

# Convergence to Lévy stable processes under some weak dependence conditions

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## Abstract

For a strictly stationary sequence of random vectors in  $\mathbb{R}^d$  we study convergence of partial sums processes to a Lévy stable process in the Skorohod space with  $J_1$ -topology. We identify necessary and sufficient conditions for such convergence and provide sufficient conditions when the stationary sequence is strongly mixing.

*Key words:* Lévy stable processes, Poisson point processes, functional limit theorem, Skorohod topology, strong mixing

*2000 MSC:* 60F17, 60F05, 60G51, 60G55

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## 1. Introduction

Let  $\{Z_j : j \geq 1\}$  be a strictly stationary sequence of  $\mathbb{R}^d$ -valued random vectors defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . If the  $Z_j$  are i.i.d. then according to Rvačeva [41], there exist sequences of constants  $b_n > 0$  and  $c_n$  such that

$$\frac{1}{b_n} \sum_{j=1}^n Z_j - c_n \xrightarrow{d} \zeta_\alpha \quad \text{in } \mathbb{R}^d \quad (1.1)$$

for some non-degenerate  $\alpha$ -stable random vector  $\zeta_\alpha$  with  $\alpha \in (0, 2)$  if and only if  $Z_1$  is *regularly varying with index*  $\alpha \in (0, 2)$ : there exists a probability measure  $\sigma$  on  $\mathcal{B}(\mathbb{S}^{d-1})$ , the Borel  $\sigma$ -algebra of the unit sphere  $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ , such that as  $x \rightarrow \infty$

$$\frac{\mathbb{P}(|Z_1| > rx, Z_1/|Z_1| \in A)}{\mathbb{P}(|Z_1| > x)} \rightarrow r^{-\alpha} \sigma(A) \quad (1.2)$$

for all  $r > 0$  and  $A \in \mathcal{B}(\mathbb{S}^{d-1})$  such that  $\sigma(\partial A) = 0$  (the notation ‘ $\xrightarrow{d}$  in  $\mathbb{X}$ ’ refers to weak convergence of distributions of given random elements with values in the space  $\mathbb{X}$  and  $|\cdot|$  denotes the Euclidean norm). The sequences  $c_n$  and  $b_n$  can be chosen as

$$c_n = \frac{n}{b_n} \mathbb{E}(Z_1 I(|Z_1| \leq b_n)) \quad \text{and} \quad nP(|Z_1| > b_n) \rightarrow 1. \quad (1.3)$$

A result of Ibragimov [20], and its extension to random vectors by Phillip [37], tells us that if for a strongly mixing sequence the normalized partial sums in (1.1) converge to a non-degenerate random vector  $\zeta$ , then necessarily  $\zeta = \zeta_\alpha$  for some  $\alpha \in (0, 2]$ ; the case of  $\alpha = 2$  refers to a Gaussian distributed random vector.

For the functional generalization of (1.1) define the partial sum processes

$$X_n(t) = \frac{1}{b_n} \sum_{1 \leq j \leq nt} Z_j - tc_n, \quad t \geq 0. \quad (1.4)$$

For each  $\omega$ ,  $X_n(\cdot)$  is an element of the Skorohod space  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued functions on  $[0, \infty)$  that have finite left-hand limits and are continuous from the right. In this paper we study weak convergence of distributions of the partial sum processes in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  with the Skorohod  $J_1$  topology (see Section 2.4). In the i.i.d. case if

$$X_n \xrightarrow{d} X \quad \text{in } \mathbb{D}([0, \infty), \mathbb{R}^d), \quad (1.5)$$

then necessarily  $X$  is a Lévy  $\alpha$ -stable process with  $\alpha \in (0, 2]$ , whose increments are stationary, independent, and  $X(1)$  has the same distribution as  $\zeta_\alpha$ .

In the case of  $\alpha = 2$  in (1.1), a substantial amount of work has been devoted to extend the central limit theorem and the Donsker's invariance principle to weakly dependent random variables; see the recent review by Merlevède, Peligrad, and Utev [31] for sequences of random variables with finite variances and Bradley [7], Shao [45] for random variables with infinite variances. In the case  $\alpha \in (0, 2)$  it was shown by Avram and Taqqu [3] that for some  $m$ -dependent random variables weak- $J_1$  convergence cannot hold despite the fact that (1.1) holds. However, for  $\varphi$ -mixing sequences there is a characterization of convergence in (1.5) by Samur [42] in terms of convergence in (1.1) and some additional conditions. Both [3] and [42] use the Skorohod approach [48] in  $\mathbb{D}([0, 1])$  via tightness plus convergence of finite dimensional distributions.

Since the case of  $\alpha = 2$  is to some extent well understood we shall focus on the less studied case of  $\alpha < 2$ . Our method of proof of (1.5) is based on point process techniques used by Durrett and Resnick [15] for convergence of dependent random variables. For a comprehensive account on this subject in the independent case, we refer the reader to the expository article by Resnick [38] and to his recent monograph [40]. We recall relevant notation and background in Section 2. In Section 3 we study, in a somewhat more general setting, the problem of convergence  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  with  $X$  being a Lévy process without Gaussian component. Theorem 3.1 gives necessary and sufficient conditions for such convergence. Roughly speaking these are a convergence of point processes  $N_n$ , consisting of the jump points of  $X_n$ , to the corresponding point process  $N$  of jumps of  $X$ , which necessarily is a Poisson point process, and a condition which allows one to neglect accumulation of small jumps. In the case of a strictly stationary strongly mixing sequence  $\{Z_j: j \geq 1\}$  of random variables, the class of possible limiting processes for  $N_n$  were discussed first by Mori [32]. Any element of this class must be infinitely divisible and invariant under certain transformations. Then Hsing [18] derived cluster representations of the limiting processes under somewhat

weaker distributional mixing conditions. Thus we need to impose extra condition to obtain a Poisson process in the limit. We now describe an application of Theorem 3.1 to a strictly stationary sequence  $\{Z_j : j \geq 1\}$  under strong mixing conditions.

There exist several coefficients 'measuring' the dependence between two  $\sigma$ -algebras  $\mathcal{A}$  and  $\mathcal{B} \subset \mathcal{F}$ , the most usual ones being

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{A}, B \in \mathcal{B}\},$$

$$\varphi(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(B|A) - \mathbb{P}(B)| : A \in \mathcal{A}, \mathbb{P}(A) > 0, B \in \mathcal{B}\},$$

and the maximal coefficient of correlations

$$\rho(\mathcal{A}, \mathcal{B}) = \sup\{|\text{Corr}(f, g)| : f \in L^2(\mathcal{A}), g \in L^2(\mathcal{B})\};$$

see the review paper by Bradley [8] as a general reference for mixing conditions.

Given the sequence  $\{Z_j : j \geq 1\}$ , we define  $\mathcal{F}_m^n = \sigma\{Z_j : m \leq j \leq n\}$  and, for every  $n \geq 1$ ,

$$\phi_0(n) = \sup_{k \geq 1} \alpha(\mathcal{F}_1^k, \mathcal{F}_{n+k}^\infty),$$

$$\phi_1(n) = \sup_{k \geq 1} \varphi(\mathcal{F}_1^k, \mathcal{F}_{n+k}^\infty),$$

$$\rho(n) = \sup_{k \geq 1} \rho(\mathcal{F}_1^k, \mathcal{F}_{n+k}^\infty).$$

The sequence  $\{Z_j : j \geq 1\}$  is said to be *mixing with rate function*  $\phi_s$  if  $\phi_s(n) \rightarrow 0$  as  $n \rightarrow \infty$ ; the case of  $s = 0$  ( $s = 1$ ) refers to *strongly* (*uniformly* or  $\varphi$ -) *mixing* sequence.

Our main result for strongly mixing sequences is the following functional limit theorem.

**Theorem 1.1.** *Let a strictly stationary sequence  $\{Z_j : j \geq 1\}$  be mixing with rate function  $\phi_0$ . Assume that  $Z_1$  is regularly varying with index  $\alpha \in (0, 2)$  and that one of the following conditions is satisfied:*

- (1)  $\alpha \in (0, 1)$ ;
- (2)  $\alpha \in [1, 2)$  and for every  $\delta > 0$

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq k \leq n} \left| \sum_{j=1}^k (Z_j I(|Z_j| \leq \varepsilon b_n) - \mathbb{E}(Z_1 I(|Z_1| \leq \varepsilon b_n))) \right| \geq \delta b_n\right) = 0. \quad (1.6)$$

Then  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$ , where  $X_n$  is as in (1.3)–(1.4), and  $X$  is a Lévy  $\alpha$ -stable process if and only if the following local dependence condition holds:

**LD**( $\phi_0$ ) *For any  $\varepsilon > 0$  there exist sequences of integers  $r_n = r_n(\varepsilon), l_n = l_n(\varepsilon) \rightarrow \infty$  such that*

$$r_n = o(n), \quad l_n = o(r_n), \quad n\phi_0(l_n) = o(r_n), \quad \text{as } n \rightarrow \infty, \quad (1.7)$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{2 \leq j \leq r_n} |Z_j| > \varepsilon b_n \mid |Z_1| > \varepsilon b_n\right) = 0. \quad (1.8)$$

Note that if  $\phi_0(n) \rightarrow 0$  then one can always find sequences  $r_n, l_n$  satisfying (1.7). Thus, if for any  $\varepsilon > 0$

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\max_{2 \leq j \leq [n/k]} |Z_j| > \varepsilon b_n | |Z_1| > \varepsilon b_n) = 0,$$

then condition  $\mathbf{LD}(\phi_0)$  holds. Therefore,  $\mathbf{LD}(\phi_0)$  is implied by (1.9) which is the local dependence condition  $D'$  of Davis [12], since  $n\mathbb{P}(|Z_1| > \varepsilon b_n) \rightarrow \varepsilon^{-\alpha}$  by (1.2) and (1.3).

**Corollary 1.2.** *Let a strictly stationary sequence  $\{Z_j : j \geq 1\}$  be mixing with rate function  $\phi_0$ . Assume that  $Z_1$  is regularly varying with index  $\alpha \in (0, 2)$ . If for any  $\varepsilon > 0$*

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{j=2}^{[n/k]} \mathbb{P}(|Z_j| > \varepsilon b_n, |Z_1| > \varepsilon b_n) = 0, \quad (1.9)$$

*then condition  $\mathbf{LD}(\phi_0)$  holds.*

For uniformly mixing sequences we have the following result.

**Corollary 1.3.** *Let a strictly stationary sequence  $\{Z_j : j \geq 1\}$  be mixing with rate function  $\phi_1$ . Assume that  $Z_1$  is regularly varying. Then condition  $\mathbf{LD}(\phi_0)$  is equivalent to*

$\mathbf{LD}(\phi_1)$  *For any  $\varepsilon > 0$  and  $j \geq 2$*

$$\lim_{n \rightarrow \infty} \mathbb{P}(|Z_j| > \varepsilon b_n | |Z_1| > \varepsilon b_n) = 0. \quad (1.10)$$

With the notation as in Theorem 1.1 we have the following characterization for  $m$ -dependent sequences.

**Corollary 1.4.** *Assume that  $\{Z_j : j \geq 1\}$  is  $m$ -dependent. Then  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  if and only if  $Z_1$  is regularly varying and for any  $\varepsilon > 0$  and  $j = 2, \dots, m$  condition (1.10) is satisfied.*

The proofs of these results are presented in Section 4. The reader is referred to Theorem 4.1 and Remark 4.3 for necessity of conditions (1.6) and (1.8) without the strong mixing assumption and to Remark 4.5 for a relation between (1.8) and the extremal index of the sequence  $\{|Z_j| : j \geq 1\}$ . We show in Lemma 4.8 that condition (2) of Theorem 1.1 holds if  $\sum_j \rho(2^j) < \infty$ . Thus, Theorem 1.1 complements the results of Bradley [7] and Shao [45]. Theorem 1.1 together with Corollary 1.3 establishes Corollary 5.9 of Kobus [27], which was proved using the results of Samur [42].

The methods and results of this paper were used in [49] to prove functional limit theorems for particular examples of stationary sequences arising from dynamical systems such as continued fractions, Gibbs-Markov maps, and piecewise expanding maps [1, 2, 21, 30, 43, 51]. In that setting condition (1.8) has a nice interpretation in terms of hitting times and it can be also used without the strong mixing assumption, see [49, Sections 3 and 4].

We should also point out that proving the weak convergence of the partial sums of a strictly stationary sequence to an infinite variance  $\alpha$ -stable random vector in (1.1) might

require less restrictive assumptions as opposed to the weak convergence in (1.5). The recent paper [4] contains a detailed study of sufficient conditions for the convergence in (1.1) and a comparison of various conditions used to prove stable limits under the assumption that the stationary sequence is jointly regularly varying, which means that all finite dimensional distributions are regularly varying with index  $\alpha \in (0, 2)$ . Here we comment on the approach through point processes. A number of authors studied the point processes  $N'_n$  consisting of the points  $b_n^{-1}Z_j$ ,  $j = 1, \dots, n$ , in order to obtain convergence to  $\alpha$ -stable random vectors. The one dimensional case with  $N'_n$  converging to a Poisson process was studied in [12]. A systematic application of point process techniques for obtaining limit theorems for arrays of dependent random vectors has been developed in [23, 27]. Their results for strongly mixing stationary sequences are obtained under stronger assumptions than our condition  $\mathbf{LD}(\phi_0)$ . For a jointly regularly varying stationary sequence of dependent random variables sufficient conditions for convergence in (1.1) can be found in [13] and their multivariate extensions in [14]; here the limiting process for  $N'_n$  might not be a Poisson process, so that their examples provide a large class of processes for which the functional limit theorem does not hold in the  $J_1$ -topology. It would be interesting to obtain corresponding results in one of the weaker Skorohod's topologies  $M_1$  or  $M_2$  as defined in [47]; see [3] for a result in the  $M_1$  topology. After the submission of this paper, we became aware of [5], where sufficient conditions for  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, 1], \mathbb{R})$  with the  $M_1$  topology were obtained by building upon the approach and assumptions in [13]; one condition is the same as our condition (2) in Theorem 1.1.

## 2. Preliminaries

In this section we collect some basic tools and notions to be used throughout this paper.

### 2.1. Point processes

We begin by introducing some background on point processes. We follow the point process theory as presented in Kallenberg [24] and Resnick [39]. Let  $E$  be a locally compact Hausdorff topological space with a countable basis for its topology. For our purposes,  $E$  is a subset of either  $\overline{\mathbb{R}}_0^d := \overline{\mathbb{R}}^d \setminus \{0\}$  or  $[0, \infty) \times \overline{\mathbb{R}}_0^d$ , where  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ . The topology on  $\overline{\mathbb{R}}_0^d$  is chosen so that the Borel  $\sigma$ -algebras  $\mathcal{B}(\overline{\mathbb{R}}_0^d)$  and  $\mathcal{B}(\mathbb{R}^d)$  coincide on  $\mathbb{R}^d \setminus \{0\}$ . Moreover,  $B \subset \overline{\mathbb{R}}_0^d$  is relatively compact (or bounded) if and only if  $B \cap \mathbb{R}^d$  is bounded away from zero in  $\mathbb{R}^d$ , i.e.,  $0 \notin \overline{B \cap \mathbb{R}^d}$ .

Let  $M(E)$  be the set of all Radon measures on  $\mathcal{B}(E)$ , i.e., nonnegative Borel measures which are finite on relatively compact subsets of  $E$ . The space  $M(E)$  is a Polish space when considered with the topology of vague convergence. Recall that  $m_n$  converges vaguely to  $m$

$$m_n \xrightarrow{v} m \quad \text{iff} \quad m_n(f) \rightarrow m(f) \quad \text{for all } f \in C_K^+(E),$$

where  $m(f) = \int_E f(x)m(dx)$  and  $C_K^+(E)$  is the space of nonnegative continuous functions on  $E$  with compact support. We have  $m_n \xrightarrow{v} m$  if and only if  $m_n(B) \rightarrow m(B)$  for all relatively compact  $B$  for which  $m(\partial B) = 0$ .

The set  $M_p(E)$  of point measures on  $E$  is a closed subspace of  $M(E)$  consisting of all integer-valued measures in  $M(E)$ . Denote by  $\epsilon_x$  the unit measure concentrated at  $x \in E$ . Any point measure  $m \in M_p(E)$  is of the form  $m = \sum_i \epsilon_{x_i}$ , where  $\{x_i\}$  is at most a countable collection of points of  $E$ . The measure  $m$  is called *simple* if the points  $\{x_i\}$  are all distinct.

A point process  $N$  on  $E$  is an  $M_p(E)$ -valued random variable, defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The measure  $Q$  defined by  $Q(A) = \mathbb{E}(N(A))$ ,  $A \in \mathcal{B}(E)$ , is called a *mean measure of  $N$* . The process  $N$  is called *simple* if almost all its realizations are simple. A point process  $N$  is called a *Poisson process* with mean measure  $Q \in M(E)$  if  $N(A_1), \dots, N(A_l)$  are independent random variables for any disjoint sets  $A_1, \dots, A_l \in \mathcal{B}(E)$  and  $N(A)$  is a Poisson random variable with mean  $Q(A)$  for  $A \in \mathcal{B}(E)$  with  $Q(A) < \infty$ . The Poisson process is simple if its mean measure is non-atomic. The Laplace functional of Poisson process  $N$  is of the form

$$\mathbb{E}[e^{-N(f)}] = \exp \left\{ - \int_E (1 - e^{-f(x)}) Q(dx) \right\}$$

for nonnegative measurable  $f$ . Given a sequence of point processes  $N_n$  we have  $N_n \xrightarrow{d} N$  in  $M_p(E)$ , by [24, Theorem 4.2], if and only if  $\mathbb{E}[e^{-N_n(f)}] \rightarrow \mathbb{E}[e^{-N(f)}]$  for all  $f \in C_K^+(E)$ .

## 2.2. Infinitely divisible and stable random vectors

An infinitely divisible  $\mathbb{R}^d$ -valued random vector  $\zeta$  is uniquely determined through the Lévy-Khintchine formula, which states that its characteristic function is of the form

$$\mathbb{E}e^{i\langle u, \zeta \rangle} = \exp \left( i\langle a, u \rangle - \frac{1}{2} \langle \Sigma u, u \rangle + \int_{\mathbb{R}^d} (e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle I(|x| \leq 1)) \Pi(dx) \right),$$

where  $a$  is a  $d$ -dimensional vector,  $\Sigma$  is a symmetric nonnegative definite  $d \times d$  matrix, and  $\Pi$  is a *Lévy measure*, i.e., a  $\sigma$ -finite Borel measure on  $\mathbb{R}^d$  such that  $\Pi(\{0\}) = 0$  and

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2) \Pi(dx) < \infty.$$

Here  $\langle x, u \rangle$  denotes the usual inner product in  $\mathbb{R}^d$ . We have  $\Pi(B) < \infty$  for any set  $B \in \mathcal{B}(\mathbb{R}^d)$  bounded away from 0. We can extend  $\Pi$  on  $\mathcal{B}(\overline{\mathbb{R}}_0^d)$  in such a way that  $\Pi(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ . The generating triplet  $(\Sigma, \Pi, a)$  uniquely determines a given infinitely divisible random vector.

A particular class of infinitely divisible random vectors without Gaussian component, i.e., with  $\Sigma = 0$ , are stable random vectors. Recall that a vector  $\zeta$  with values in  $\mathbb{R}^d$  is said to be  $\alpha$ -stable for some  $\alpha \in (0, 2)$  if its characteristic function is given by

$$\mathbb{E}e^{i\langle u, \zeta \rangle} = \begin{cases} \exp(- \int_{\mathbb{S}^{d-1}} |\langle u, s \rangle|^\alpha (1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn} \langle u, s \rangle) \lambda_1(ds) + i\langle \tau, u \rangle), & \alpha \neq 1, \\ \exp(- \int_{\mathbb{S}^{d-1}} |\langle u, s \rangle| (1 + i \frac{2}{\pi} \operatorname{sgn} \langle u, s \rangle \log |\langle u, s \rangle|) \lambda_1(ds) + i\langle \tau, u \rangle), & \alpha = 1, \end{cases}$$

where  $\tau \in \mathbb{R}^d$  and  $\lambda_1$  is a finite nonzero measure on  $\mathcal{B}(\mathbb{S}^{d-1})$ ; see [44, Theorem 14.3]. Its Lévy measure is of the form

$$\Pi_\alpha(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(rs) r^{-\alpha-1} dr \lambda(ds), \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (2.1)$$

where  $\lambda$  is a finite nonzero measure on  $\mathcal{B}(\mathbb{S}^{d-1})$  and it is a constant multiple of  $\lambda_1$ .

### 2.3. Regularly varying vectors

If  $\zeta$  is  $\alpha$ -stable for some  $\alpha \in (0, 2)$ , it is regularly varying with index  $\alpha$  as defined in Introduction. The concept of vague convergence in  $M(\mathbb{S}^{d-1})$  allows us to rewrite condition (1.2) in the form

$$\frac{\mathbb{P}(|\zeta| > tx, \zeta/|\zeta| \in \cdot)}{\mathbb{P}(|\zeta| > x)} \xrightarrow{v} t^{-\alpha} \sigma(\cdot) \quad \text{as } x \rightarrow \infty. \quad (2.2)$$

The measure  $\sigma$  is called the *spectral measure*. See [39, Chapter 5] for background on multivariate regular variation.

Alternatively, (2.2) is equivalent to: there exists  $Q \in M(\overline{\mathbb{R}}_0^d)$  such that  $Q(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$  and

$$\frac{\mathbb{P}(x^{-1}\zeta \in \cdot)}{\mathbb{P}(|\zeta| > x)} \xrightarrow{v} Q \quad \text{in } M(\overline{\mathbb{R}}_0^d), \quad \text{as } x \rightarrow \infty,$$

and to the sequential definition of regular variation: there exists  $b_n \rightarrow \infty$  such that

$$n\mathbb{P}(b_n^{-1}\zeta \in \cdot) \xrightarrow{v} Q, \quad n \rightarrow \infty. \quad (2.3)$$

One can always choose  $b_n$  such that  $n\mathbb{P}(|\zeta| > b_n) \sim 1$ . The measure  $Q$  necessarily has the property  $Q(rB) = r^{-\alpha}Q(B)$ ,  $r > 0$ , for some  $\alpha > 0$  and all  $b \in \mathcal{B}(\overline{\mathbb{R}}_0^d)$ , which clarifies the relation with the index  $\alpha$ .

If (2.3) holds with  $Q = \Pi_\alpha$  where  $\Pi_\alpha$  is as in (2.1), then for the finite measure  $\lambda$  in (2.1) we have  $\lambda = \alpha\sigma$ . Note that the property of regular variation does not depend on a given norm  $|\cdot|$  in  $\mathbb{R}^d$ , however, the spectral measure and the limit measure  $Q$  are different for distinct norms.

Regular variation of  $\zeta$  implies that the function  $x \mapsto \mathbb{P}(|\zeta| > x)$  is regularly varying:

$$\mathbb{P}(|Z_1| > x) = x^{-\alpha}L(x), \quad (2.4)$$

where  $L$  is a *slowly varying function*, i.e.,  $L(rx)/L(x) \rightarrow 1$  as  $x \rightarrow \infty$  for every  $r > 0$ . In the special case of  $d = 1$  a random variable  $\zeta$  is regularly varying with index  $\alpha \in (0, 2)$  if and only if (2.4) holds for a slowly varying  $L$  and the tails are balanced: there exist  $p, q \geq 0$  with  $p + q = 1$  such that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(Z_1 > x)}{\mathbb{P}(|Z_1| > x)} = p \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(Z_1 < -x)}{\mathbb{P}(|Z_1| > x)} = q. \quad (2.5)$$

From (2.5) and (2.4) it follows that

$$\lim_{n \rightarrow \infty} \frac{nL(b_n)}{b_n^\alpha} = 1$$

and for  $r > 0$  we have

$$\lim_{n \rightarrow \infty} n\mathbb{P}(Z_1 > b_n r) = r^{-\alpha}p \quad \text{and} \quad \lim_{n \rightarrow \infty} n\mathbb{P}(Z_1 < -b_n r) = r^{-\alpha}q.$$

Hence

$$n\mathbb{P}(b_n^{-1}Z_1 \in \cdot) \xrightarrow{v} \Pi_\alpha(\cdot),$$

where  $\Pi_\alpha$  is an absolutely continuous measure on  $\mathbb{R}$  with density

$$\Pi_\alpha(dx) = (p\alpha 1_{(0, \infty)}(x) + q\alpha 1_{(-\infty, 0)}(x)) |x|^{-\alpha-1} dx.$$

#### 2.4. Skorohod $J_1$ topology

Let  $\mathbb{D}([0, T], \mathbb{R}^d)$  for  $T > 0$  be the space of all  $\mathbb{R}^d$ -valued functions  $\psi$  on  $[0, T]$  that are right continuous on  $[0, T)$  and have finite left-hand limits  $\psi(t-)$  for all  $t \in (0, T]$ . Consider the set  $\Lambda_T$  of strictly increasing, continuous mappings  $\lambda$  of  $[0, T]$  onto itself such that  $\lambda(0) = 0$  and  $\lambda(T) = T$ . The Skorohod  $J_1$  metric on  $\mathbb{D}([0, T], \mathbb{R}^d)$  is defined as

$$d_T(\psi_1, \psi_2) = \inf_{\lambda \in \Lambda_T} \left( \sup_{0 \leq s \leq T} |\psi_1(\lambda(s)) - \psi_2(s)| \vee \sup_{0 \leq s \leq T} |\lambda(s) - s| \right)$$

for  $\psi_1, \psi_2 \in \mathbb{D}([0, T], \mathbb{R}^d)$ , where  $a \vee b = \max\{a, b\}$ . Then

$$d_\infty(\psi_1, \psi_2) = \int_0^\infty e^{-t} (d_t(\psi_1, \psi_2) \wedge 1) dt, \quad \psi_1, \psi_2 \in \mathbb{D}([0, \infty), \mathbb{R}^d),$$

defines a metric in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  which induces the Skorohod  $J_1$ -topology. The metric spaces  $(\mathbb{D}([0, T], \mathbb{R}^d), d_T)$  and  $(\mathbb{D}([0, \infty), \mathbb{R}^d), d_\infty)$  are separable. For more details see [50] and [22, Section 6].

We have the following characterization of weak convergence in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$ . If  $X_n, X$  are stochastic processes with sample paths in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  then  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  if and only if  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, T], \mathbb{R}^d)$  for all  $T \in T_X = \{t > 0 : \mathbb{P}(X(t) \neq X(t-)) = 0\}$ .

### 3. Necessary and sufficient conditions for weak convergence in the Skorohod topology

In this section we study convergence in distributions of partial sum processes to Lévy processes without Gaussian component in the Skorohod space  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  with  $J_1$ -topology.

Let  $X = \{X(t) : t \geq 0\}$  be an  $\mathbb{R}^d$ -valued Lévy process, i.e.,  $X(0) = 0$  a.s.,  $X$  has stationary independent increments and sample paths in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$ . We assume that  $X$  is such that

$$\mathbb{E}(e^{i\langle u, X(1) \rangle}) = \exp\left(\int (e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle I(|x| \leq 1)) \Pi(dx)\right), \quad (3.1)$$

where  $\Pi$  is a Lévy measure with  $\Pi(\{x : |x| = 1\}) = 0$ . The jump process  $\Delta X(t) := X(t) - X(t-)$ ,  $t > 0$ , determines a Poisson point process  $N$  on  $[0, \infty) \times \bar{\mathbb{R}}_0^d$ , which can be represented as

$$N = \sum_{\{t: \Delta X(t) \neq 0\}} \epsilon_{(t, \Delta X(t))}, \quad (3.2)$$

and  $\text{Leb} \times \Pi$  is the mean measure of  $N$ ; see [44] for details.

We also consider a family  $\{X_{n,j} : j, n \geq 1\}$  of  $\mathbb{R}^d$ -valued random vectors such that  $\mathbb{P}(0 < |X_{n,j}| < \infty) = 1$ . Define the partial sum process

$$X_n(t) = \sum_{1 \leq j \leq nt} X_{n,j} - tc_n, \quad t \geq 0, n \geq 1,$$



where  $\{c_n: n \geq 1\}$  is a sequence of vectors in  $\mathbb{R}^d$ . It should be emphasized that we are not assuming any dependence structure for the random vectors  $\{X_{n,j}: j, n \geq 1\}$  in this section.

The following result extends [29, Theorem 2.10.1], [15, Theorem 4.1], [38, Proposition 3.4]. It describes the connection between convergence in the Skorohod space with  $J_1$ -topology and convergence of the corresponding point processes of jumps in  $M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$ .

**Theorem 3.1.** *Let  $X$  be a Lévy process satisfying (3.1) and let  $N$  be the corresponding Poisson point process as in (3.2). Then*

$$X_n \xrightarrow{d} X \quad \text{in } \mathbb{D}([0, \infty), \mathbb{R}^d) \quad (3.3)$$

*if and only if*

$$N_n := \sum_{j \geq 1} \epsilon_{(\frac{j}{n}, X_{n,j})} \xrightarrow{d} N \quad \text{in } M_p((0, \infty) \times \overline{\mathbb{R}}_0^d) \quad (3.4)$$

*and for every  $\delta > 0$  and  $T > 0$*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{0 \leq t \leq T} \left| \sum_{j \leq nt} X_{n,j} I(|X_{n,j}| \leq \varepsilon) - t(c_n - \int_{\{x: \varepsilon < |x| \leq 1\}} x \Pi(dx)) \right| \geq \delta\right) = 0, \quad (3.5)$$

*where the limit is taken over all  $\varepsilon$  with  $\Pi(\{x: |x| = \varepsilon\}) = 0$ .*

We shall prove this result in Section 5 using the continuous mapping theorem and properties of Lévy processes. In the next section we will use this result to prove limit theorems for stationary sequences. For completeness we also provide the following sufficient conditions for convergence of marginal distributions.

**Theorem 3.2.** *Let  $N'$  be a Poisson point process in  $M_p(\overline{\mathbb{R}}_0^d)$  with mean measure  $\Pi$ . If*

$$N'_n := \sum_{j=1}^n \epsilon_{X_{n,j}} \xrightarrow{d} N' \quad \text{in } M_p(\overline{\mathbb{R}}_0^d) \quad (3.6)$$

*and for any  $\delta > 0$*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\left| \sum_{j=1}^n X_{n,j} I(|X_{n,j}| \leq \varepsilon) - c_n + \int_{\{x: \varepsilon < |x| \leq 1\}} x \Pi(dx) \right| \geq \delta\right) = 0, \quad (3.7)$$

*where the limit is taken over all  $\varepsilon$  with  $\Pi(\{x: |x| = \varepsilon\}) = 0$ , then*

$$X_n(1) \xrightarrow{d} X(1) \quad \text{in } \mathbb{R}^d. \quad (3.8)$$

**Remark 3.3.** Note that Poisson convergence in (3.6) is not necessary for (3.8). There are many examples of dependent random variables for which the latter holds, but in (3.6) we have convergence to a non-Poisson point process [13].

Theorem 3.1 can be used to disprove weak convergence in Skorohod  $J_1$ -topology. Since condition (3.4) implies (3.6), the convergence in (3.3) is impossible in the  $J_1$ -topology for all examples where  $N'$  was shown to be non-Poisson. These include moving average processes [10] for which lack of convergence in the  $J_1$ -topology was shown in [3] using the finite-dimensional plus tightness technique.

We conclude this section with a discussion of some of our conditions. A situation when condition (3.5) is not needed at all is described in the following.

**Corollary 3.4.** *Let the Lévy measure  $\Pi$  be such that  $\int (1 \wedge |x|) \Pi(dx) < \infty$  and let  $c_n = 0$  for all  $n \geq 1$ . Suppose that for any  $T > 0$  we have*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \sum_{j \leq nT} \mathbb{E}(|X_{n,j}| I(|X_{n,j}| \leq \varepsilon)) = 0. \quad (3.9)$$

Then  $N_n \xrightarrow{d} N$  in  $M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$  if and only if  $X_n \xrightarrow{d} \tilde{X}$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$ , where

$$\tilde{X}(t) = X(t) + t \int x I(|x| \leq 1) \Pi(dx), \quad t \geq 0.$$

This result is a consequence of Theorem 3.1 and the following maximal inequality from [28, Theorem 1], the proof of which extends directly to random vectors. Note that if  $\Pi = \Pi_\alpha$ , where  $\Pi_\alpha$  is as in (2.1) with  $\alpha \in (0, 2)$ , then  $\int (1 \wedge |x|) \Pi(dx) < \infty$  precisely when  $\alpha < 1$ .

**Lemma 3.5.** *If  $\zeta_i$  are  $\mathbb{R}^d$ -valued random variables with  $E|\zeta_i| < \infty$ ,  $i = 1, \dots, k$ , then for any  $\delta > 0$  we have*

$$\mathbb{P}(\max_{1 \leq j \leq k} |\zeta_1 + \dots + \zeta_j| \geq \delta) \leq \frac{1}{\delta} \sum_{i=1}^k \mathbb{E}|\zeta_i|.$$

**Remark 3.6.** Condition (3.5) can be equivalently replaced by

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{0 \leq t \leq T} \left| \sum_{j \leq nt} (X_{n,j} I(|X_{n,j}| \leq \varepsilon) - \mathbb{E}(X_{n,j} I(|X_{n,j}| \leq \varepsilon))) \right| \geq \delta\right) = 0 \quad (3.10)$$

if for example, for any  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \left| \sum_{j \leq nt} \mathbb{E}(X_{n,j} I(|X_{n,j}| \leq \varepsilon)) - t(c_n - \int_{\{x: \varepsilon < |x| \leq 1\}} x \Pi(dx)) \right| = 0.$$

In order to check condition (3.4) we may use Kallenberg's characterization of convergence to simple point processes. In particular, the following lemma is a direct consequence of [24, Theorems 4.7 and 4.8] with later improvements from [25].

**Lemma 3.7.** *Let  $N$  be a Poisson point process with mean measure  $\text{Leb} \times \Pi$  where the Lévy measure  $\Pi$  is non-atomic and let  $\mathcal{U}$  be the class of all finite unions of rectangles in  $\overline{\mathbb{R}}_0^d$  bounded away from 0 and with boundary of zero  $\Pi$  measure.*

(1) *We have*

$$N_n \xrightarrow{d} N \quad \text{in } M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$$

*if and only if*

$$N_n((s, t] \times \cdot) \xrightarrow{d} N((s, t] \times \cdot) \quad \text{in } M_p(\overline{\mathbb{R}}_0^d) \quad (3.11)$$

for all  $t > s \geq 0$  and

$$\lim_{n \rightarrow \infty} \mathbb{E}(e^{-N_n(B)}) = \mathbb{E}(e^{-N(B)})$$

for any set  $B$  of the form  $\bigcup_{j=1}^k (s_j, t_j] \times U_j$ , where  $0 \leq s_1 < t_1 < \dots < s_k < t_k$ , and  $U_j \in \mathcal{U}$ ,  $j = 1, \dots, k$ ,  $k \geq 2$ .

(2) Let  $t > s \geq 0$ . If  $\mathbb{E}N_n((s, t] \times \cdot) \xrightarrow{v} \mathbb{E}N((s, t] \times \cdot)$  in  $M(\overline{\mathbb{R}}_0^d)$  and

$$\lim_{n \rightarrow \infty} \mathbb{P}(N_n((s, t] \times U) = 0) = \mathbb{P}(N((s, t] \times U) = 0)$$

for every  $U \in \mathcal{U}$ , then (3.11) holds.

#### 4. Limit theorems for stationary sequences

In this section we study limit theorems for stationary sequences of random vectors in  $\mathbb{R}^d$ . Let  $X$  be a Lévy  $\alpha$ -stable process as in (3.1) with Lévy measure  $\Pi = \Pi_\alpha$  defined by (2.1) and let  $N$  be a Poisson point process on  $[0, \infty) \times \overline{\mathbb{R}}_0^d$  with mean measure  $\text{Leb} \times \Pi$ .

We assume throughout this section that  $\{Z_j: j \geq 1\}$  is a strictly stationary sequence of random vectors in  $\mathbb{R}^d$  such that  $Z_1$  is regularly varying with index  $\alpha \in (0, 2)$ . Then we have

$$n\mathbb{P}(b_n^{-1}Z_1 \in \cdot) \xrightarrow{v} \Pi(\cdot), \quad (4.1)$$

where the normalizing constants  $b_n$  are such that

$$\lim_{n \rightarrow \infty} n\mathbb{P}(|Z_1| > b_n) = 1.$$

We define

$$X_n(t) = \frac{1}{b_n} \left( \sum_{j \leq nt} Z_j - tn\mathbb{E}(Z_1 I(|Z_1| \leq b_n)) \right) \quad \text{and} \quad N_n = \sum_{j \geq 1} \epsilon_{(\frac{j}{n}, \frac{Z_j}{b_n})}.$$

**Theorem 4.1.** *Suppose that  $Z_1$  is regularly varying with index  $\alpha \in (0, 2)$ . Then  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  if and only if  $N_n \xrightarrow{d} N$  in  $M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$  and (1) or (2) of Theorem 1.1 holds.*

*Proof.* Let  $X_{n,j} = Z_j/b_n$ ,  $j \geq 1$ , and  $c_n = n\mathbb{E}(X_{n,1}I(|X_{n,1}| \leq 1))$ ,  $n \geq 1$ . From (4.1) it follows that, for any  $\varepsilon \in (0, 1)$ ,

$$\lim_{n \rightarrow \infty} n\mathbb{E}(X_{n,1}I(\varepsilon < |X_{n,1}| \leq 1)) = \int_{\{x: \varepsilon < |x| \leq 1\}} x\Pi(dx),$$

which together with  $\mathbb{E}(|X_{n,1}|I(|X_{n,1}| \leq \varepsilon)) \rightarrow 0$ , as  $n \rightarrow \infty$ , implies that

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \left| t \int_{\{x: \varepsilon < |x| \leq 1\}} x\Pi(dx) - tc_n + [nt]\mathbb{E}(X_{n,1}I(|X_{n,1}| \leq \varepsilon)) \right| = 0,$$

for all  $T > 0$  and  $\varepsilon \in (0, 1)$ . Now observe that, by stationarity, condition (1.6) holds for all  $\delta > 0$  if and only if condition (3.10) holds for all  $T > 0$  and  $\delta > 0$ . Consequently, by Theorem 3.1 and Remark 3.6, we obtain  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  if and only if  $N_n \xrightarrow{d} N$  in  $M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$  and condition (1.6) holds for every  $\delta > 0$ .

It remains to show that if  $\alpha < 1$  then (4.1) implies (1.6). By Lemma 3.5, we have

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \left| \sum_{j=1}^k (Z_j I(|Z_j| \leq \varepsilon b_n) - \mathbb{E}(Z_1 I(|Z_1| \leq \varepsilon b_n))) \right| \geq \delta b_n\right) \leq \frac{2n}{\delta b_n} \mathbb{E}(|Z_1| I(|Z_1| \leq \varepsilon b_n))$$

for all  $n \geq 1$ ,  $\delta, \varepsilon > 0$ . The rest of the argument is standard. From Karamata's theorem, it follows that

$$\mathbb{E}(|Z_1| I(|Z_1| \leq \varepsilon b_n)) \sim \frac{\alpha}{1 - \alpha} \varepsilon b_n \mathbb{P}(|Z_1| \geq \varepsilon b_n) \sim \frac{\alpha}{1 - \alpha} (\varepsilon b_n)^{1-\alpha} L(\varepsilon b_n),$$

where  $L$  is a slowly varying function such that  $nb_n^{-\alpha} L(b_n) \rightarrow 1$ . Consequently, we obtain

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n} \mathbb{E}(|Z_1| 1_{\{|Z_1| \leq \varepsilon b_n\}}) = \frac{\alpha}{1 - \alpha} \varepsilon^{1-\alpha}$$

for every  $\varepsilon > 0$ , which completes the proof.  $\square$

The next result gives necessary conditions for convergence of point processes to the Poisson process  $N$ .

**Theorem 4.2.** *Suppose that  $Z_1$  is regularly varying. If  $N_n \xrightarrow{d} N$  in  $M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$  then for any  $t, \varepsilon > 0$  we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq j \leq nt} |Z_j| \leq \varepsilon b_n\right) = e^{-t\Pi(\{x: |x| > \varepsilon\})}$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{2 \leq j \leq nt} |Z_j| > \varepsilon b_n \mid |Z_1| > \varepsilon b_n\right) = 1 - e^{-t\Pi(\{x: |x| > \varepsilon\})}. \quad (4.2)$$

The first statement is a consequence of the assumption and the identity

$$\mathbb{P}(N_n((0, t] \times \{x: |x| > \varepsilon\}) = 0) = \mathbb{P}\left(\max_{1 \leq j \leq nt} |Z_j| \leq \varepsilon b_n\right).$$

Condition (4.2) follows from the next lemma.

**Remark 4.3.** Observe that the convergence in (4.2) is locally uniform with respect to  $t$ . Hence, condition (1.8) holds for every  $\varepsilon > 0$  and all sequences  $r_n$  such that  $r_n = o(n)$ .

**Lemma 4.4.** *Let  $\{\xi_j: j \geq 1\}$  be a strictly stationary sequence of random variables. Suppose that  $\lambda > 0, \theta > 0$ , and  $u_n, n \geq 1$ , are such that*

$$\lim_{n \rightarrow \infty} n\mathbb{P}(\xi_1 > u_n) = \lambda \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq j \leq nt} \xi_j \leq u_n\right) = e^{-\theta\lambda t} \quad (4.3)$$

for all  $t > 0$ . Then

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{2 \leq j \leq nt} \xi_j > u_n \mid \xi_1 > u_n\right) = 1 - \theta e^{-\theta\lambda t}, \quad t > 0. \quad (4.4)$$

*Proof.* Define for  $n \geq 1$ ,  $t \geq 0$

$$F_n(t) = \mathbb{P}(\max_{2 \leq j \leq nt} \xi_j > u_n)$$

and

$$G_n(t) = \mathbb{P}(\max_{2 \leq j \leq nt} \xi_j > u_n | \xi_1 > u_n),$$

where we set  $F_n(t) = G_n(t) = 0$  for  $0 \leq t < 2/n$ . Both functions are nondecreasing and piecewise constant. We first show that

$$1 - G_n(t) = \frac{F_n(t + \frac{1}{n}) - F_n(t)}{\mathbb{P}(\xi_1 > u_n)}, \quad t \geq 0, n \geq 1. \quad (4.5)$$

Observe that equality (4.5) holds for  $t \in [0, 2/n)$ . Let  $k \geq 2$  and  $t \in [k/n, (k+1)/n)$ . We have

$$1 - G_n(t) = \mathbb{P}(\max_{2 \leq j \leq k} \xi_j \leq u_n | \xi_1 > u_n),$$

which leads to

$$1 - G_n(t) = \frac{\mathbb{P}(\max_{2 \leq j \leq k} \xi_j \leq u_n) - \mathbb{P}(\max_{1 \leq j \leq k} \xi_j \leq u_n)}{\mathbb{P}(\xi_1 > u_n)},$$

and, by stationarity, concludes the proof of (4.5).

To complete the proof it suffices to show that

$$n(F_n(t + \frac{1}{n}) - F_n(t)) \rightarrow \theta \lambda e^{-\theta \lambda t}.$$

We proceed similarly to [17, pp. 2047-2048]. Define piecewise linear functions  $\tilde{F}_n$  by  $\tilde{F}_n(t) = F_n(t)$  for  $t = k/n$  and  $\tilde{F}_n$  linear on  $[k/n, (k+1)/n]$ ,  $k \geq 0$ . Then the right-hand derivative  $\tilde{F}'_n(t+)$  at every point  $t$  is given by

$$\tilde{F}'_n(t+) = n(F_n(t + \frac{1}{n}) - F_n(t)).$$

Note that  $F_n(t) \rightarrow 1 - e^{-\theta \lambda t}$ , as  $n \rightarrow \infty$ , for all  $t \geq 0$ , and, by (4.5), we have

$$\sup_{t \geq 0} |\tilde{F}_n(t) - F_n(t)| \leq \mathbb{P}(\xi_1 > u_n) \rightarrow 0.$$

Since the functions  $\tilde{F}_n$  are concave, we obtain  $\tilde{F}'_n(t+) \rightarrow \theta \lambda e^{-\theta \lambda t}$  for all  $t$ . □

**Remark 4.5.** Note that the constant  $\theta$  in (4.3) might be referred to as the extremal index of the sequence  $\{\xi_j : j \geq 1\}$ ; see e.g. [29, 34] and [16, Chapter 8.1] for the definition and properties. In particular, if the  $\xi_j$  are i.i.d. then  $\theta = 1$ . Dependent random variables have the extremal index equal to 1 when they satisfy the extreme mixing conditions  $D(u_n)$  and  $D'(u_n)$ . This will be also the case for the sequence  $\{|Z_j| : j \geq 1\}$  in Theorem 4.1.

We now provide sufficient conditions for convergence to Poisson processes for strongly mixing sequences. The mixing condition  $D^*$  of Davis and Resnick [11, p. 47] is implied by strong mixing. Hence, from [11, Theorem 2.1] it follows that if the local dependence condition (1.9) of Davis [12], then  $N_n \xrightarrow{d} N$  in  $M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$ . Although condition (1.9) is sufficient for Poisson convergence, it is not necessary. We now prove that our condition  $\mathbf{LD}(\phi_0)$  from Theorem 1.1, which is necessary, is also sufficient.

**Theorem 4.6.** *Suppose that the sequence  $\{Z_j: j \geq 1\}$  is strongly mixing and  $Z_1$  is regularly varying. If condition  $\mathbf{LD}(\phi_0)$  holds then  $N_n \xrightarrow{d} N$  in  $M_p((0, \infty) \times \overline{\mathbb{R}}_0^d)$ .*

*Proof.* By part (1) of Lemma 3.7, we have  $N_n \xrightarrow{d} N$  if and only if for any  $t > s \geq 0$  condition (3.11) holds, since  $\{Z_j: j \geq 1\}$  is strongly mixing. For any  $f \in C_K^+(\overline{\mathbb{R}}_0^d)$  and  $t > s \geq 0$  we have, by stationarity,

$$|\mathbb{E}(e^{-\sum_{ns < j \leq nt} f(b_n^{-1} Z_j)}) - \mathbb{E}(e^{-\sum_{0 < j \leq n(t-s)} f(b_n^{-1} Z_j)})| \leq \mathbb{E}(1 - e^{-f(b_n^{-1} Z_1)}),$$

which converges to 0 as  $n \rightarrow \infty$ , since  $f(b_n^{-1} Z_1) \rightarrow 0$  a.s. Consequently, we have  $N_n \xrightarrow{d} N$  if and only if for any  $t > 0$

$$N_n((0, t] \times \cdot) \xrightarrow{d} N((0, t] \times \cdot) \quad \text{in } M_p(\overline{\mathbb{R}}_0^d).$$

Let  $t > 0$ . From (4.1) it follows that  $\mathbb{E}(N_n((0, t] \times \cdot)) \xrightarrow{v} \mathbb{E}(N((0, t] \times \cdot))$ . Hence, it suffices to show, by part (2) of Lemma 3.7, that

$$\lim_{n \rightarrow \infty} \mathbb{P}(N_n((0, t] \times U) = 0) = e^{-t\Pi(U)} \quad (4.6)$$

for every finite union  $U$  of rectangles with  $\Pi(U) < \infty$  and  $\Pi(\partial U) = 0$ .

Let  $\varepsilon > 0$  be such that  $U \subset \{x: |x| > \varepsilon\}$ . Take  $r_n, l_n$  as in (1.7) and (1.8). Since  $l_n = o(r_n)$  we may assume that  $l_n < r_n$ . Let the integers  $k_n, s_n$  be given by the Euclidean division of  $[nt]$  by  $r_n$ ,  $[nt] = k_n r_n + s_n$  and  $0 \leq s_n < r_n$ . We will prove the following two statements

$$|\mathbb{P}(N_n((0, t] \times U) = 0) - (1 - \mathbb{P}(\bigcup_{j=1}^{r_n - l_n} \{Z_j \in b_n U\}))^{k_n}| \rightarrow 0 \quad (4.7)$$

and

$$\frac{\mathbb{P}(\bigcup_{j=1}^{r_n - l_n} \{Z_j \in b_n U\})}{r_n \mathbb{P}(Z_1 \in b_n U)} \rightarrow 1. \quad (4.8)$$

Since  $k_n r_n \mathbb{P}(A_1) \rightarrow t\Pi(U)$ , conditions (4.7) and (4.8) imply (4.6).

To prove (4.7) we use the standard big-little block technique. Write  $A_j = \{Z_j \in b_n U\}$ ,  $j \geq 1$ , and observe that

$$|\mathbb{P}(\bigcap_{j=1}^{[nt]} A_j^c) - \mathbb{P}(\bigcap_{j=1}^{k_n r_n} A_j^c)| \leq r_n \mathbb{P}(A_1).$$

Let us divide the integers  $1, \dots, k_n r_n$ , into blocks of the form

$$I_j = \{(j-1)r_n + 1, \dots, jr_n - l_n\}, \quad I_j^* = \{(j-1)r_n - l_n + 1, \dots, jr_n\}, \quad j = 1, \dots, k_n.$$

We have

$$|\mathbb{P}(\bigcap_{j=1}^{k_n r_n} A_j^c) - \mathbb{P}(\bigcap_{j=1}^{k_n} \bigcap_{i \in I_j} A_i^c)| \leq \mathbb{P}(\bigcup_{j=1}^{k_n} \bigcup_{i \in I_j^*} A_i^c) \leq k_n l_n \mathbb{P}(A_1)$$

and, by strong mixing,

$$|\mathbb{P}(\bigcap_{j=1}^{k_n} \bigcap_{i \in I_j} A_i^c) - \mathbb{P}(\bigcap_{i \in I_1} A_i^c)^{k_n}| \leq (k_n - 1)\phi_0(l_n).$$

Summarizing

$$|\mathbb{P}(\bigcap_{j=1}^{[nt]} A_j^c) - \mathbb{P}(\bigcap_{j \in I_1} A_j^c)^{k_n}| \leq (r_n + k_n l_n) \mathbb{P}(A_1) + (k_n - 1)\phi_0(l_n)$$

and, by the choice of the sequences, the right-hand side in the last inequality goes to 0 as  $n \rightarrow \infty$ , which completes the proof of (4.7). Now observe that

$$|\mathbb{P}(\bigcup_{j=1}^{r_n} A_j) - \mathbb{P}(\bigcup_{j=1}^{r_n - l_n} A_j)| \leq 2l_n \mathbb{P}(A_1).$$

Hence, it remains to show that

$$\frac{\mathbb{P}(\bigcup_{j=1}^{r_n} A_j)}{r_n \mathbb{P}(A_1)} \rightarrow 1.$$

Since  $A_j = \{Z_j \in b_n U\} \subset \{|Z_j| > \varepsilon b_n\}$ , we have

$$\mathbb{P}(\bigcup_{j=2}^{r_n} A_j | A_1) \leq \mathbb{P}(\max_{2 \leq j \leq r_n} |Z_j| > \varepsilon b_n | |Z_1| > \varepsilon b_n) \frac{\mathbb{P}(|Z_1| > \varepsilon b_n)}{\mathbb{P}(Z_1 \in b_n U)},$$

which shows that the left-hand side in the last inequality goes to 0 as  $n \rightarrow \infty$ . Consequently,

$$\mathbb{P}(\bigcup_{j=3}^{r_n} A_j | A_1) \rightarrow 0 \quad \text{and} \quad \mathbb{P}(A_2 | A_1) \rightarrow 0. \quad (4.9)$$

We have

$$\mathbb{P}(\bigcup_{j=1}^{r_n} A_j) = \sum_{j=1}^{r_n-2} \mathbb{P}(A_j \cap A_{j+1}^c \cap \bigcap_{i=j+2}^{r_n} A_i^c) + \mathbb{P}(A_{r_n-1} \cap A_{r_n}^c) + \mathbb{P}(A_{r_n})$$

and thus, by stationarity,

$$\begin{aligned} |r_n \mathbb{P}(A_1 \cap A_2^c) - \mathbb{P}(\bigcup_{j=1}^{r_n} A_j)| &\leq \sum_{j=1}^{r_n-2} \mathbb{P}(A_j \cap A_{j+1}^c \cap \bigcup_{i=j+2}^{r_n} A_i) + \mathbb{P}(A_1 \cap A_2) \\ &\leq r_n \mathbb{P}(A_1 \cap \bigcup_{i=3}^{r_n} A_i) + \mathbb{P}(A_1 \cap A_2), \end{aligned}$$

since  $\mathbb{P}(A_1 \cap A_2^c) = \mathbb{P}(A_j \cap A_{j+1}^c)$  and  $\mathbb{P}(A_j \cap A_{j+1}^c \cap \bigcup_{i=j+2}^{r_n} A_i) = \mathbb{P}(A_1 \cap A_2^c \cap \bigcup_{i=3}^{r_n+1-j} A_i)$  for each  $j = 1, \dots, r_n - 2$ , which completes the proof by (4.9).  $\square$

Theorem 1.1 is a direct consequence of Theorems 4.1, 4.2, and 4.6. For the proof of Corollary 1.3 we need the following result of Novak [33, Corollary 2.2].

**Lemma 4.7.** *Let  $\{\xi_j: j \geq 1\}$  be a strictly stationary and uniformly mixing sequence of random variables. If the sequence  $u_n$  is such that*

$$0 < \liminf_{n \rightarrow \infty} n \mathbb{P}(\xi_1 > u_n) \leq \limsup_{n \rightarrow \infty} n \mathbb{P}(\xi_1 > u_n) < \infty$$

and, for every  $j \geq 2$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\xi_j > u_n | \xi_1 > u_n) = 0,$$

then

$$\mathbb{P}(\max_{1 \leq j \leq n} \xi_j \leq u_n) - \exp(-n \mathbb{P}(\xi_1 > u_n)) \rightarrow 0.$$

*Proof of Corollary 1.3.* Let  $\varepsilon > 0$  and  $t > 0$ . Define  $u_n = \varepsilon b_{[n/t]}$ ,  $n \geq 1$ . Set  $\lambda = \Pi(\{x: |x| > \varepsilon\})$  and observe that  $n \mathbb{P}(|Z_1| > \varepsilon b_n) \rightarrow \lambda$  and  $n \mathbb{P}(|Z_1| > u_n) \rightarrow \lambda t$ , as  $n \rightarrow \infty$ . From (1.10) it follows that the sequences  $u_n$  and  $\{|Z_j|: j \geq 1\}$  satisfy all assumptions of Lemma 4.7. Hence,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\max_{1 \leq j \leq n} |Z_j| \leq u_n) = e^{-\lambda t},$$

and, consequently,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\max_{1 \leq j \leq nt} |Z_j| \leq \varepsilon b_n) = e^{-\lambda t}.$$

From Lemma 4.4 it follows that

$$\lim_{n \rightarrow \infty} \mathbb{P}(\max_{2 \leq j \leq nt} |Z_j| > \varepsilon b_n | |Z_1| > \varepsilon b_n) = 1 - e^{-\lambda t},$$

which implies condition (1.8), by Remark 4.3, and concludes the proof.  $\square$

**Lemma 4.8.** *Suppose that the maximal correlation coefficient  $\rho(n) = \rho(\mathcal{F}_1^1, \mathcal{F}_{n+1}^\infty)$  of the sequence  $\{Z_j: j \geq 1\}$  satisfies  $\sum_j \rho(2^j) < \infty$ . If  $Z_1$  is regularly varying with index  $\alpha \in [1, 2)$  then condition (1.6) holds for all  $\delta > 0$ .*



*Proof.* First observe that it suffices to show that for every  $\delta > 0$  there exists a constant  $C > 0$  such that for any  $\varepsilon > 0$  and  $n \geq 1$

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \left| \sum_{j=1}^k (Z_j I(|Z_j| \leq \varepsilon b_n) - \mathbb{E}(Z_j I(|Z_j| \leq \varepsilon b_n))) \right| \geq \delta b_n\right) \leq \frac{Cn}{b_n^2} \mathbb{E}(|Z_1|^2 I(|Z_1| \leq \varepsilon b_n)),$$

since  $|Z_1|$  is regularly varying with index  $\alpha < 2$  and  $nb_n^{-2} \mathbb{E}(|Z_1|^2 I(|Z_1| \leq \varepsilon b_n)) \rightarrow \frac{\alpha}{2-\alpha} \varepsilon^{2-\alpha}$ , by Karamata's theorem. When  $d = 1$  then this type of bound follows from the  $L^2$ -maximal inequality from [46]. We now outline how to get a similar bound in the multivariate case.

Write  $Z_j(a) = Z_j I(|Z_j| \leq a) - \mathbb{E}(Z_j I(|Z_j| \leq a))$  for  $a > 0$ ,  $j \geq 1$ . For every  $a$  and  $j$  the random vector  $Z_j(a)$  has zero mean and is bounded. The proof of the  $L^2$ -maximal inequality for stationary sequences of random variables as given in [36, pp. 544-555] still works for random vectors and we can deduce the following

$$\mathbb{E}\left(\max_{1 \leq k \leq n} |S_k|^2\right) \leq 2n \left(2\|Z_1(a)\|_2 + 4 \sum_{j=0}^{\lfloor \log_2 n \rfloor} 2^{-j/2} \|\mathbb{E}(S_{2^j} | Z_1(a))\|_2\right)^2,$$

where  $S_k = \sum_{j=2}^{k+1} Z_j(a)$  for  $k \geq 1$  and  $\|Y\|_2 = \sqrt{\mathbb{E}\langle Y, Y \rangle}$ . By Chebyshev's inequality, it remains to show that there exists a constant  $C_1$  such that for any  $a > 0$

$$\sum_{j=0}^{\infty} 2^{-j/2} \|\mathbb{E}(S_{2^j} | Z_1(a))\|_2 \leq C_1 \|Z_1(a)\|_2,$$

since  $\|Z_1(a)\|_2^2 \leq 2\mathbb{E}(|Z_1|^2 I(|Z_1| \leq a))$ . We have

$$\|\mathbb{E}(S_{2n} | Z_1(a))\|_2 \leq \|\mathbb{E}(S_n | Z_1(a))\|_2 + \|\mathbb{E}(S_{2n} - S_n | Z_1(a))\|_2, \quad n \geq 1$$

and, by using [9, Theorem 4.2],

$$\begin{aligned} \|\mathbb{E}(S_{2n} - S_n | Z_1(a))\|_2^2 &= \mathbb{E}\langle S_{2n} - S_n, \mathbb{E}(S_{2n} - S_n | Z_1(a)) \rangle \\ &\leq \rho(n) \|S_{2n} - S_n\|_2 \|\mathbb{E}(S_{2n} - S_n | Z_1(a))\|_2, \end{aligned}$$

which implies

$$\|\mathbb{E}(S_{2n} - S_n | Z_1(a))\|_2 \leq \rho(n) \|S_{2n} - S_n\|_2 = \rho(n) \|S_n\|_2.$$

The proof of [35, Lemma 3.4] extends directly to random vectors. Thus, there exists a constant  $C_2$  such that for every  $a > 0$  and  $n \geq 1$  we have

$$\|S_n\|_2 \leq C_2 \sqrt{n} \|Z_1(a)\|_2,$$

which gives, as in [36, Lemma 1], the following estimate

$$\sum_{j=0}^{\infty} 2^{-j/2} \|\mathbb{E}(S_{2^j} | Z_1(a))\|_2 \leq 4C_2 \|Z_1(a)\|_2 \sum_{j=0}^{\infty} \rho(2^j),$$

and completes the proof. □

**Remark 4.9.** Note that since the sequence  $\rho(n)$ ,  $n \geq 1$ , is nonincreasing we have

$$\sum_{j=0}^{\infty} \rho(2^j) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{\rho(n)}{n} < \infty.$$

Corollary 1.4 follows from Theorems 1.1, 3.1, Lemma 4.8, and [19, Theorem 1] or [27, Theorem 1.1].

## 5. Proofs of Theorems 3.1 and 3.2

*Proof of Theorem 3.1.* Since the Lévy process has no fixed points of discontinuity, it follows that  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$  if and only if  $X_n \xrightarrow{d} X$  in  $\mathbb{D}([0, T], \mathbb{R}^d)$  for any  $T > 0$ .

First assume that (3.4) and (3.5) hold. For the proof of (3.3) we adapt the arguments of [15, Section 4] (see also [40, Section 7.2]). Let us define

$$X_{\varepsilon}^{(1)}(t) = \int_{[0, t] \times \{x: |x| > \varepsilon\}} x N(ds, dx)$$

for  $\varepsilon > 0$  and for every  $\varepsilon \in [0, 1)$

$$X_{\varepsilon}^{(2)}(t) = \int_{[0, t] \times \{x: \varepsilon < |x| \leq 1\}} x (N(ds, dx) - ds \Pi(dx)), \quad t \geq 0.$$

By the Lévy-Itô integral representation, we can rewrite  $X$  almost surely as

$$X(t) = X_1^{(1)}(t) + \lim_{\varepsilon \downarrow 0} X_{\varepsilon}^{(2)}(t). \quad (5.1)$$

The terms in (5.1) are independent and the convergence in the last term is a.s. and uniform in  $t$  on any bounded interval. Hence, we obtain

$$X_1^{(1)} + X_{\varepsilon}^{(2)} \xrightarrow{d} X \quad \text{in } \mathbb{D}([0, \infty), \mathbb{R}^d), \quad \text{as } \varepsilon \rightarrow 0. \quad (5.2)$$

From (3.4) and the continuous mapping theorem it follows that

$$X_{n, \varepsilon}^{(1)} \xrightarrow{d} X_{\varepsilon}^{(1)} \quad \text{in } \mathbb{D}([0, \infty), \mathbb{R}^d), \quad \text{as } n \rightarrow \infty,$$

for all  $\varepsilon \in (0, 1)$  such that  $\Pi(\{x : |x| = \varepsilon\}) = 0$ , where

$$X_{n, \varepsilon}^{(1)}(t) := \sum_{j \leq nt} X_{n, j} I(|X_{n, j}| > \varepsilon), \quad t \geq 0,$$

since the mapping  $R_{0, \varepsilon}: M_p((0, \infty) \times \overline{\mathbb{R}}_0^d) \rightarrow \mathbb{D}([0, \infty), \mathbb{R}^d)$  defined by

$$R_{0, \varepsilon}(m)(t) = \int_{[0, t] \times \{x: |x| > \varepsilon\}} x m(ds, dx), \quad m \in M_p((0, \infty) \times \overline{\mathbb{R}}_0^d),$$

is a.s. continuous with respect to the distribution of the Poisson point process  $N$  for all such  $\varepsilon \in (0, 1)$  (see e.g. [38, p.84] or [40, Section 7.2]). Hence, for

$$X_{n,\varepsilon}(t) = X_{n,\varepsilon}^{(1)}(t) - t \int_{\{x:\varepsilon < |x| \leq 1\}} x \Pi(dx), \quad t \geq 0,$$

we obtain  $X_{n,\varepsilon} \xrightarrow{d} X_1^{(1)} + X_\varepsilon^{(2)}$  in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$ . The function  $\varepsilon \mapsto \Pi(\{x : |x| > \varepsilon\})$  is monotonic. Therefore, we may chose a sequence  $\varepsilon_k \in (0, 1)$  such that  $\Pi(\{x : |x| = \varepsilon_k\}) = 0$  and  $\varepsilon_k \downarrow 0$ . From (5.2) and the converging together theorem [6, Theorem 4.2], it suffices to show that, for any  $\delta > 0$ ,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d_T(X_n, X_{n,\varepsilon_k}) \geq \delta) = 0.$$

This is a consequence of (3.5), since

$$X_n(t) - X_{n,\varepsilon}(t) = \sum_{j \leq nt} X_{n,j} I(|X_{n,j}| \leq \varepsilon) - tc_n + t \int_{\{x:\varepsilon < |x| \leq 1\}} x \Pi(dx)$$

and the Skorohod metric  $d_T$  on  $\mathbb{D}([0, T], \mathbb{R}^d)$  is bounded above by the uniform metric on  $\mathbb{D}([0, T], \mathbb{R}^d)$ .

Now assume that (3.3) holds. To prove (3.4) it suffices to show that for every  $f \in C_K^+([0, \infty) \times \overline{\mathbb{R}}_0^d)$  we have

$$\mathbb{E}(e^{-N_n(f)}) \rightarrow \mathbb{E}(e^{-N(f)}).$$

Let  $U(X) = \{r > 0 : \mathbb{P}(|\Delta X(t)| = r \text{ for some } t > 0) > 0\}$ . The set  $U(X)$  is at most countable [22, Lemma VI.3.12]. Let  $\varepsilon \notin U(X)$  and  $E_{T,\varepsilon} = [0, T] \times \{x : |x| > \varepsilon\}$  for  $T > 0$ . Define the mapping  $R_{1,\varepsilon} : \mathbb{D}([0, \infty), \mathbb{R}^d) \rightarrow M_p(E_{T,\varepsilon})$  by

$$R_{1,\varepsilon}\psi = \sum_{\{t \leq T : |\Delta\psi(t)| > \varepsilon\}} \epsilon_{(t, \Delta\psi(t))}.$$

Since for every  $\psi \in \mathbb{D}([0, \infty), \mathbb{R}^d)$  the set  $\{t \leq T : |\Delta\psi(t)| > \varepsilon\}$  is finite, the mapping  $R_{1,\varepsilon}$  is well defined. Moreover,  $R_{1,\varepsilon}$  is continuous at all  $\psi$  such that  $\varepsilon \notin \{r > 0 : |\Delta\psi(t)| = r \text{ for some } t > 0\}$  (see e.g. [22, Section 6.2]). Hence, the mapping  $R_{1,\varepsilon}$  is almost surely continuous with respect to the distribution of  $X$ . From the continuous mapping theorem it follows that

$$R_{1,\varepsilon}X_n \xrightarrow{d} R_{1,\varepsilon}X \quad \text{in } M_p(E_{T,\varepsilon}).$$

Thus

$$\mathbb{E}(e^{-R_{1,\varepsilon}X_n(f)}) \rightarrow \mathbb{E}(e^{-R_{1,\varepsilon}X(f)}) \quad \text{for all } f \in C_K^+(E_{T,\varepsilon}).$$

Observe that we have  $\Delta X_n(t) \neq 0$  if and only if  $t = j/n$  for some  $j$ . Since for every  $f \in C_K^+([0, \infty) \times \overline{\mathbb{R}}_0^d)$  we can find  $T > 0$  and  $\varepsilon > 0$  such that the support of  $f$  is contained in  $E_{T,\varepsilon}$ , we obtain

$$\mathbb{E}(e^{-N_n(f)}) = \mathbb{E}(e^{-R_{1,\varepsilon}X_n(f)}) \rightarrow \mathbb{E}(e^{-R_{1,\varepsilon}X(f)}) = \mathbb{E}(e^{-N(f)}),$$

by the definition of  $N_n$  in (3.4) and that of  $N$  in (3.2), which completes the proof of (3.4).

To prove (3.5), we first show that

$$X_n - \sum_{s \leq \cdot} \Delta X_n(s) I(|\Delta X_n(s)| > \varepsilon) \xrightarrow{d} X - \sum_{s \leq \cdot} \Delta X(s) I(|\Delta X(s)| > \varepsilon) \quad \text{in } \mathbb{D}([0, \infty), \mathbb{R}^d)$$

for all  $\varepsilon > 0$  such that  $\Pi(\{x : |x| = \varepsilon\}) = 0$ . Define the mapping  $R_{2,\varepsilon} : \mathbb{D}([0, \infty), \mathbb{R}^d) \rightarrow \mathbb{D}([0, \infty), \mathbb{R}^d)$  by

$$R_{2,\varepsilon}\psi(t) = \psi(t) - \sum_{s \leq t} \Delta\psi(s) I(|\Delta\psi(s)| > \varepsilon), \quad t \geq 0.$$

By [22, Proposition VI.2.7],  $R_{2,\varepsilon}$  is continuous at  $\psi$  if  $\varepsilon \notin \{r : |\Delta\psi(t)| = r \text{ for some } t > 0\}$ . Observe that  $\{\varepsilon > 0 : \Pi(\{x : |x| = \varepsilon\}) = 0\} \subseteq \mathbb{R}_+ \setminus U(X)$ , thus the claim follows from the continuous mapping theorem. For  $\varepsilon < 1$  and  $\psi \in \mathbb{D}([0, \infty), \mathbb{R}^d)$  define

$$R_\varepsilon\psi(t) := R_{2,\varepsilon}\psi(t) + t \int_{\{x : \varepsilon < |x| \leq 1\}} x \Pi(dx).$$

We have

$$R_\varepsilon X_n(t) = \sum_{j \leq nt} X_{n,j} I(|X_{n,j}| \leq \varepsilon) - t(c_n - \int_{\{x : \varepsilon < |x| \leq 1\}} x \Pi(dx))$$

and

$$R_\varepsilon X(t) = X(t) - X_1^{(1)}(t) - X_\varepsilon^{(2)}(t), \quad t \geq 0.$$

The set  $F_\delta = \{\psi \in \mathbb{D}([0, \infty), \mathbb{R}^d) : \sup_{0 \leq t \leq T} |\psi(t)| \geq \delta\}$  is closed in  $\mathbb{D}([0, \infty), \mathbb{R}^d)$ . Since  $R_\varepsilon X_n \xrightarrow{d} R_\varepsilon X$ , we obtain

$$\limsup_{n \rightarrow \infty} \mathbb{P}(R_\varepsilon X_n \in F_\delta) \leq \mathbb{P}(R_\varepsilon X \in F_\delta),$$

by Portmanteau's theorem. From (5.1) it follows that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(R_\varepsilon X \in F_\delta) = 0$$

which completes the proof of (3.5). □

*Proof of Theorem 3.2.* Let  $\varepsilon \in (0, 1)$  be such that  $\Pi(\{x : |x| = \varepsilon\}) = 0$  and let  $f_\varepsilon(x) = xI(|x| > \varepsilon)$ . Since  $\Pi(1 \wedge |f_\varepsilon|) < \infty$ , the random vector  $N'(f_\varepsilon)$  has the characteristic function of the form (see e.g. [26, Lemma 12.2])

$$\mathbb{E}(e^{i\langle u, N'(f_\varepsilon) \rangle}) = \exp\left(\int (e^{i\langle u, x \rangle} - 1) I(|x| > \varepsilon) \Pi(dx)\right).$$

From (3.6) and the continuous mapping theorem it follows that

$$N'_n(f_\varepsilon) \xrightarrow{d} N'(f_\varepsilon) \quad \text{in } \mathbb{R}^d.$$

With the notation as in the proof of Theorem 3.1, observe that  $X_1^{(1)} + X_\varepsilon^{(2)}$  is a Lévy process such that

$$\mathbb{E}(e^{i\langle u, X_1^{(1)}(1) + X_\varepsilon^{(2)}(1) \rangle}) = \exp\left(\int (e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle I(\varepsilon < |x| \leq 1)) \Pi(dx)\right)$$

and, by (5.2),

$$X_1^{(1)}(1) + X_\varepsilon^{(2)}(1) \xrightarrow{d} X(1) \quad \text{in } \mathbb{R}^d.$$

Since  $N'(f_\varepsilon) - \int_{\{x: \varepsilon < |x| \leq 1\}} x \Pi(dx)$  has the same distribution as  $X_1^{(1)}(1) + X_\varepsilon^{(2)}(1)$ , the result follows from (3.7) and [6, Theorem 4.2].  $\square$

## Acknowledgments

This work was supported by Polish MNiSW grant N N201 0211 33 and by EPSRC grant EP/F031807/1 *Anomalous Diffusion in Deterministic Systems*. Parts of this work were completed while the author was visiting the University of Surrey, Guildford, United Kingdom, and the hospitality of the Department of Mathematics is gratefully acknowledged. We thank the referees for constructive remarks which led to an improved presentation of the paper.

## References

- [1] J. Aaronson, Random  $f$ -expansions, *Ann. Probab.* 14 (1986) 1037–1057.
- [2] J. Aaronson, H. Nakada, On the mixing coefficients of piecewise monotonic maps, *Israel J. Math.* 148 (2005) 1–10.
- [3] F. Avram, M. S. Taqqu, Weak convergence of sums of moving averages in the  $\alpha$ -stable domain of attraction, *Ann. Probab.* 20 (1992) 483–503.
- [4] K. Bartkiewicz, A. Jakubowski, T. Mikosch, O. Wintenberger, Infinite variance stable limits for sums of dependent random variables, URL <http://arxiv.org/abs/0906.2717>.
- [5] B. Basrak, D. Krizmanic, J. Segers, A functional limit theorem for partial sums of dependent random variables with infinite variance, URL <http://arxiv.org/abs/1001.1345>.
- [6] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968.
- [7] R. C. Bradley, A central limit theorem for stationary  $\rho$ -mixing sequences with infinite variance, *Ann. Probab.* 16 (1988) 313–332.
- [8] R. C. Bradley, Basic properties of strong mixing conditions. A survey and some open questions, *Probab. Surv.* 2 (2005) 107–144.
- [9] R. C. Bradley, W. Bryc, Multilinear forms and measures of dependence between random variables, *J. Multivariate Anal.* 16 (1985) 335–367.
- [10] R. Davis, S. Resnick, Limit theory for moving averages of random variables with regularly varying tail probabilities, *Ann. Probab.* 13 (1985) 179–195.
- [11] R. Davis, S. Resnick, Extremes of moving averages of random variables from the domain of attraction of the double exponential distribution, *Stochastic Process. Appl.* 30 (1988) 41–68.
- [12] R. A. Davis, Stable limits for partial sums of dependent random variables, *Ann. Probab.* 11 (1983) 262–269.
- [13] R. A. Davis, T. Hsing, Point process and partial sum convergence for weakly dependent random variables with infinite variance, *Ann. Probab.* 23 (1995) 879–917.
- [14] R. A. Davis, T. Mikosch, The sample autocorrelations of heavy-tailed processes with applications to ARCH, *Ann. Statist.* 26 (1998) 2049–2080.

- [15] R. Durrett, S. I. Resnick, Functional limit theorems for dependent variables, *Ann. Probab.* 6 (1978) 829–846.
- [16] P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling extremal events, Applications of Mathematics* (New York), vol. 33, Springer-Verlag, Berlin, 1997.
- [17] N. Haydn, Y. Lacroix, S. Vaienti, Hitting and return times in ergodic dynamical systems, *Ann. Probab.* 33 (2005) 2043–2050.
- [18] T. Hsing, On the characterization of certain point processes, *Stochastic Process. Appl.* 26 (1987) 297–316.
- [19] W. N. Hudson, H. G. Tucker, J. A. Veeh, Limit distributions of sums of  $m$ -dependent Bernoulli random variables, *Probab. Theory Related Fields* 82 (1989) 9–17.
- [20] I. A. Ibragimov, Some limit theorems for stationary processes, *Teor. Veroyatnost. i Primenen.* 7 (1962) 361–392.
- [21] M. Iosifescu, C. Kraaikamp, *Metrical theory of continued fractions, Mathematics and its Applications*, vol. 547, Kluwer Academic Publishers, Dordrecht, 2002.
- [22] J. Jacod, A. N. Shiryaev, *Limit theorems for stochastic processes, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 288, 2nd ed., Springer-Verlag, Berlin, 2003.
- [23] A. Jakubowski, M. Kobus,  $\alpha$ -stable limit theorems for sums of dependent random vectors, *J. Multivariate Anal.* 29 (1989) 219–251.
- [24] O. Kallenberg, *Random measures, Akademie-Verlag, Berlin, 1975.*
- [25] O. Kallenberg, Improved criteria for distributional convergence of point processes, *Stochastic Process. Appl.* 64 (1996) 93–102.
- [26] O. Kallenberg, *Foundations of modern probability, Probability and its Applications* (New York), 2nd ed., Springer-Verlag, New York, 2002.
- [27] M. Kobus, Generalized Poisson distributions as limits of sums for arrays of dependent random vectors, *J. Multivariate Anal.* 52 (1995) 199–244.
- [28] E. G. Kounias, T.-S. Weng, An inequality and almost sure convergence, *Ann. Math. Statist.* 40 (1969) 1091–1093.
- [29] M. R. Leadbetter, H. Rootzén, Extremal theory for stochastic processes, *Ann. Probab.* 16 (1988) 431–478.
- [30] I. Melbourne, M. Nicol, A vector-valued almost sure invariance principle for hyperbolic dynamical systems, *Ann. Probab.* 37 (2009) 478–505.
- [31] F. Merlevède, M. Peligrad, S. Utev, Recent advances in invariance principles for stationary sequences, *Probab. Surv.* 3 (2006) 1–36 (electronic).
- [32] T. Mori, Limit distributions of two-dimensional point processes generated by strong-mixing sequences, *Yokohama Math. J.* 25 (1977) 155–168.
- [33] S. Y. Novak, Extreme values in stationary sequences, *Siberian Adv. Math.* 6 (1996) 68–80.
- [34] G. L. O’Brien, Extreme values for stationary and Markov sequences, *Ann. Probab.* 15 (1987) 281–291.
- [35] M. Peligrad, Invariance principles for mixing sequences of random variables, *Ann. Probab.* 10 (1982) 968–981.
- [36] M. Peligrad, S. Utev, W. B. Wu, A maximal  $L_p$ -inequality for stationary sequences and its applications, *Proc. Amer. Math. Soc.* 135 (2007) 541–550.
- [37] W. Philipp, Weak and  $L^p$ -invariance principles for sums of  $B$ -valued random variables, *Ann. Probab.* 8 (1980) 68–82. Corrigendum: *Ann. Probab.* 14 (1986) 1095–1101.
- [38] S. I. Resnick, Point processes, regular variation and weak convergence, *Adv. in Appl. Probab.* 18 (1986) 66–138.
- [39] S. I. Resnick, Extreme values, regular variation, and point processes, *Applied Probability. A Series of the Applied Probability Trust*, vol. 4, Springer-Verlag, New York, 1987.
- [40] S. I. Resnick, *Heavy-tail phenomena, Springer Series in Operations Research and Financial Engineering*, Springer, New York, 2007.
- [41] E. L. Rvačeva, On domains of attraction of multi-dimensional distributions, in: *Select. Transl. Math.*

- Statist. and Probability, Vol. 2, American Mathematical Society, Providence, R.I., 1962, 183–205.
- [42] J. D. Samur, On the invariance principle for stationary  $\phi$ -mixing triangular arrays with infinitely divisible limits, *Probab. Theory Related Fields* 75 (1987) 245–259.
  - [43] J. D. Samur, On some limit theorems for continued fractions, *Trans. Amer. Math. Soc.* 316 (1989) 53–79.
  - [44] K.-I. Sato, Lévy processes and infinitely divisible distributions, *Cambridge Studies in Advanced Mathematics*, vol. 68, Cambridge University Press, Cambridge, 1999.
  - [45] Q. M. Shao, An invariance principle for stationary  $\rho$ -mixing sequences with infinite variance, *Chinese Ann. Math. Ser. B* 14 (1993) 27–42.
  - [46] Q. M. Shao, Maximal inequalities for partial sums of  $\rho$ -mixing sequences, *Ann. Probab.* 23 (1995) 948–965.
  - [47] A. V. Skorohod, Limit theorems for stochastic processes, *Teor. Veroyatnost. i Primenen.* 1 (1956) 289–319.
  - [48] A. V. Skorohod, Limit theorems for stochastic processes with independent increments., *Teor. Veroyatnost. i Primenen.* 2 (1957) 145–177.
  - [49] M. Tyran-Kamińska, Weak convergence to Lévy stable processes in dynamical systems, to appear in *Stochastics and Dynamics* (2010).
  - [50] W. Whitt, Some useful functions for functional limit theorems, *Math. Oper. Res.* 5 (1980) 67–85.
  - [51] R. Zweimüller, Stable limits for probability preserving maps with indifferent fixed points, *Stoch. Dyn.* 3 (2003) 83–99.