Lemma, p. 579)

$$\lim_{x \rightarrow \infty} \frac{x^{2-\beta} \mathbb{E}(|\xi_1|^\beta I(|\xi_1| > x))}{\mathbb{E}(\xi_1^2 I(|\xi_1| \leq x))} = \frac{2-\alpha}{\alpha-\beta}$$

for all  $\beta < \alpha$ , which combined with (8) gives

$$\limsup_{n \rightarrow \infty} nb_n^{-\beta} \mathbb{E}(|\xi_1|^\beta I(|\xi_1| > b_n)) < \infty \quad \text{for } \beta < \alpha. \quad (14)$$

We first assume that  $\alpha > 1$ . We have

$$\mathbb{E} \left| \sum_{|k|>m} a_k \xi_{j-k} I(|\xi_{j-k}| > b_n) \right| \leq \mathbb{E}(|\xi_1| I(|\xi_1| > b_n)) \sum_{|k|>m} |a_k| < \infty.$$

From Lemma 1 it follows that

$$\mathbb{P} \left( \max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k|>m} a_k \xi_{j-k} I(|\xi_{j-k}| > b_n) \right| > \frac{\delta}{2} \right) \leq \frac{2nT}{\delta b_n} \mathbb{E}(|\xi_1| I(|\xi_1| > b_n)) \sum_{|k|>m} |a_k|.$$

Applying (14) with  $\beta = 1$ , we can find a constant  $C_2$  such that

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k|>m} a_k \xi_{j-k} I(|\xi_{j-k}| > b_n) \right| > \frac{\delta}{2} \right) \leq C_2 \sum_{|k|>m} |a_k|,$$

which combined with (12) and (13) gives

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \sup_{0 \leq t \leq T} |X_n(t) - X_n^{(m)}(t)| > \delta \right) \leq C_1 \left( \sum_{|k|>m} |a_k| \right)^s + C_2 \sum_{|k|>m} |a_k|.$$

This shows that (11) holds when  $\alpha > 1$ , since the series  $\sum_k |a_k|$  converges.

Next, assume that  $\alpha \leq 1$ . Let  $r < \alpha$  be as in (6). Since  $r \leq 1$ , we have

$$\mathbb{E} \left| \sum_{|k|>m} a_k \xi_{j-k} I(|\xi_{j-k}| > b_n) \right|^r \leq \mathbb{E}(|\xi_1|^r I(|\xi_1| > b_n)) \sum_{|k|>m} |a_k|^r.$$

Applying again Lemma 1 and (14) with  $\beta = r$ , we can find a constant  $C_3$  such that

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k|>m} a_k \xi_{j-k} I(|\xi_{j-k}| > b_n) \right| > \frac{\delta}{2} \right) \leq C_3 \sum_{|k|>m} |a_k|^r,$$

which combined with (13) completes the proof of (11) under the assumption that (9) holds.

To prove the second part of the lemma it suffices to check that (13) remains valid with  $s = 1$ , if  $\alpha < 1$  and  $\sup_n n b_n^{-1} |c_n| < \infty$ . We have

$$\mathbb{E} \left| \sum_{|k|>m} a_k \xi_{n,j-k} \right| \leq \sum_{|k|>m} |a_k| \mathbb{E} |\xi_{n,j-k}| \leq (\mathbb{E}(|\xi_1| I(|\xi_1| \leq b_n)) + |c_n|) \sum_{|k|>m} |a_k|.$$

Therefore, we can apply Lemma 1, which gives

$$\mathbb{P} \left( \max_{1 \leq l \leq [nT]} \left| \frac{1}{b_n} \sum_{j=1}^l \sum_{|k|>m} a_k \xi_{n,j-k} \right| > \frac{\delta}{2} \right) \leq \frac{2nT}{\delta b_n} (\mathbb{E}(|\xi_1| I(|\xi_1| \leq b_n)) + |c_n|) \sum_{|k|>m} |a_k|.$$

Since  $\alpha < 1$ , we obtain, by (5) and Karamata's theorem (see Feller, 1971, Lemma, p. 579),

$$\lim_{n \rightarrow \infty} \frac{x \mathbb{E}(|\xi_1| I(|\xi_1| \leq x))}{\mathbb{E}(\xi_1^2 I(|\xi_1| \leq x))} = \frac{2 - \alpha}{1 - \alpha}.$$

This together with (8) shows that  $\limsup_{n \rightarrow \infty} n b_n^{-1} \mathbb{E}(|\xi_1| I(|\xi_1| \leq b_n)) < \infty$ , which completes the proof.  $\square$

We shall now recall (Whitt, 2002, Chapter 12) that the sequence  $\{\psi_n : n \in \mathbb{N}\}$  converges to  $\psi$  as  $n \rightarrow \infty$  in  $\mathbb{D}[0, \infty)$  with some Skorohod topology if and only if the restrictions of  $\psi_n$  to  $[0, T]$  converge to the restriction of  $\psi$  to  $[0, T]$  in  $\mathbb{D}[0, T]$  with the same topology for all  $T > 0$  that are continuity points of  $\psi$ . Each of the Skorohod topologies in  $\mathbb{D}[0, T]$  is metrizable with a metric  $d_T$  such that  $(\mathbb{D}[0, T], d_T)$  is a separable metric space and weaker than the uniform metric

$$d_T(\psi_1, \psi_2) \leq \sup_{0 \leq t \leq T} |\psi_1(t) - \psi_2(t)| \quad \text{for } \psi_1, \psi_2 \in D[0, T].$$

**Lemma 3.** Suppose that  $\{\xi_j : j \in \mathbb{Z}\}$ ,  $\{b_n, c_n : n \in \mathbb{N}\}$ , and  $X$  are such that  $b_n \rightarrow \infty$ ,  $b_n^{-1}c_n \rightarrow 0$ , and (10) holds in  $\mathbb{D}[0, \infty)$  with one of the Skorohod topologies. Then for each  $k \in \mathbb{Z}$

$$\frac{1}{b_n} \sum_{j=1}^{\lfloor nt \rfloor} (\xi_{j-k} - c_n) \Longrightarrow X(t), \quad \text{as } n \rightarrow \infty, \quad (15)$$

in  $\mathbb{D}[0, \infty)$  with the same topology.

*Proof.* Let  $k \in \mathbb{Z}$ . Define  $h_n : \mathbb{D}[0, \infty) \rightarrow \mathbb{D}[0, \infty)$  by  $h_n(\psi)(t) = \psi(s_n(t))$  for  $\psi \in \mathbb{D}[0, \infty)$ , where  $s_n(t) = \max\{0, t - kn^{-1}\}$ . Since  $h_n(\psi) \rightarrow \psi$  in  $\mathbb{D}[0, \infty)$  with any of the Skorohod topologies, (10) implies that

$$\frac{1}{b_n} \sum_{j=1}^{\lfloor ns_n(t) \rfloor} (\xi_j - c_n) \Longrightarrow X(t),$$

by Theorem 5.5 of Billingsley (1968). If  $k < 0$ , we have

$$\sup_{t \geq 0} \left| \frac{1}{b_n} \sum_{j=1}^{\lfloor nt \rfloor} (\xi_{j-k} - c_n) - \frac{1}{b_n} \sum_{j=1}^{\lfloor ns_n(t) \rfloor} (\xi_j - c_n) \right| \leq \frac{1}{b_n} \left| \sum_{j=1}^{1+k} (\xi_j - c_n) \right| \rightarrow 0$$

in probability as  $n \rightarrow \infty$ , since  $b_n^{-1}\xi_j \rightarrow 0$  in probability for each  $j$  and  $b_n^{-1}c_n \rightarrow 0$ . Similarly, if  $k > 0$ , we have

$$\sup_{t \geq 0} \left| \frac{1}{b_n} \sum_{j=1}^{\lfloor nt \rfloor} (\xi_{j-k} - c_n) - \frac{1}{b_n} \sum_{j=1}^{\lfloor ns_n(t) \rfloor} (\xi_j - c_n) \right| \leq \frac{1}{b_n} \left| \sum_{j=1-k}^0 (\xi_j - c_n) \right| \rightarrow 0$$

in probability as  $n \rightarrow \infty$ . Consequently, the result follows from Slutsky's theorem (Billingsley, 1968, Theorem 4.1).  $\square$

**Lemma 4.** Let  $\{\xi_j : j \in \mathbb{Z}\}$  and  $\{b_n, c_n : n \in \mathbb{N}\}$  be arbitrary sequences. Suppose that there exists a process  $X$  such that for each  $k \in \mathbb{Z}$  condition (15) holds in  $\mathbb{D}[0, \infty)$  with the topology  $J_1$  or  $M_1$ . Then

$$\frac{1}{b_n} \sum_{j=1}^{\lfloor nt \rfloor} \sum_{|k| \leq m} a_k (\xi_{j-k} - c_n) \Longrightarrow \left( \sum_{|k| \leq m} a_k \right) X(t), \quad \text{as } n \rightarrow \infty, \quad (16)$$

in  $\mathbb{D}[0, \infty)$  with the  $M_1$  topology for all constants  $a_k \geq 0$ ,  $|k| \leq m$ , and all  $m \geq 0$ .

*Proof.* The constants  $a_k$  are nonnegative, thus any two limiting processes  $a_{k_1}X$  and  $a_{k_2}X$  have jumps of common sign. Since addition is continuous in the  $M_1$  topology for limiting processes with this property (see e.g. Whitt, 2002, Theorem 12.7.3), the result follows.  $\square$

*Remark 3.* Note that addition is continuous in the  $J_1$  topology if the limiting processes almost surely have no common discontinuities. Consequently, if  $X$  has continuous sample paths and the convergence in (15) holds in  $\mathbb{D}[0, \infty)$  with the  $J_1$  topology, then (16) holds with the same topology for any real constants  $a_k$ ,  $|k| \leq m$ , and all  $m \geq 0$ .

*Proof of Theorem 1.* Lemma 3 combined with Lemma 4 and Remark 3 implies that  $X_n^{(m)}(t) \xrightarrow{M_1} A_m X(t)$  as  $n \rightarrow \infty$  for all  $m \geq 0$ , where  $A_m = \sum_{|k| \leq m} a_k$ . We also have  $A_m X(t) \xrightarrow{M_1} AX(t)$ , which is a consequence of  $A_m \rightarrow A$  as  $m \rightarrow \infty$ . We can clearly assume that  $A \neq 0$  when  $X$  has discontinuous sample paths. Observe that for each  $m$  sufficiently large the processes  $A_m X$  being a constant multiple of  $AX$  have the same set of continuity points  $T_X = \{t > 0 : \mathbb{P}(X(t) = X(t-)) = 1\}$  with  $[0, \infty) \setminus T_X$  being at most countable. From Lemma 2 it follows that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d_T(X_n, X_n^{(m)}) > \delta) = 0 \quad \text{for all } \delta > 0$$

for all  $T > 0$ , where  $d_T$  is the metric in  $\mathbb{D}[0, T]$  which induces the  $M_1$  topology. Therefore, by Theorem 4.2 of Billingsley (1968), we conclude that  $X_n(t) \xrightarrow{M_1} AX(t)$  in  $\mathbb{D}[0, T]$  for all  $T \in T_X$ , which completes the proof. □

### 3. Final remarks

Observe that the nonnegativity of the coefficients  $\{a_k : k \in \mathbb{Z}\}$  was only used to deduce (16) from (10). Thus, we have the following result in one of the Skorohod topologies  $J_1, M_1, J_2, M_2$ .

**Theorem 2.** *Let a linear process  $\{Z_j : j \in \mathbb{Z}\}$  be defined by (1), where  $\{\xi_j : j \in \mathbb{Z}\}$  and  $\{a_k : k \in \mathbb{Z}\}$  satisfy (5) and (6) with  $\alpha \in (0, 2]$ . Assume that  $\{b_n, c_n : n \in \mathbb{N}\}$  are sequences satisfying (8) and (9). If there exists a process  $X$  such that for each sufficiently large  $m \geq 0$ , as  $n \rightarrow \infty$ , (16) holds in  $\mathbb{D}[0, \infty)$  with one of the Skorohod topologies, then*

$$\frac{1}{b_n} \sum_{j=1}^{[nt]} (Z_j - Ac_n) \implies AX(t), \quad \text{where } A = \sum_{k \in \mathbb{Z}} a_k,$$

in  $\mathbb{D}[0, \infty)$  with the same topology. Moreover, (9) can be omitted, if  $\alpha < 1$  and  $\limsup_{n \rightarrow \infty} nb_n^{-1}|c_n| < \infty$ .

After submission of this paper, the author has learned of a recent result by Basrak et al. (2010) which gives some sufficient conditions for (10) in the  $M_1$  topology.

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