WEAK CONVERGENCE TO LÉVY STABLE PROCESSES IN DYNAMICAL SYSTEMS

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ABSTRACT. We study convergence of normalized ergodic sum processes to Lévy stable process in the Skorohod space with J_1-topology. Our necessary and sufficient conditions allow us to prove or disprove such convergence for specific examples.

1. INTRODUCTION

Let $T\colon Y \to Y$ be a measurable transformation on a probability space $(Y, \mathcal{B}, \nu)$ and let $h\colon Y \to \mathbb{R}$ be measurable. Under appropriate assumptions about the transformation T and the function h there exist sequences $b_n > 0$, c_n, and a non-degenerate random variable ζ such that the distributional limit holds

$$\frac{1}{b_n}\sum_{j=0}^{n-1} h\circ T^j - c_n \xrightarrow{d} \zeta \quad \text{in } \mathbb{R} \tag{1.1}$$

(the notation '$\xrightarrow{d}$ in $\mathbb{X}$' refers to weak convergence of distributions of given random elements with values in the space $\mathbb{X}$). The most studied case is the central limit theorem when ζ is Gaussian distributed (see [8, 24, 37] and the references therein). In particular, examples of dynamical systems which display convergence to stable laws have been given [3, 11, 12, 14, 40].

A stronger result than the limit theorem in (1.1) is its functional version, called a *functional limit theorem (FLT)* or *weak invariance principle (WIP)*. We define the processes $\{X_n(t)\colon t \geq 0\}$, $n \geq 1$, by

$$X_n(t) = \frac{1}{b_n}\sum_{j=0}^{[nt]-1} h\circ T^j - tc_n \quad \text{for } t \geq 0 \tag{1.2}$$

(where the sum from 0 to -1 is set to be equal to 0). Then the X_n are random elements with values in the Skorohod space $\mathbb{D}[0,\infty)$, i.e., the space of all functions ψ on $[0,\infty)$ that are right-continuous and have left-hand limits $\psi(t-)$ for every $t > 0$. We consider $\mathbb{D}[0,\infty)$ with the Skorohod J_1-topology: if $\psi_n, \psi \in \mathbb{D}[0,\infty)$ then ψ_n converges to ψ in the J_1-topology if and only if there exists a sequence $\{\lambda_n\} \subset \Lambda$ such that

$$\sup_s |\lambda_n(s) - s| \to 0 \quad \text{and} \quad \sup_{s\leq m} |\psi_n(\lambda_n(s)) - \psi(s)| \to 0$$

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for all $m \in \mathbb{N}$, where Λ is the family of strictly increasing, continuous mappings λ of $[0, \infty]$ onto itself such that $\lambda(0) = 0$ and $\lambda(\infty) = \infty$ (see e.g. [18, Section 6]).

The functional version of (1.1) takes the form of

$$X_n \xrightarrow{d} X \quad \text{in } \mathbb{D}[0, \infty), \tag{1.3}$$

where X has sample paths in $\mathbb{D}[0, \infty)$. In the case when the random variables $h \circ T^j$ are independent and identically distributed, (1.1) holds if and only if (1.3) holds [29, 36], where necessarily X is a Lévy α-stable process with $\alpha \in (0, 2]$. Recall that X is a Lévy α-stable process if $X(0) = 0$, X has stationary independent increments, and $X(1)$ has an α-stable distribution. If $\alpha = 2$ then X is a Brownian motion and has continuous sample paths; see [24, 37] for results when (1.3) holds in the context of dynamical systems. If $\alpha \in (0, 2)$ then the paths of X are purely discontinuous and proving or disproving (1.3) seems to be much harder if one tries the typical approach using tightness arguments and convergence of finite dimensional distributions. Instead, we make use of necessary and sufficient conditions from [38] for convergence to Lévy processes in $\mathbb{D}[0, \infty)$ with J_1-topology, which are based on point process techniques and have their origin in [9].

For $\alpha \in (0, 2)$ and $\beta \in [-1, 1]$, we will denote by $\Xi_{\alpha,\beta}$ a random variable with characteristic function given by

$$\mathbb{E}e^{iu\Xi_{\alpha,\beta}} = \begin{cases} \exp(-\sigma^\alpha |u|^\alpha (1 - i\beta \operatorname{sign}(u) \tan(\pi\alpha/2))), & \alpha \neq 1, \\ \exp(iu\beta(1-\gamma) - \sigma^\alpha |u| (1 + i\beta(2/\pi) \operatorname{sign}(u) \ln(u)), & \alpha = 1, \end{cases} \tag{1.4}$$

where γ is Euler's constant, i.e., the limit of $\sum_{j=1}^n 1/j - \log n$, and the scale constant σ^α is

$$\sigma^\alpha = \begin{cases} \frac{\Gamma(2-\alpha)}{1-\alpha} \cos(\alpha\pi/2), & \alpha \neq 1, \\ \pi/2, & \alpha = 1. \end{cases}$$

Any α-stable random variable can be represented as $b\Xi_{\alpha,\beta} + a$ for some $a, b \in \mathbb{R}$. The Lévy-Khintchine representation for $\Xi_{\alpha,\beta}$ takes the form

$$\mathbb{E}e^{iu\Xi_{\alpha,\beta}} = \exp\big[iua_\alpha + \int (e^{iux} - 1 - iux1_{[-1,1]}(x)\Pi_\alpha(dx)\big],$$

where

$$a_\alpha = \begin{cases} \beta\frac{\alpha}{1-\alpha}, & \alpha \neq 1, \\ 0, & \alpha = 1, \end{cases}$$

and Π_α is a Lévy measure given by

$$\Pi_\alpha(dx) = \alpha \left(p1_{(0,\infty)}(x) + (1-p)1_{(-\infty,0)}(x)\right) |x|^{-\alpha-1} dx, \quad p = \frac{1+\beta}{2}.$$

It is often convenient to denote by $I(A)$ the indicator function 1_A of the set A.

Let $X_{(\alpha)}$ be a Lévy α-stable process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with sample paths in $\mathbb{D}[0, \infty)$ and with $X_{(\alpha)}(1) \stackrel{d}{=} \Xi_{\alpha,\beta}$. The process of discontinuities $\Delta X_{(\alpha)}(t) := X_{(\alpha)}(t) - X_{(\alpha)}(t-)$, $t > 0$, determines Poisson random measures. For $B \in \mathcal{B}((0, \infty) \times (\mathbb{R} \setminus \{0\}))$, we define the random variable by

$$N_{(\alpha)}(B) := \#\{s > 0 \colon (s, \Delta X_{(\alpha)}(s)) \in B\}.$$

We have $\mathbb{P}(N_{(\alpha)}(B) < \infty) = 1$ if and only if $\text{Leb} \times \Pi_\alpha(B) < \infty$, where Leb denotes the Lebesgue measure. In that case, $N_{(\alpha)}(B)$ has a Poisson distribution with mean $\text{Leb} \times \Pi_\alpha(B)$ (see e.g. [33, Chapter 4]).

Let T be a measurable transformation on a probability space $(Y,\mathcal{B},\nu)$ and $h\colon Y\to\mathbb{R}$ be measurable. Let X_n, $n\ge 1$, be as in (1.2), where $b_n>0$, c_n are some constants. We define

$$N_n(B):=\#\Big\{j\ge 1:\Big(\frac{j}{n},\frac{h\circ T^{j-1}}{b_n}\Big)\in B\Big\},\quad n\ge 1,$$

for $B\in\mathcal{B}((0,\infty)\times(\mathbb{R}\setminus\{0\}))$ and we will write

$$N_n\xrightarrow{d}N_{(\alpha)}$$

if and only if $N_n(B)\xrightarrow{d}N_{(\alpha)}(B)$ in $\mathbb{R}$ for all $B\in\mathcal{B}((0,\infty)\times(\mathbb{R}\setminus\{0\}))$ with $\mathrm{Leb}\times\Pi_\alpha(B)<\infty$ and $\mathrm{Leb}\times\Pi_\alpha(\partial B)=0$, where ∂ denotes the boundary of a given set. Let h be such that $\nu(h\circ T^j\neq 0)=1$ for all $j\ge 0$ and let us observe that $\Delta X_n(s):=X_n(s)-X_n(s-)\neq 0$ if and only if $s=j/n$ and $h\circ T^{j-1}\neq 0$ for some $j\ge 1$, in which case we have $\Delta X_n(s)=h\circ T^{j-1}/b_n$ and

$$N_n(B)=\#\big\{s>0:(s,\Delta X_n(s))\in B\big\}.$$

Thus, N_n counts the number of discontinuities of the process X_n and the condition $N_n(B)\xrightarrow{d}N_{(\alpha)}(B)$ means that this number is asymptotically Poisson distributed.

Theorem 1.1. *Suppose that $\nu(h\circ T^j\neq 0)=1$ for all $j\ge 0$. Then $X_n\xrightarrow{d}X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ if and only if $N_n\xrightarrow{d}N_{(\alpha)}$ and*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu\Big(\sup_{0\le t\le m}\Big|\frac{1}{b_n}\sum_{j=0}^{[nt]-1}h\circ T^jI(|h\circ T^j|\le\varepsilon b_n)-t(c_n-c_\alpha(\varepsilon))\Big|\ge\delta\Big)=0 \tag{1.5}$$

for all $\delta>0$, $m>0$, where $c_\alpha(\varepsilon)=\varepsilon^{1-\alpha}\beta\alpha/(\alpha-1)$ for $\alpha\in(1,2)$, $c_1(\varepsilon)=-\beta\ln\varepsilon$, and $c_\alpha(\varepsilon)=0$ for $\alpha\in(0,1)$.

If the $h\circ T^j$ are independent and identically distributed then $N_n\xrightarrow{d}N_{(\alpha)}$ (see e.g. [29]) if and only if h is *regularly varying with index* $\alpha\in(0,2)$: there exists $p\in[0,1]$ such that

$$\lim_{x\to\infty}\frac{\nu(h>x)}{\nu(|h|>x)}=p\quad\text{and}\quad\lim_{x\to\infty}\frac{\nu(|h|>x)}{x^{-\alpha}L(x)}=1, \tag{1.6}$$

where L is a *slowly varying function at* ∞, i.e., $L(rx)/L(x)\to 1$ as $x\to\infty$ for every $r>0$. In that case, condition (1.5) also holds for all $\delta>0$, $m>0$, where b_n, c_n, are such that

$$\lim_{n\to\infty}n\nu(|h|>b_n)=1\quad\text{and}\quad c_n=\begin{cases}0, & 0<\alpha<1,\\ nb_n^{-1}\mathbb{E}_\nu(hI(|h|\le b_n)), & \alpha=1,\\ nb_n^{-1}\mathbb{E}_\nu(h), & 1<\alpha<2.\end{cases} \tag{1.7}$$

Note that h satisfying condition (1.6) is also called ([3, 11]) to be in the domain of attraction of a stable law with index α.

Under the additional assumptions that T is measure preserving and h is regularly varying we obtain the following result.

Theorem 1.2. *Let T be a measure preserving transformation on $(Y,\mathcal{B},\nu)$. Suppose that h is regularly varying with index $\alpha\in(0,2)$, the sequences b_n,c_n, are as in (1.7), and one of the following two conditions holds:*

(1) *$\alpha\in(0,1)$;*

(2) *$\alpha \in [1,2)$ and, for any $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu\Big(\max_{1\le k\le n}\Big|\frac{1}{b_n}\sum_{j=0}^{k-1}\big(h\circ T^j I(|h\circ T^j|\le \varepsilon b_n)-\mathbb{E}_\nu(hI(|h|\le\varepsilon b_n))\big)\Big|\ge\delta\Big)=0.$$

If $N_n \xrightarrow{d} N_{(\alpha)}$ then $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$.

The condition $N_n \xrightarrow{d} N_{(\alpha)}$ implies that $N_n((0,1]\times B) \xrightarrow{d} N_{(\alpha)}((0,1]\times B)$ for all $B \in \mathcal{B}(\mathbb{R}\setminus\{0\})$ with $\Pi_\alpha(B)<\infty$ and $\Pi_\alpha(\partial B)=0$, which we will denote by

$$N_n((0,1]\times\cdot) \xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot).$$

We have the following result for convergence to stable laws.

Theorem 1.3. *Let T be a measure preserving transformation on $(Y,\mathcal{B},\nu)$. Suppose that h is regularly varying with index $\alpha\in(0,2)$, the sequences b_n, c_n, are as in (1.7), and one of the following two conditions holds:*

(1) *$\alpha\in(0,1)$;*

(2) *$\alpha\in[1,2)$ and for any $\delta>0$, we have*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu\Big(\Big|\frac{1}{b_n}\sum_{j=0}^{n-1}\big(h\circ T^j I(|h\circ T^j|\le \varepsilon b_n)-\mathbb{E}_\nu(hI(|h|\le\varepsilon b_n))\big)\Big|\ge\delta\Big)=0.$$

If $N_n((0,1]\times\cdot) \xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot)$, then

$$\frac{1}{b_n}\sum_{j=0}^{n-1}h\circ T^j - c_n \xrightarrow{d} \Xi_{\alpha,\beta} \quad \text{in } \mathbb{R}. \tag{1.8}$$

Theorems 1.1–1.3 are proved in Section 2 using results from [38]. We now give one example when (1.8) holds but the convergence to the Lévy process $X_{(\alpha)}$ with $X_{(\alpha)}(1) \stackrel{d}{=} \Xi_{\alpha,\beta}$ in $\mathbb{D}[0,\infty)$ with J_1-topology fails; see also Example 2.1 for a similar conclusion.

Example 1.1. Consider the map $T_\gamma\colon[0,1]\to[0,1]$ given by

$$T_\gamma(y)=\begin{cases} y(1+2^\gamma y^\gamma), & 0\le y\le\frac12,\\ 2y-1, & \frac12<y\le 1,\end{cases}$$

where $\gamma\in(0,1)$. The transformation T_γ has a unique absolutely continuous invariant probability measure ν_γ. It is shown in [11] that if $\gamma>1/2$ and h is Hölder continuous with $h(0)\neq 0$ and $\mathbb{E}_{\nu_\gamma}(h)=0$, then for $\alpha=1/\gamma$ and $b_n=bn^{1/\alpha}$, where b is a positive constant, we have

$$\frac{1}{b_n}\sum_{j=0}^{n-1}h\circ T^j \xrightarrow{d} \Xi_{\alpha,\mathrm{sign}(h(0))} \quad \text{in } \mathbb{R}.$$

Since h is bounded and $b_n\to\infty$, there exists $\varepsilon>0$ such that $\sup_j|h\circ T^j|\le\varepsilon b_n$ for all n sufficiently large. Thus

$$\lim_{n\to\infty}\nu\big(N_n((0,1]\times B)=0\big)=1$$

for all $B\subset\mathbb{R}\setminus[-\varepsilon,\varepsilon]$, but

$$\mathbb{P}\big(N_{(\alpha)}((0,1]\times B)=0\big)=e^{-\Pi_\alpha(B)},$$

which is equal to 1 if and only if $\Pi_\alpha(B) = 0$. This shows that the condition $N_n((0,1]\times\cdot) \xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot)$ does not hold and that it is not necessary for (1.8). From Theorem 1.1 it also follows that the distributional limit theorem in this example has no functional version in the Skorohod space with J_1-topology.

The main difficulty in proving convergence to Lévy stable processes for specific examples is to show that $N_n \xrightarrow{d} N_{(\alpha)}$. Thus, in Section 3 we provide two sufficient conditions (Theorems 3.1 and 3.3) for $N_n \xrightarrow{d} N_{(\alpha)}$, which are expressed with the help of hitting times. These are our main tools in Section 4, where we show how Theorem 1.2 can be applied to particular examples of maps and functions. We hope that our approach can be improved to give more examples where there is convergence to Lévy stable processes in $\mathbb{D}[0,\infty)$ with J_1-topology. In Section 4.1 we consider exponentially continued fraction mixing sequences [1, 5], which extend the standard example of the Gauss continued fraction map for which distributional limit theorems were studied in [22] and their functional versions in [32]; examples of such sequences can also be constructed via Gibbs-Markov maps. Section 4.2 is devoted to weakly mixing piecewise monotonic maps of the interval which are uniformly expanding and satisfy Adler's and finite images conditions. Here we prove FLT when the function h is locally constant on the dynamical partition, which allows us to study distributional behavior of the digits of Japanese continued fractions [26]. We also provide a simple sufficient condition for $N_n \xrightarrow{d} N_{(\alpha)}$ when the function h is piecewise monotonic with finitely many branches (Theorem 4.4). We now give one example in this setting where we have convergence to Lévy stable processes.

Example 1.2. Consider the tent map $T(y) = 1 - 2|y|$, $y \in [-1,1]$, where ν is the normalized Lebesgue measure on $[-1,1]$. Let $h(y) = y^{-1/\alpha}$ for $y > 0$ and $h(y) = -h(-y)$ for $y < 0$. Then $b_n = n^{1/\alpha}$, $c_n = 0$, and $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ with $X_{(\alpha)}(1) \stackrel{d}{=} \Xi_{\alpha,0}$.

As an application of Theorem 1.3 we give a positive solution to a recent question of Sinai [35].

Example 1.3. Let T be the doubling map $T(y) = 2y \mod 1$ on $[0,1]$ preserving the Lebesgue measure. Consider the non-integrable function

$$h(y) = \frac{1}{y - y_0},$$

where $y_0 \in (0,1)$ has a finite dyadic expansion. Observe that h is regularly varying with index $\alpha = 1$, $p = 1/2$, and the sequences b_n, c_n, are of the form

$$b_n = 2n, \quad c_n = \frac{1}{2}\ln\frac{1-y_0}{y_0}.$$

We will show in Section 4.3 that Theorem 1.3 applies. Hence we obtain

$$\frac{1}{b_n}\sum_{j=0}^{n-1} h\circ T^j - c_n \xrightarrow{d} \Xi_{1,0} \quad \text{in } \mathbb{R}.$$

Consequently, for every integrable function h_1 we have

$$\frac{1}{n}\sum_{j=0}^{n-1}(h+h_1)\circ T^j \xrightarrow{d} \zeta_c \quad \text{in } \mathbb{R},$$

where ζ_c is the Cauchy distribution, whose density is

$$\frac{1}{(x-c)^2+\pi^2}, \quad \text{where } c = \ln\frac{1-y_0}{y_0} + \int_0^1 h_1(y)dy.$$

2. Necessary and sufficient conditions for FLT

We begin by introducing some background on point processes. We follow point process theory as presented in Kallenberg [19] and Resnick [30]. For our purposes, let E be either $\overline{\mathbb{R}}_0 = \overline{\mathbb{R}}\setminus\{0\}$ or $(0,\infty)\times\overline{\mathbb{R}}_0$, where $\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,\infty\}$. The topology on $\overline{\mathbb{R}}_0$ is chosen so that the Borel σ-algebras $\mathcal{B}(\overline{\mathbb{R}}_0)$ and $\mathcal{B}(\mathbb{R})$ coincide on $\mathbb{R}\setminus\{0\}$. Moreover, $B\subset\overline{\mathbb{R}}_0$ is relatively compact (or bounded) if and only if $B\cap\mathbb{R}$ is bounded away from zero in $\mathbb{R}$, i.e., $0\notin\overline{B\cap\mathbb{R}}$. The set of all Radon measures $M(E)$ on $\mathcal{B}(E)$, i.e., nonnegative Borel measures which are finite on relatively compact subsets of E, is a Polish space when considered with the topology of vague convergence. Recall that m_n *converges vaguely* to m

$$m_n \xrightarrow{v} m \quad \text{iff} \quad m_n(f)\to m(f) \quad \text{for all } f\in C_K^+(E),$$

where $m(f) = \int_E f(x)m(dx)$ and $C_K^+(E)$ is the space of nonnegative continuous functions on E with compact support. We have $m_n \xrightarrow{v} m$ if and only if $m_n(B)\to m(B)$ for all relatively compact B for which $m(\partial B)=0$. The set $M_p(E)$ of all integer-valued measures in $M(E)$, called point measures on E, is a closed subspace of $M(E)$. A point process N on E is an $M_p(E)$-valued random variable, defined on some probability space. Given a sequence of point processes N_n we have $N_n\xrightarrow{d} N$ in $M_p(E)$, by [19, Theorem 4.2], if and only if $\mathbb{E}[e^{-N_n(f)}]\to\mathbb{E}[e^{-N(f)}]$ for all $f\in C_K^+(E)$. A point process N is called a *Poisson process with mean measure* $\Pi\in M(E)$ if $N(B_1),\dots,N(B_l)$ are independent random variables for any disjoint sets $B_1,\dots,B_l\in\mathcal{B}(E)$ and $N(B)$ is a Poisson random variable with mean $\Pi(B)$ for $B\in\mathcal{B}(E)$ with $\Pi(B)<\infty$.

Proof of Theorem 1.1. We will apply [38, Theorem 3.1]. Let X be a Lévy process with characteristic function of $X(1)$ given by

$$\mathbb{E}e^{iuX(1)} = \exp\Big[\int(e^{iux}-1-iuxI(|x|\le 1))\Pi_\alpha(dx)\Big], \quad u\in\mathbb{R},$$

and let N be a Poisson point process on $(0,\infty)\times\overline{\mathbb{R}}_0$ with mean measure $\mathrm{Leb}\times\Pi_\alpha$, where we extend Π_α on $\mathcal{B}(\overline{\mathbb{R}}_0)$ by setting $\Pi_\alpha(\overline{\mathbb{R}}_0\setminus\mathbb{R})=0$. We define the processes $\{\widetilde{X}_n(t)\colon t\ge 0\}$, $n\ge 1$, by

$$\widetilde{X}_n(t) = \sum_{j\le nt} X_{n,j} - t\widetilde{c}_n, \quad t\ge 0, \quad \text{where } X_{n,j} = \frac{1}{b_n}h\circ T^{j-1}, \quad j\ge 1,$$

and $\widetilde{c}_n = c_n + a_\alpha$, $n\ge 1$. The corresponding point process $\widetilde{N}_n$ on $(0,\infty)\times\overline{\mathbb{R}}_0$ is given by

$$\widetilde{N}_n(B) := \#\big\{s>0 : (s,\widetilde{X}_n(s)-\widetilde{X}_n(s-))\in B\big\}, \quad B\in\mathcal{B}((0,\infty)\times\overline{\mathbb{R}}_0).$$

From [38, Theorem 3.1] it follows that $\widetilde{X}_n \xrightarrow{d} X$ in $\mathbb{D}[0,\infty)$ with J_1-topology if and only if $\widetilde{N}_n \xrightarrow{d} N$ in $M_p((0,\infty)\times\overline{\mathbb{R}}_0)$ and, for any $\delta>0$, $m>0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu\Big(\sup_{0\le t\le m}\Big|\sum_{j\le nt}X_{n,j}I(|X_{n,j}|\le\varepsilon)-t(\widetilde{c}_n-a(\varepsilon))\Big|\ge\delta\Big)=0, \tag{2.1}$$

where $a(\varepsilon)=\int_{\{x\colon \varepsilon<|x|\le 1\}}x\Pi_\alpha(dx)$.

First, we observe that for $\alpha\neq 1$ we have

$$a_\alpha=\begin{cases}\int_{\{x\colon |x|\le 1\}}x\Pi_\alpha(dx), & \alpha\in(0,1),\\ \int_{\{x\colon |x|>1\}}x\Pi_\alpha(dx), & \alpha\in(1,2).\end{cases}$$

Thus, if $\alpha\in[1,2)$ then $\widetilde{c}_n-a(\varepsilon)=c_n-c_\alpha(\varepsilon)$ and condition (2.1) is equivalent to (1.5). If $\alpha\in(0,1)$ then $a_\alpha-a(\varepsilon)=\beta\alpha\varepsilon^{1-\alpha}/(1-\alpha)\to 0$ as $\varepsilon\to 0$, which shows that (2.1) holds for all $\delta,m>0$ if and only if condition (1.5) holds for all $\delta,m>0$.

Since $X_n(t)-\widetilde{X}_n(t)=ta_\alpha$, $t\ge 0$, and $X(1)+a_\alpha\overset{d}{=}\Xi_{\alpha,\beta}$, we obtain $\widetilde{X}_n\xrightarrow{d}X$ in $\mathbb{D}[0,\infty)$ if and only if $X_n\xrightarrow{d}X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$. Hence, it remains to show that $\widetilde{N}_n\xrightarrow{d}N$ in $M_p((0,\infty)\times\overline{\mathbb{R}}_0)$ if and only if $N_n\xrightarrow{d}N_{(\alpha)}$. Since the measure $\mathrm{Leb}\times\Pi_\alpha$ is non-atomic, we have, by [20, Theorem 16.16], $\widetilde{N}_n\xrightarrow{d}N$ in $M_p((0,\infty)\times\overline{\mathbb{R}}_0)$ if and only if $\widetilde{N}_n(B)\xrightarrow{d}N(B)$ in $\mathbb{R}$ for all $B\in\mathcal{B}((0,\infty)\times\overline{\mathbb{R}}_0)$ with $\mathrm{Leb}\times\Pi_\alpha(B)<\infty$ and $\mathrm{Leb}\times\Pi_\alpha(\partial B)=0$. Note that

$$\widetilde{N}_n(B)=\#\{s>0\colon (s,\Delta X_n(s))\in B\}=N_n(B)$$

for all $B\in\mathcal{B}((0,\infty)\times(\mathbb{R}\setminus\{0\}))$. Moreover, $\nu(\widetilde{N}_n(B)=0)=1$ and $\mathbb{P}(N(B)=0)=1$ for all $B\in\mathcal{B}((0,\infty)\times\overline{\mathbb{R}}_0)\setminus\mathcal{B}((0,\infty)\times(\mathbb{R}\setminus\{0\}))$, which completes the proof. □

Remark 2.1. A closer look at the proof of Theorem 3.1 in [38] shows that $N_n\xrightarrow{d}N_{(\alpha)}$ and condition (1.5) imply $X_n\xrightarrow{d}X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ without the assumption that $\nu(h\circ T^j\neq 0)=1$ for all $j\ge 0$, which is needed only for the converse implication.

With the notation as in the Introduction we have the following.

Lemma 2.2. *Suppose that $N_n((0,1]\times\cdot)\xrightarrow{d}N_{(\alpha)}((0,1]\times\cdot)$.*

(1) *For every $x>0$ we have*

$$\lim_{n\to\infty}\nu\big(\max\{h,h\circ T,\dots,h\circ T^{n-1}\}\le xb_n\big)=e^{-\Pi_\alpha((x,\infty))}.$$

(2) *If*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu\Big(\Big|\frac{1}{b_n}\sum_{j=0}^{n-1}h\circ T^jI(|h\circ T^j|\le\varepsilon b_n)+c_\alpha(\varepsilon)-c_n\Big|\ge\delta\Big)=0 \tag{2.2}$$

for all $\delta>0$, then

$$\frac{1}{b_n}\sum_{j=0}^{n-1}h\circ T^j-c_n\xrightarrow{d}\Xi_{\alpha,\beta}\quad \text{in } \mathbb{R}.$$

Proof. To prove part (1) let $x>0$. Since $\Pi_\alpha((x,\infty))<\infty$ and $\Pi_\alpha(\{x\})=0$, we obtain

$$N_n((0,1]\times(x,\infty))\xrightarrow{d}N_{(\alpha)}((0,1]\times(x,\infty)),$$

where $N_{(\alpha)}((0,1]\times(x,\infty))$ has a Poisson distribution with mean $\Pi_\alpha((x,\infty))$. Hence,

$$\nu\big(N_n((0,1]\times(x,\infty))=0\big)\xrightarrow{d}\mathbb{P}\big(N_{(\alpha)}((0,1]\times(x,\infty))=0\big)=e^{-\Pi_\alpha((x,\infty))},$$

and the left hand-side is equal to $\nu(\max\{h,h\circ T,\dots,h\circ T^{n-1}\}\le xb_n)$, which completes the proof of part (1).

Part (2) follows from [38, Theorem 3.2] similarly to the proof of Theorem 1.1. □

We now provide one more example when (1.8) holds but $X_n \overset{d}{\not\to} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ with J_1-topology.

Example 2.1. Let $T\colon[0,1]\to[0,1]$ be the doubling map $T(y)=2y \mod 1$ on $[0,1]$. Consider the invariant measure $\nu=\text{Leb}$ and the function $h(y)=y^{-1/\alpha}$, $y\in(0,1]$, $\alpha\in(0,2)$. It is shown in [13] that there exists a sequence c_n such that

$$\frac{2^{1/\alpha}-1}{n^{1/\alpha}}\sum_{j=0}^{n-1}h\circ T^j-c_n\xrightarrow{d}\Xi_{\alpha,1}\quad\text{in }\mathbb{R}.$$

Let us suppose that $X_n\xrightarrow{d}X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$, where $b_n=n^{1/\alpha}/(2^{1/\alpha}-1)$. From Theorem 1.1 and Lemma 2.2 it follows that

$$\lim_{n\to\infty}\nu\big(\max\{h,\dots,h\circ T^{n-1}\}\le b_nx\big)=e^{-\Pi_\alpha((x,\infty))}=e^{-x^{-\alpha}},\quad x>0.$$

We now show that this is not true. By the result of [17], we have

$$\lim_{\varepsilon\to0}\nu\Big(\bigcup_{0\le j\le\frac{t}{\varepsilon}}T^{-j}([0,\varepsilon))\Big)=1-e^{-t/2}\quad\text{for }t>0,$$

which can be rewritten as

$$\lim_{\varepsilon\to0}\nu\big(\varepsilon\sigma_\varepsilon\le t\big)=1-e^{-t/2},$$

where $\sigma_\varepsilon(y)=\inf\{j\ge0\colon T^j(y)\in[0,\varepsilon)\}$. For $x>0$, we have

$$\nu\big(\max\{h,\dots,h\circ T^{n-1}\}\le b_nx\big)=1-\nu(\sigma_{\varepsilon_n}\le n-1),$$

where $\varepsilon_n:=(b_nx)^{-\alpha}\to0$. Hence, $\varepsilon_n(n-1)\to(2^{1/\alpha}-1)^\alpha x^{-\alpha}$ and, consequently,

$$\lim_{n\to\infty}\nu\big(\max\{h,\dots,h\circ T^{n-1}\}\le b_nx\big)=e^{-x^{-\alpha}(1-2^{-1/\alpha})^\alpha},\quad x>0.$$

We now turn to the proofs of Theorems 1.2 and 1.3. From (1.6) it follows that the sequence b_n satisfies

$$\lim_{n\to\infty}\frac{nL(b_n)}{b_n^\alpha}=1,$$

and for $x>0$ we have

$$\lim_{n\to\infty}n\nu(h>b_nx)=x^{-\alpha}p\quad\text{and}\quad\lim_{n\to\infty}n\nu(h<-b_nx)=x^{-\alpha}(1-p).\tag{2.3}$$

From Karamata's theorem (see e.g. [10]) we obtain the following asymptotic behavior of truncated moments:

$$\mathbb{E}_\nu(|h|I(|h|\le\varepsilon b_n))\sim\frac{\alpha}{1-\alpha}\varepsilon b_n\nu(|h|>\varepsilon b_n)\tag{2.4}$$

and

$$\mathbb{E}_\nu(h^2I(|h|\le\varepsilon b_n))\sim\frac{\alpha}{2-\alpha}(\varepsilon b_n)^2\nu(|h|>\varepsilon b_n).\tag{2.5}$$

Proof of Theorem 1.3. We apply Lemma 2.2. For $\alpha \in (0,1)$, we obtain, by the Markov inequality,

$$\nu\big(\big|\sum_{j=0}^{n-1} h\circ T^j I(|h\circ T^j| \le \varepsilon b_n)\big| \ge \delta b_n\big) \le \frac{n}{\delta b_n}\mathbb{E}_\nu(|h|I(|h|\le \varepsilon b_n)).$$

From (2.4) it follows that

$$\lim_{n\to\infty} nb_n^{-1}\mathbb{E}_\nu(|h|I(|h|\le\varepsilon b_n)) = \frac{\alpha}{1-\alpha}\varepsilon^{1-\alpha},$$

which shows that (2.2) holds in this case, since $1-\alpha>0$.

Let $\varepsilon\in(0,1)$. If $\alpha=1$ then $c_n = nb_n^{-1}\mathbb{E}_\nu(hI(|h|\le b_n))$ and

$$\lim_{n\to\infty} nb_n^{-1}\mathbb{E}_\nu(hI(\varepsilon b_n < |h|\le b_n)) = c_\alpha(\varepsilon),$$

if $\alpha\in(1,2)$ then $c_n = nb_n^{-1}\mathbb{E}_\nu(h)$ and

$$\lim_{n\to\infty} nb_n^{-1}\mathbb{E}_\nu(hI(|h|>\varepsilon b_n)) = c_\alpha(\varepsilon).$$

Consequently,

$$\lim_{n\to\infty}\big(nb_n^{-1}\mathbb{E}_\nu(hI(|h|\le\varepsilon b_n)) + c_\alpha(\varepsilon) - c_n\big) = 0,$$

which shows that condition (2) implies condition (2) in Lemma 2.2. □

Proof of Theorem 1.2. We apply Theorem 1.1. First suppose that $\alpha\in(0,1)$. By the maximal inequality from [21, Theorem 1], we obtain

$$\nu\big(\sup_{0\le t\le m}\big|\frac{1}{b_n}\sum_{j=0}^{[nt]-1} h\circ T^j I(|h\circ T^j|\le\varepsilon b_n)\big|\ge\delta\big) \le \frac{[nm]}{\delta b_n}\mathbb{E}_\nu(|h|I(|h|\le\varepsilon b_n)),$$

which shows, as above, that condition (1.5) holds for all $\delta>0$, $m>0$.

Now suppose that $\alpha\in[1,2)$. Observe that we have

$$\lim_{n\to\infty}\sup_{0\le t\le m}\big|[nt]b_n^{-1}\mathbb{E}_\nu(hI(|h|\le\varepsilon b_n)) + tc_\alpha(\varepsilon) - tc_n\big| = 0$$

for all $\varepsilon\in(0,1)$, $m>0$. Thus condition (1.5) holds if and only if

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\nu\big(\sup_{0\le t\le m}\big|\frac{1}{b_n}\sum_{j=0}^{[nt]-1}\big(h\circ T^j I(|h\circ T^j|\le\varepsilon b_n) - \mathbb{E}_\nu(hI(|h|\le\varepsilon b_n))\big)\big|\ge\delta\big) = 0$$

for all $\delta>0$, $m>0$, which is implied by condition (2), since ν is invariant for T. □

To conclude this section we use the notion of transfer operator to provide sufficient conditions for condition (2) of Theorem 1.2 and, respectively, Theorem 1.3. Given a measurable transformation T on $(Y,\mathcal{B})$ and a σ-finite measure μ on $(Y,\mathcal{B})$ with respect to which T is *nonsingular*, i.e., $\mu(T^{-1}(A))=0$ for all $A\in\mathcal{B}$ with $\mu(A)=0$, the *Perron-Frobenius* (or *transfer*) *operator* $P\colon L^1(Y,\mu)\to L^1(Y,\mu)$ is defined by the relation

$$\int_A Pf(y)\mu(dy) = \int_{T^{-1}(A)} f(y)\mu(dy) \qquad \text{for all } A\in\mathcal{B}.$$

This in turn gives rise to different operators for different underlying measures on $\mathcal{B}$. Thus if ν is invariant for T, then T is nonsingular and the transfer operator $\mathcal{P}_T\colon L^1(Y,\nu)\to L^1(Y,\nu)$ is well defined. Here we write $\mathcal{P}_T$ to emphasize that the underlying measure ν is invariant under T. The following is a consequence of [23, Proposition 1].

Proposition 2.3. *If T is a non-invertible measure preserving transformation on the probability space $(Y,\mathcal{B},\nu)$, then*

$$\big\|\max_{1\le k\le n}\big|\sum_{j=0}^{k-1} f\circ T^j\big|\,\big\|_2\le\sqrt{n}\big(3\|f-\mathcal{P}_T f\circ T\|_2+4\sqrt{2}\sum_{j=0}^{\log_2 n}2^{-j/2}\big\|\sum_{k=1}^{2^j}\mathcal{P}_T^k f\big\|_2\big)$$

for all $f\in L^2(Y,\nu)$ and $n\ge 1$.

Let us define

$$h_x=hI(|h|\le x)-\mathbb{E}_\nu(hI(|h|\le x)),\quad x>0.$$

Corollary 2.4. *Let T be a non-invertible measure preserving transformation on $(Y,\mathcal{B},\nu)$. Suppose that h is regularly varying with index $\alpha\in[1,2)$ and the sequence b_n is such that $n\nu(|h|>b_n)\to 1$ as $n\to\infty$. If*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\sqrt{n}}{b_n}\sum_{j=0}^{\log_2 n}2^{-j/2}\big\|\sum_{k=1}^{2^j}\mathcal{P}_T^k h_{\varepsilon b_n}\big\|_2=0, \tag{2.6}$$

then condition (2) of Theorem 1.2 holds.

Proof. From Proposition 2.3 it follows that

$$\big\|\max_{1\le k\le n}\big|\sum_{j=0}^{k-1} h_x\circ T^j\big|\,\big\|_2\le 6\sqrt{n}\|h_x\|_2+4\sqrt{2}\sqrt{n}\sum_{j=0}^{\log_2 n}2^{-j/2}\big\|\sum_{k=1}^{2^j}\mathcal{P}_T^k h_x\big\|_2$$

for all $x>0$ and $n\ge 1$. We have

$$nb_n^{-2}\|h_{\varepsilon b_n}\|_2^2\le nb_n^{-2}\mathbb{E}_\nu(h^2I(|h|\le\varepsilon b_n))$$

and, by (2.5), we obtain

$$\lim_{n\to\infty}nb_n^{-2}\mathbb{E}_\nu(h^2I(|h|\le\varepsilon b_n))=\frac{\alpha}{2-\alpha}\varepsilon^{2-\alpha},\quad\varepsilon>0.$$

Hence the result follows, since $2-\alpha>0$. □

Corollary 2.5. *Let T be a non-invertible measure preserving transformation on $(Y,\mathcal{B},\nu)$. Suppose that h is regularly varying with index $\alpha\in[1,2)$ and the sequence b_n is such that $n\nu(|h|>b_n)\to 1$ as $n\to\infty$. If*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{b_n^2}\sum_{k=1}^{n-1}(n-k)\int h_{\varepsilon b_n}\mathcal{P}_T^k h_{\varepsilon b_n}d\nu=0, \tag{2.7}$$

then condition (2) of Theorem 1.3 holds.

Proof. Making use of the identity

$$\mathbb{E}_\nu(\sum_{j=0}^{n-1}h_{\varepsilon b_n}\circ T^j)^2=n\mathbb{E}_\nu(h_{\varepsilon b_n}^2)+2\sum_{j=1}^{n-1}(n-j)\mathbb{E}_\nu(h_{\varepsilon b_n}h_{\varepsilon b_n}\circ T^j),$$

the result follows from the Markov inequality and (2.5). □

3. Hitting times and Poisson laws

In this section we provide sufficient conditions for $N_n \xrightarrow{d} N_{(\alpha)}$ in terms of hitting time statistics. We assume throughout that T is a measure preserving transformation on a probability space $(Y, \mathcal{B}, \nu)$. For any set $U \in \mathcal{B}$ with $\nu(U) > 0$ we define the *return/hitting time* function τ_U by

$$\tau_U(y) = \inf\{k \geq 1 \colon T^k(y) \in U\},$$

where $\inf \emptyset := \infty$. When restricted to U, τ_U is the return time function of U, while it is usually called the hitting time when considered as a function on the whole Y. If ν is ergodic then τ_U is finite a.e. If $U_n \in \mathcal{B}$ are sets of positive measure such as shrinking balls or cylinders with $\nu(U_n) \to 0$ as $n \to \infty$, then it is known that $\nu(U_n)\tau_{U_n}$ may converge in distribution to an exponential distribution (see [7, 15, 16]). The next result also provides examples of such asymptotically rare events.

Theorem 3.1. *Let h be regularly varying with index α and let the sequence b_n be such that $n\nu(|h| > b_n) \to 1$ as $n \to \infty$.*

(1) *We have $N_n((0,1] \times \cdot) \xrightarrow{d} N_{(\alpha)}((0,1] \times \cdot)$ if and only if*

$$\lim_{n\to\infty} \nu\big(\tau_{h^{-1}(b_n J)} > n\big) = e^{-\Pi_\alpha(J)} \tag{3.1}$$

for all sets $J \in \mathcal{J}$, where $\mathcal{J}$ is the family of all finite unions of intervals of the form $(x, y]$, where $-\infty \leq x < y \leq \infty$ and $0 \notin [x, y]$.

(2) *If*

$$\lim_{n\to\infty} \sup_{B\in\mathcal{B}} \big|\nu\big(\{\tau_{h^{-1}(b_n J)} > [ns]\} \cap T^{-[nt]}(B)\big) - e^{-s\Pi_\alpha(J)}\nu(B)\big| = 0 \tag{3.2}$$

for all $J \in \mathcal{J}$ and $0 \leq s < t$, then $N_n \xrightarrow{d} N_{(\alpha)}$.

(3) *If $N_n \xrightarrow{d} N_{(\alpha)}$ then*

$$\nu(h^{-1}(b_n J))\tau_{h^{-1}(b_n J)} \xrightarrow{d} \mathrm{Exp}(1)$$

for all $J \in \mathcal{J}$, where Exp(1) *is an exponentially distributed random variable with mean* 1.

Proof. We first prove part (2). Let $\mathcal{R}$ be the class of all finite unions of disjoint rectangles of the form $(s,t] \times (x,y]$ where $0 \leq s < t$ and $0 \notin [x,y]$. By Kallenberg's theorem [19, Theorem 4.7] (see also [30, Proposition 3.22]) we have $N_n \xrightarrow{d} N_{(\alpha)}$ if the following holds: for any $R \in \mathcal{R}$

$$\lim_{n\to\infty} \nu\big(N_n(R) = 0\big) = \mathbb{P}\big(N_{(\alpha)}(R) = 0\big) \tag{3.3}$$

and

$$\lim_{n\to\infty} \mathbb{E}_\nu N_n(R) = \mathbb{E}N_{(\alpha)}(R). \tag{3.4}$$

Any set $R \in \mathcal{R}$ can be rewritten as

$$R = \bigcup_{i=1}^{k} (s_i, t_i] \times J_i, \tag{3.5}$$

where $0 \leq s_1 < t_1 < \ldots < s_k < t_k$, and $J_i \in \mathcal{J}$, $i = 1, \ldots, k$, $k \geq 1$. We have

$$\mathbb{E}N_{(\alpha)}(R) = \sum_{i=1}^{k} (\mathrm{Leb} \times \Pi_\alpha)((s_i, t_i] \times J_i) = \sum_{i=1}^{k} (t_i - s_i)\Pi_\alpha(J_i)$$

and

$$\mathbb{E}_\nu N_n(R) = \sum_{i=1}^{k} \int N_n((s_i, t_i] \times J_i) d\nu = \sum_{i=1}^{k} ([nt_i] - [ns_i])\nu(h^{-1}(b_n J_i)).$$

From (2.3) it follows that $n\nu(h^{-1}(b_n J_i)) \to \Pi_\alpha(J_i)$, as $n \to \infty$, for each i, which completes the proof of (3.4). To prove (3.3), we use induction on the number of sets in the union (3.5). Let $R = (s_1, t_1] \times J_1$, $0 \leq s_1 < t_1$, $J_1 \in \mathcal{J}$. Define $U_n = h^{-1}(b_n J_1)$, $n \geq 1$. We have

$$\begin{aligned} \nu\big(N_n(R) = 0\big) &= \nu\big(\{y : T^j(y) \notin U_n, ns_1 < j + 1 \leq nt_1\}\big) \\ &= \nu\big(\{y : T^j(y) \notin U_n, 0 \leq j \leq [nt_1] - [ns_1] - 1\}\big). \end{aligned}$$

Hence,

$$\big|\nu\big(N_n(R) = 0\big) - \nu\big(\tau_{U_n} > [n(t_1 - s_1)]\big)\big| \leq 2\nu(U_n) \to 0, \tag{3.6}$$

which proves the claim for such sets, since $\nu(\tau_{U_n} > [ns]) \to e^{-s\Pi_\alpha(J_1)}$ for $s = t_1 - s_1$ by (3.2). Now let $0 \leq s_1 < t_1 < \ldots < s_k < t_k$ and $J_i \in \mathcal{J}$ for $i = 1, \ldots, k$. Observe that

$$\Big|\nu\big(N_n(\bigcup_{i=1}^{k}(s_i, t_i] \times J_i) = 0\big) - \nu\big(N_n(\bigcup_{i=1}^{k}(s_i', t_i'] \times J_i) = 0\big)\Big| \leq 2\sum_{i=j}^{k} \nu(h^{-1}(b_n J_i)) \to 0,$$

where $s_i' = s_i - s_1$, $t_i' = t_i - s_1$, $i = 1, \ldots, k$, $k \geq 2$. Thus, we can assume that $s_1 = 0$. Write $R_1 = (0, t_1] \times J_1$,

$$R_2 = \bigcup_{i=2}^{k} (s_i, t_i] \times J_i, \quad \text{and} \quad R_2' = \bigcup_{i=2}^{k} (s_i - s_2, t_i - s_2] \times J_i.$$

Since

$$\Big|\nu\big(N_n(R_1 \cup R_2) = 0\big) - \nu\big(\{\tau_{U_n} > [nt_1]\} \cap T^{-[ns_2]}(\{N_n(R_2') = 0\})\big)\Big| \to 0,$$

it follows from (3.2) that

$$\nu\big(N_n(R) = 0\big) - e^{-t_1\Pi_\alpha(J_1)}\nu\big(N_n(R_2') = 0\big) \to 0,$$

which, by the induction hypothesis, implies

$$\nu\big(N_n(R) = 0\big) \to e^{-t_1\Pi_\alpha(J_1)}\mathbb{P}\big(N_{(\alpha)}(R_2') = 0\big) = \mathbb{P}\big(N_{(\alpha)}(R) = 0\big).$$

For the proof of part (1) note that, by (3.6), we have

$$\lim_{n\to\infty} \nu\big(N_n((0, 1] \times J) = 0\big) = \mathbb{P}\big(N_{(\alpha)}((0, 1] \times J) = 0\big) \tag{3.7}$$

if and only if (3.1) holds for $J \in \mathcal{J}$. By Kallenberg's theorem, this and (3.4) imply $N_n((0, 1] \times \cdot) \xrightarrow{d} N_{(\alpha)}((0, 1] \times \cdot)$. Conversely, $N_n((0, 1] \times \cdot) \xrightarrow{d} N_{(\alpha)}((0, 1] \times \cdot)$ implies (3.7) for every $\mathcal{J} \in \mathcal{J}$.

Finally, to prove part (3) let $J \in \mathcal{J}$ and $U_n = h^{-1}(b_n J)$, $n \geq 1$. We have to show that for all $s > 0$

$$\lim_{n\to\infty} \nu\big(\nu(U_n)\tau_{U_n} > s\big) = e^{-s}.$$

Since $\Pi_\alpha(J) < \infty$ and $\mathrm{Leb} \times \Pi_\alpha(\partial((0,t] \times J)) = 0$, where $t = s/\Pi_\alpha(J)$, we obtain

$$N_n((0,t] \times J) \xrightarrow{d} N_{(\alpha)}((0,t] \times J).$$

Hence,

$$\nu\big(N_n((0,t] \times J) = 0\big) \to \mathbb{P}\big(N_{(\alpha)}((0,t] \times J) = 0\big) = e^{-s}.$$

Since $\nu(U_n)[nt] \to s$ as $n \to \infty$, the result follows as in (3.6). □

The conditional measure $\nu(\cdot|U)$ on U is defined for $B \in \mathcal{B}$ by

$$\nu(B|U) = \begin{cases} \dfrac{\nu(B \cap U)}{\nu(U)}, & \nu(U) > 0, \\ 0, & \nu(U) = 0. \end{cases}$$

For the next result we will need the following consequence of [16, Lemma 2.4.].

Lemma 3.2. *Let $U \in \mathcal{B}$ be such that $\nu(U) > 0$. Then for each $k \geq 0$*

$$|\nu(\tau_U > k) - (1 - \nu(U))^k| \leq \inf\{m\nu(U) + \nu(\tau_U \leq m|U) + \beta_m(U) : m \in \mathbb{N}\}, \tag{3.8}$$

where

$$\beta_m(U) = \sup_{B \in \mathcal{B}} |\nu(T^{-m}(B)|U) - \nu(B)|. \tag{3.9}$$

Let $\mathcal{Q}$ be a countable measurable partition of Y in the sense that $\nu(\bigcup_{A \in \mathcal{Q}} A) = 1$. We denote by $\mathcal{Q}_k = \bigvee_{j=0}^{k-1} T^{-j}\mathcal{Q}$ the family of all k-cylinders and by $\sigma(\mathcal{Q}_k)$ the σ-algebra generated by $\mathcal{Q}_k$. The partition $\mathcal{Q}$ is called *mixing with rate function ϑ* if $\vartheta(n) \to 0$ as $n \to \infty$, where

$$\vartheta(n) := \sup\{|\nu(A \cap T^{-(n+k)}(B)) - \nu(A)\nu(B)| \colon A \in \sigma(\mathcal{Q}_k),\ B \in \mathcal{B},\ k \geq 1\}.$$

Theorem 3.3. *Let h be regularly varying with index α and measurable with respect to $\sigma(\mathcal{Q})$. Suppose that the partition $\mathcal{Q}$ is mixing with rate function ϑ. If for every $\varepsilon > 0$ there exists a sequence of integers $k_n = k_n(\varepsilon)$ such that*

$$k_n = o(n), \quad n\vartheta(k_n) \to 0, \quad \text{as } n \to \infty, \tag{3.10}$$

and

$$\lim_{n \to \infty} \nu\big(\tau_{\{|h| > \varepsilon b_n\}} \leq k_n \big| |h| > \varepsilon b_n\big) = 0, \tag{3.11}$$

then $N_n \xrightarrow{d} N_{(\alpha)}$.

Proof. To prove that $N_n \xrightarrow{d} N_{(\alpha)}$ we make use of Theorem 3.1. Let $J \in \mathcal{J}$ and $0 \leq s < t$. Since $\{\tau_{h^{-1}(b_n J)} > [ns]\} \in \sigma(\mathcal{Q}_{[ns]})$, we obtain

$$\sup_{B \in \mathcal{B}} \big|\nu\big(\{\tau_{h^{-1}(b_n J)} > [ns]\} \cap T^{-[nt]}(B)\big) - \nu(\tau_{h^{-1}(b_n J)} > [ns])\nu(B)\big| \leq \vartheta([nt] - [ns]).$$

Hence, to check condition (3.2) it suffices to show that

$$\lim_{n \to \infty} \nu(\tau_{h^{-1}(b_n J)} > [ns]) = e^{-s\Pi_\alpha(J)}.$$

Since $\Pi_\alpha(J) < \infty$, there is $\varepsilon > 0$ such that $J \subset \{x \colon |x| > \varepsilon\}$. Take k_n as in (3.10) such that (3.11) holds. We have $n\nu(h^{-1}(b_n J)) \to \Pi_\alpha(J)$ and $n\nu(|h| > \varepsilon b_n) \to \Pi_\alpha(\{x \colon |x| > \varepsilon\})$. Since $h^{-1}(b_n J) \subset \{|h| > \varepsilon b_n\}$, we obtain

$$\nu\big(\tau_{h^{-1}(b_n J)} \leq k_n \big| h^{-1}(b_n J)\big) \leq \nu\big(\tau_{\{|h| > \varepsilon b_n\}} \leq k_n \big| |h| > \varepsilon b_n\big) \frac{\nu(|h| > \varepsilon b_n)}{\nu(h^{-1}(b_n J))},$$

which shows that the left-hand side in the last inequality goes to 0 as $n \to \infty$. We also have

$$\lim_{n\to\infty} k_n \nu(h^{-1}(b_n J)) + \vartheta(k_n)\nu(h^{-1}(b_n J))^{-1} = 0.$$

From Lemma 3.2 we conclude that

$$\lim_{n\to\infty} |\nu(\tau_{h^{-1}(b_n J)} > [ns]) - (1 - \nu(h^{-1}(b_n J))^{[ns]}| = 0,$$

which completes the proof. □

Remark 3.4. It is shown in [38, Theorem 4.2] that if h is regularly varying with index α and $N_n \xrightarrow{d} N_{(\alpha)}$ then (3.11) holds for all sequences k_n such that $k_n = o(n)$. Condition (3.11) as well as part (3) of Theorem 3.1 can be used to construct more examples where convergence to Lévy stable processes fails in $\mathbb{D}[0, \infty)$ with J_1-topology.

4. Examples

In this section we collect a number of examples where there is convergence to Lévy stable processes in $\mathbb{D}[0, \infty)$ with J_1-topology. In Corollaries 4.1 and 4.3 we make the simplifying assumption that h is locally constant on the dynamical partition. Then we can apply Theorem 3.3 to show that $N_n \xrightarrow{d} N_{(\alpha)}$ and the maximal inequality of [34] to show that part (2) of Theorem 1.2 holds. In Theorem 4.4 we show how the decay of correlations for weakly mixing AFU-maps can be combined with Theorem 3.3 to obtain a simpler sufficient condition for $N_n \xrightarrow{d} N_{(\alpha)}$. Here we assume that h is piecewise monotonic with finitely many branches. In the last subsection we show how Theorem 1.3 applies to Example 1.3.

4.1. Continued fraction mixing maps.

Let T be a measure preserving map on a probability space $(Y, \mathcal{B}, \nu)$ and let $\mathcal{Q} \subset \mathcal{B}$ be a countable partition. Recall (see [5] or [2]) that $(T, \mathcal{Q})$ is called *continued fraction mixing* if there exists a constant $C > 0$ such that

$$\nu(A \cap T^{-k}(B)) \leq C\nu(A)\nu(B), \quad A \in \mathcal{Q}_k,\ B \in \mathcal{B},\ k \geq 1, \tag{4.1}$$

and there is $n_1 \geq 1$ and a sequence $\{\epsilon_n\}_{n \geq n_1}$, $\epsilon_n \to 0$, such that

$$(1 - \epsilon_n)\nu(A)\nu(B) \leq \nu(A \cap T^{-(n+k)}(B)) \leq (1 + \epsilon_n)\nu(A)\nu(B) \tag{4.2}$$

for all $A \in \mathcal{Q}_k$, $B \in \mathcal{B}$, $n \geq n_1$, $k \geq 1$. If $\epsilon_n \to 0$ exponentially, i.e., there exist constants $C_1 > 0$ and $r \in (0, 1)$ such that $\epsilon_n \leq C_1 r^n$, $n \geq n_1$, then $(T, \mathcal{Q})$ is called *exponentially* continued fraction mixing.

Corollary 4.1. *Suppose that $(T, \mathcal{Q})$ is exponentially continued fraction mixing. If h is $\mathcal{Q}$ measurable and regularly varying with index $\alpha \in (0, 2)$, then $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0, \infty)$.*

Proof. First we apply Theorem 3.3 to show that $N_n \xrightarrow{d} N_{(\alpha)}$. From (4.2) it follows that $\phi_-(n) \leq \epsilon_n$, $n \geq n_1$, where

$$\phi_-(n) = \sup\Big\{\Big|\frac{\nu(A \cap T^{-(n+k)}(B))}{\nu(B)} - \nu(A)\Big| : A \in \sigma(\mathcal{Q}_k), B \in \mathcal{B}, \nu(B) > 0, k \geq 1\Big\}.$$

In particular, the partition $\mathcal{Q}$ is mixing with rate function $\vartheta(n) \leq \phi_-(n)$, $n \geq 1$. Since $\epsilon_n \to 0$ exponentially, we can find a sequence $k_n = o(n)$ such that (3.10) holds. To check (3.11) let $\varepsilon > 0$ and $U_n = \{|h| > \varepsilon b_n\}$, $n \geq 1$. We have $n\nu(U_n) \to \Pi_\alpha(\{x\colon |x| > \varepsilon\})$ and, by (4.1),

$$\nu\big(\tau_{U_n} \leq k_n \big| U_n\big) \leq \sum_{j=1}^{k_n} \nu(T^{-j}(U_n)\big|U_n) \leq Ck_n\nu(U_n) \to 0.$$

Consequently, $N_n \xrightarrow{d} N_{(\alpha)}$. To check condition (2) of Theorem 1.2 we recall the maximal correlation coefficients

$$\rho(n) = \sup\{|\mathrm{Corr}(f,g)| : f \in L^2(\mathcal{F}^k), g \in L^2(\mathcal{F}_{n+k}), k \geq 1\},$$

where $\mathcal{F}^k = \sigma(\{h \circ T^{j-1} \colon j \leq k\})$ and $\mathcal{F}_{n+k} = \sigma(\{h \circ T^{j-1} \colon j \geq n+k\})$. From [28] it follows that

$$\rho(n) \leq 2\sqrt{\phi_-(n)}, \quad n \geq 1.$$

Let $\varepsilon > 0$ and $h_{\varepsilon b_n} = hI(|h| \leq \varepsilon b_n) - \mathbb{E}_\nu(hI(|h| \leq \varepsilon b_n)$ for $n \geq 1$. The stationary sequence $\{h_{\varepsilon b_n} \circ T^j \colon j \geq 0\}$ is ρ-mixing. By [34, Theorem 1.1], there exists a constant K_1 such that

$$\mathbb{E}_\nu\big(\max_{1\leq k\leq n} |\frac{1}{b_n}\sum_{j=0}^{k-1} h_{\varepsilon b_n} \circ T^j|^2\big) \leq K_1 \frac{n}{b_n^2}\mathbb{E}_\nu(|h_{\varepsilon b_n}|^2)$$

for all $\varepsilon > 0$ and $n \geq 1$. From (2.5) it follows that

$$\lim_{n\to\infty} \frac{n}{b_n^2}\mathbb{E}_\nu(|h|^2 I(|h| \leq \varepsilon b_n)) = \frac{\alpha}{2-\alpha}\varepsilon^{2-\alpha},$$

which completes the proof. □

Example 4.1. (Gauss' continued fraction map) This is the map $T\colon [0,1) \to [0,1)$ given by $T(y) = 1/y \bmod 1$. Let ν be the Gauss measure with density $g_*(y) = 1/\ln 2(y+1)$. Then the partition $\mathcal{Q} = \{(1/(j+1), 1/j) : j \geq 1\}$ is exponentially continued fraction mixing. Consider the function $h(y) = a_1(y) := \big[1/y\big]$. It is regularly varying with index 1 and we have

$$b_n = n/\ln 2 \quad \text{and} \quad c_n = \sum_{1\leq j\leq b_n} j\ln\Big(1 + \frac{1}{j(j+2)}\Big).$$

By Corollary 4.1, we have $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ with $X_{(\alpha)}(1) \stackrel{d}{=} \Xi_{\alpha,1}$.

Examples of maps with exponentially continued fraction mixing partitions are provided by Gibbs-Markov maps [2, 3]. Let $(Y, \mathcal{B}, m, T)$ denote a nonsingular transformation of a standard probability space. It is called a *Markov map* if there is a measurable partition $\mathcal{Q}$ such that $TA \in \sigma(\mathcal{Q}) \bmod m$, which generates $\mathcal{B}$ under T in the sense that $\sigma(\{T^{-n}\mathcal{Q}\colon n \geq 1\}) = \mathcal{B}$ and which satisfies $T_{|A}$ is invertible and nonsingular for $A \in \mathcal{Q}$ (Markov maps are called Markov fibred systems in [5]). For $n \geq 1$, inverse branches of T denoted by $v_A\colon T^n(A) \to A$, $A \in \mathcal{Q}_n$, are nonsingular with respect to m and have Radon-Nikodym derivatives

$$v_A' := \frac{dm \circ v_A}{dm}.$$

Let $\theta \in (0,1)$. We define the metric d_θ on Y by $d(x,y) = \theta^{s(x,y)}$, where $s(x,y)$ is the greatest integer n such that x, y lie in the same n-cylinder.

A Markov map T is *Gibbs-Markov* if the following two additional conditions hold:

(1) *Big images property*: $\inf\{m(TA)\colon A\in\mathcal{Q}\}>0$.
(2) *Distortion*: there exists a constant $c>0$ such that

$$\Big|\frac{v_A'(x)}{v_A'(y)}-1\Big|\le cd(x,y),\quad x,y\in T^nA,\ A\in\mathcal{Q}_n,\ n\ge 1.$$

A topologically mixing Gibbs-Markov map has a probability invariant measure ν equivalent to m and $(T,\mathcal{Q})$ is exponentially continued fraction mixing. A particular class of Gibbs-Markov maps are Rényi maps as in [40].

Example 4.2. (First return time for intermittent maps) Let T_γ be as in Example 1.1. Let $Y=(1/2,1]$ and $\nu(\cdot)=\nu_\gamma(\cdot|Y)$, where ν_γ is the unique absolutely continuous invariant measure for T_γ. Consider the first return time function $\phi(y)=\min\{n\ge 1\colon T_\gamma^n(y)\in Y\}$, $y\in Y$, and the induced map $T=T_Y$ given by $T(y)=T_\gamma^{\phi(y)}(y)$, $y\in Y$. The map T is Gibbs-Markov for the partition $\mathcal{Q}=\{Y\cap\{\phi=j\}\colon j\ge 1\}$ and ν is invariant for T. Limit theorems for T_γ proved in [40] used the induced map T and functions of the form $h=a\phi+\psi$, where $a\neq 0$ is a constant and ψ is bounded $\mathcal{Q}$ measurable and such that $\int h\,d\nu=0$. The first return time function ϕ is regularly varying with index $\alpha=1/\gamma$, and so is h with $p=1$ in the case $a>0$ or with $p=0$, if $a<0$. Corollary 4.1 gives functional limit theorems for such h and Example 1.1 shows that the inducing technique in [25] or [40] (see also [12]) can not be used to prove functional limit theorems in $\mathbb{D}[0,\infty)$ with J_1-topology for the original map.

4.2. **Piecewise monotonic maps.** Let $I\subset\mathbb{R}$ be an interval. For every measurable $f\colon I\to\mathbb{R}$ define

$$\operatorname{var}_I(f)=\sup\sum_{i=1}^{n}|f(x_{i-1})-f(x_i)|,$$

where the supremum is taken over all finite ordered sequences, (x_j) with $x_j\in I$, and for $f\in L^1=L^1(I,\mathrm{Leb})$ set

$$\|f\|_{BV}=\|f\|_\infty+\bigvee_I f,\quad \text{where } \bigvee_I f=\inf\{\operatorname{var}_I(f^*)\colon f^*=f \text{ a.e.}\}.$$

Finally, let $BV=\{f\in L^1\colon \|f\|_{BV}<\infty\}$.

A *piecewise monotonic map of the interval* is a triple $(I,T,\mathcal{Q})$ where $\mathcal{Q}$ is a finite or countable generating partition (mod Leb) of I and $T\colon I\to I$ is a map such that $T_{|A}$ is continuous and strictly monotonic for each $A\in\mathcal{Q}$. The Perron-Frobenius operator $P\colon L^1\to L^1$ is of the form

$$Pf=\sum_{A\in\mathcal{Q}}v_A' 1_{TA} f\circ v_A,$$

where $v_A:TA\to A$ is given by $v_A=(T_{|A})^{-1}$ and $v_A'=d\mathrm{Leb}\circ v_A/d\mathrm{Leb}$.

We consider the following properties of a piecewise monotonic map $(I,T,\mathcal{Q})$:

(A) *Adler's condition*: for all $A\in\mathcal{Q}$, $T_{|A}$ extends to a C^2 map on $\overline{A}$ and $T''/(T')^2$ is bounded on I.
(F) *Finite images*: $\{TA : A\in\mathcal{Q}\}$ is finite.
(U) *Uniform expansion*: $\inf|T'|>1$.

Piecewise monotonic maps of the interval $(I,T,\mathcal{Q})$ with properties (A),(F),(U), will be called *AFU maps.* By [39, Corollary 1], every AFU map satisfies Rychlik's condition [31] for existence of absolutely continuous invariant probability measure (a.c.i.p.m.) and we have the following.

Proposition 4.2. *If $(I,T,\mathcal{Q})$ is a weakly mixing AFU map, then the unique a.c.i.p.m. ν has a density $g_* \in BV$ and there exist constants $C>0$ and $\theta \in (0,1)$ such that*

$$\|P^n f - \Big(\int_I f(x)dx\Big) g_*\|_{BV} \le C\theta^n \|f\|_{BV}, \quad f \in BV,\ n \ge 1.$$

If $(I,T,\mathcal{Q})$ is a weakly mixing AFU map, we define $Y = \{x \in I \colon g_*(x) > 0\}$ and $\mathcal{B} = \{B \cap Y \colon B \in \mathcal{B}(I)\}$. Note that g_* is bounded away from 0 and ∞ on Y.

Corollary 4.3. *Suppose that $(I,T,\mathcal{Q})$ is a weakly mixing AFU map. If h is $\mathcal{Q}$ measurable and regularly varying with index $\alpha \in (0,2)$, then $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$.*

Proof. We proceed similarly to the proof of Corollary 4.1. From [6, Theorem 1] it follows that there exists a constant $C_1 > 0$ such that $\phi_-(n) \le C_1\theta^n$. From (A) and (F) it follows that there exists a constant $C_2 > 0$ such that $\mathrm{Leb}(A \cap T^{-1}(B)) \le C_2 \mathrm{Leb}(A)\mathrm{Leb}(B)$ for all $A \in \mathcal{Q}$ and $B \in \mathcal{B}$. Since ν has a density bounded away from 0 and ∞, there exists a constant $C_3 > 0$ such that

$$\nu(A \cap T^{-1}(B)) \le C_3 \nu(A)\nu(B), \quad A \in \mathcal{Q},\ B \in \mathcal{B}.$$

The rest of the proof is the same as that of Corollary 4.1. □

Example 4.3. ("Japanese" continued fractions) For $a \in (0,1]$ define $T_a \colon [a-1,a) \to [a-1,a)$ by

$$T_a y = \Big|\frac{1}{y}\Big| - \Big[\Big|\frac{1}{y}\Big| + 1 - a\Big].$$

The map T_a is a weakly mixing AFU map. The countable partition $\mathcal{Q}$ is of the form $\mathcal{Q} = \{I_j^+\}_{j \ge j^+} \cup \{I_j^-\}_{j \ge j^-}$, where $j^+ = \big[\frac{1}{a} + 1 - a\big]$, $j^- = \max\{\big[\frac{1}{1-a} - a\big], 2\}$, and

$$I_j^+ = \Big(\frac{1}{j+a}, \frac{1}{j-1+a}\Big), \quad j > j^+, \quad I_{j^+}^+ = \Big(\frac{1}{j^+ + a}, a\Big),$$
$$I_j^- = \Big(-\frac{1}{j-1+a}, -\frac{1}{j+a}\Big), \quad j > j^-, \quad I_{j^-}^- = \Big(a-1, -\frac{1}{j^- + a}\Big).$$

It is shown in [27] that $(T,\mathcal{Q})$ is not continued fraction mixing for almost all $a \in (1/2,1)$. The map T_1 is the Gauss' map and $T_{1/2}$ is the nearest integer continued fraction map.

The unique a.c.i.p.m. $d\nu_a = dg_a(x)dx$ is known in some ranges of the parameter a. In particular, Nakada [26] computed the invariant densities g_a for $a \in [1/2,1]$:

For $(\sqrt{5}-1)/2 < a \le 1$ we have

$$g_a(y) = C_a\Big(1_{[a-1,\frac{1-a}{a}]}(y)\frac{1}{y+2} + 1_{(\frac{1-a}{a},a)}(y)\frac{1}{y+1}\Big),$$

where $C_a = 1/\ln(a+1)$, and for $1/2 \le a \le (\sqrt{5}-1)/2$

$$g_a(y) = C_a\Big(1_{[a-1,\frac{1-2a}{a}]}(y)\frac{1}{y+G+1} + 1_{(\frac{1-2a}{a},\frac{2a-1}{1-a})}(y)\frac{1}{y+2} + 1_{[\frac{2a-1}{1-a},a)}(y)\frac{1}{y+G}\Big),$$

where $C_a = 1/\ln G$ and $G = (\sqrt{5}+1)/2$.

We consider the function encoding the digits $h(y) := \text{sign}(y)\big[\big|\frac{1}{y}\big| + 1 - a\big]$, $y \in [a-1, a)$. If $a \in (1/2, 1)$ then h is regularly varying with $\alpha = 1$, $p = 1/2$, and

$$b_n = C_a n, \quad c_n = \frac{1}{C_a}\int h(y)I(|h(y)| \leq b_n)g_a(y)\,dy.$$

From Corollary 4.3 we obtain $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ with $X_{(\alpha)}(1) \overset{d}{=} \Xi_{1,0}$, which has a Cauchy distribution.

In the rest of this section we study the case when h is piecewise monotonic with a finite number of branches and $(I, T, \mathcal{Q})$ is a weakly mixing AFU map. The transfer operator $\mathcal{P}_T \colon L^1(\nu) \to L^1(\nu)$ is given by

$$g_*\mathcal{P}_T(f) = P(fg_*) \quad \text{for} \quad f \in L^1(\nu).$$

From Proposition 4.2 it follows that there exist constants $C_1 > 0$ and $\theta \in (0,1)$ such that if $fg_* \in BV$ then

$$\|\mathcal{P}_T^n(f) - \mathbb{E}_\nu(f)\|_{L^\infty(\nu)} \leq C_1\theta^n\|fg_*\|_{BV}, \quad n \geq 1. \tag{4.3}$$

For the next result we define the *return time* $\tau(U)$ *of a set* U into itself as

$$\tau(U) = \inf\{\tau_U(y) \colon y \in U\}.$$

We have $\tau(U) = \inf\{k \geq 1 \colon U \cap T^{-k}(U) \neq \emptyset\} = \inf\{k \geq 1 \colon U \cap T^k(U) \neq \emptyset\}$.

Theorem 4.4. *Let $(I, T, \mathcal{Q})$ be a weakly mixing AFU map. Suppose that h is regularly varying with index α and piecewise monotonic with a finite number of branches. If*

$$\lim_{n\to\infty} \tau(|h| > \varepsilon b_n) = \infty$$

for all $\varepsilon > 0$, then $N_n \xrightarrow{d} N_{(\alpha)}$.

Proof. We apply Theorem 3.1. Let $J \in \mathcal{J}$ and $0 \leq s < t$. It suffices to show that

$$\nu(\tau_{h^{-1}(b_nJ)} > [ns]) \to e^{-s\Pi_\alpha(J)} \tag{4.4}$$

and

$$\sup_{B\in\mathcal{B}}\big|\nu\big(\{\tau_{h^{-1}(b_nJ)} > [ns]\} \cap T^{-[nt]}(B)\big) - \nu(\tau_{h^{-1}(b_nJ)} > [ns])\nu(B)\big| \to 0. \tag{4.5}$$

Let $\varepsilon > 0$ be such that $J \subset \{x \colon |x| > \varepsilon\}$. Write

$$U_n = h^{-1}(b_nJ) \quad \text{and} \quad V_n = \{|h| > \varepsilon b_n\}, \quad n \geq 1.$$

Set $k_n = \max\{\tau(V_n), \log_\theta \nu(U_n)^2\}$, $n \geq 1$, where $\theta \in (0,1)$ is as in (4.3). Note that $k_n\nu(U_n) \to 0$ as $n \to \infty$. From Lemma 3.2 it follows that

$$\big|\nu(\tau_{U_n} > [ns]) - (1-\nu(U_n))^{[ns]}\big| \leq k_n\nu(U_n) + \nu\big(\tau_{U_n} \leq k_n\big|U_n\big) + \beta_{k_n}(U_n),$$

where, by (4.3),

$$\beta_{k_n}(U_n) \leq \frac{1}{\nu(U_n)}C_1\theta^{k_n}\|1_{U_n}g_*\|_{BV}.$$

Since h is piecewise monotonic with a finite number of branches, $\|1_{U_n}g_*\|_{BV}$ is uniformly bounded. Hence, $\beta_{k_n}(U_n) \to 0$. To prove (4.4) it remains to show that $\nu(\tau_{U_n} \leq k_n\big|U_n) \to 0$. We have

$$\nu\big(\tau_{U_n} \leq k_n\big|U_n\big) \leq \frac{\nu\big(V_n \cap \{\tau_{V_n} \leq k_n\}\big)}{\nu(U_n)} \leq \frac{1}{\nu(U_n)}\sum_{j=\tau(V_n)}^{k_n}\nu\big(V_n \cap T^{-j}(V_n)\big)$$

and

$$\nu\big(V_n \cap T^{-j}(V_n)\big) = \int_{V_n} \mathcal{P}_T^j 1_{V_n} d\nu = \int_{V_n} \mathcal{P}_T^j 1_{V_n} - \nu(V_n) d\nu + \nu(V_n)^2,$$

which, by (4.3), leads to

$$\nu\big(\tau_{U_n} \le k_n \big| U_n\big) \le \frac{\nu(V_n)}{\nu(U_n)} \big(\|1_{V_n} g_*\|_{BV} C_1 \sum_{j=\tau(V_n)}^{k_n} \theta^j + k_n \nu(V_n)\big)$$

and completes the proof of (4.4).

We now turn to the proof of (4.5). Observe that

$$\big|\nu\big(\{\tau_{U_n} > [ns]\} \cap T^{-[nt]}(B)\big) - \nu(\tau_{U_n} > [ns])\nu(B)\big| \le \|\mathcal{P}_T^{[nt]-[ns]}(f_n) - \mathbb{E}_\nu(f_n)\|_{L^\infty(\nu)},$$

where $f_n = \mathcal{P}_T^{[ns]}(1_{\{\tau_{U_n} > [ns]\}})$. By (4.3), it suffices to show that

$$\limsup_{n\to\infty} \|f_n g_*\|_{BV} = \limsup_{n\to\infty} \|P^{[ns]}(1_{\{\tau_{U_n} > [ns]\}} g_*)\|_{BV} < \infty.$$

We have $\{\tau_{U_n} > [ns]\} = \bigcap_{j=1}^{[ns]} T^{-j}(U_n^c)$ and we can write

$$1_{\{\tau_{U_n} > [ns]\}} = \prod_{j=0}^{[ns]-1} \omega \circ T^j, \quad \text{where } \omega = 1_{U_n^c} \circ T.$$

Since $\sup_n \sup_{A\in\mathcal{Q}} \operatorname{var}_A 1_{U_n^c} \circ T < \infty$, we can find $l \in \mathbb{N}$, $\theta_0 \in (0,1)$, and $C_0 > 0$, (see e.g. the proof of Proposition 4 of [4]) such that

$$\bigvee_I P^l(\omega_l f) \le \theta_0 \bigvee_I f + C_0 \|f\|_1, \quad f \in BV,$$

where $\omega_l = \prod_{j=0}^{l-1} \omega \circ T^j$. Iterating and making use of Proposition 4.2 completes the proof of (4.5). □

Example 4.4. Let $\alpha \in (0,1)$. Suppose that $(I, T, \mathcal{Q})$ is a weakly mixing AFU map and $y_0 \in I$ is a point with $g_*(y_0) \neq 0$. Assume that $h(y) = \phi(|y - y_0|)$ where $\phi\colon (0,\infty) \to (0,\infty)$ is such that $\phi(0) = \infty$, ϕ is non-increasing, and

$$\lim_{x\to\infty} \frac{\phi^{-1}(sx)}{\phi^{-1}(x)} = s^{-\alpha}$$

for all $s > 0$, where the generalized inverse ϕ^{-1} is defined by $\phi^{-1}(s) = \sup\{t \ge 0\colon \phi(t) \ge s\}$. Then h is regularly varying with index α and the sequence b_n is of the form $b_n = \phi(1/2g_*(y_0)n)$.

In particular, we have $\tau(|h| > \varepsilon b_n) \to \infty$, as $n \to \infty$, if y_0 is a point such that the return times of shrinking balls with center at y_0 diverges to ∞, i.e., $\tau(B(y_0, r)) \to \infty$ as $r \to 0$. Hence, by Theorem 4.4, we obtain $N_n \xrightarrow{d} N_{(\alpha)}$ and, by Theorem 1.2, $X_n \xrightarrow{d} X_{(\alpha)}$ with $X_{(\alpha)}(1) \stackrel{d}{=} \Xi_{\alpha,1}$.

Similarly, one can also consider functions which are piecewise monotonic and have left and right-hand limits equal to $+\infty$ or $-\infty$ at one or more points. For the case of $\alpha \in [1,2)$ we need to check condition (2) of Theorem 1.2. By Corollary 2.4, this condition holds when the function h is such that $\mathcal{P}_T h_x = 0$ for all $x > 0$, where $h_x = hI(|h| \le x) - \mathbb{E}_\nu(hI(|h| \le x))$. We illustrate this with Example 1.2. Consider the tent map $T(y) = 1 - 2|y|$, $y \in [-1,1]$, where ν is the normalized Lebesgue measure on $[-1,1]$. We have

$$\mathcal{P}_T f(y) = \frac{1}{2} f\Big(\frac{y-1}{2}\Big) + \frac{1}{2} f\Big(\frac{1-y}{2}\Big).$$

Hence, $\mathcal{P}_T f = 0$ for all f which are odd functions on $[-1,1]$. Let $h(y) = y^{-1/\alpha}$ for $y > 0$ and $h(-y) = -h(y)$. Then $\mathcal{P}_T h_x = 0$ for all $x > 0$. We have $b_n = n^{1/\alpha}$, $c_n = 0$, and $\{|h| > \varepsilon b_n\} = B(0, (\varepsilon^\alpha n)^{-1})$ for all $\varepsilon > 0$. By Theorem 4.4, $N_n \xrightarrow{d} N_{(\alpha)}$ and we conclude that $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ with $X_{(\alpha)}(1) \stackrel{d}{=} \Xi_{\alpha,0}$.

4.3. **Cauchy limiting distribution for the doubling map.** In this section we will show that Theorem 1.3 applies to the doubling map T and the function h from Example 1.3. From Theorem 4.4 it follows that $N_n \xrightarrow{d} N_{(\alpha)}$, since $\tau(B(y_0,r)) \to \infty$ as $r \to 0$. By Theorem 1.3 and Corollary 2.5, it remains to check that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{b_n^2} \sum_{k=1}^{n-1} (n-k) \int_0^1 P^k h_{\varepsilon b_n}(y) h_{\varepsilon b_n}(y) dy = 0, \tag{4.6}$$

where $h_{\varepsilon b_n} = hI(|h| \le \varepsilon b_n) - \mathbb{E}_\nu(hI(|h| \le \varepsilon b_n))$ and $P = \mathcal{P}_T$ is the Perron-Frobenius operator given by

$$Pf(y) = \frac{1}{2} f\Big(\frac{y}{2}\Big) + \frac{1}{2} f\Big(\frac{y+1}{2}\Big), \quad f \in L^1.$$

For $y_0 \in \mathbb{R}$ and $a > 0$ define

$$f_{y_0}(y) = \frac{1}{y - y_0} \quad \text{and} \quad f_{y_0,a}(y) = f_{y_0}(y) I(|y - y_0| \ge a), \quad y \in [0,1].$$

Let $a < \min\{y_0, 1 - y_0\}$ and $y_k^{\pm} = 2^k y_0 - [2^k(y_0 \pm a)]$. We first show that

$$\int_0^1 P^k f_{y_0,a}(y) f_{y_0,a}(y) dy \le \frac{\sqrt{2}}{a\sqrt{2^k}} + \frac{2}{\sqrt{a}\sqrt{|y_k^- - y_0|}} + \frac{2}{\sqrt{a}\sqrt{|y_k^+ - y_0|}} \tag{4.7}$$

for all $k \ge 1$ with $y_k^{\pm} \ne y_0$. In the case of $y_k^- = y_0$ or $y_k^+ = y_0$, the corresponding fraction in (4.7) should be replaced by $1/(2^k a)$. We have $Pf_{y_0,a} = f_{2y_0,2a} + f_{2y_0-1,2a}$. Hence,

$$P^k f_{y_0,a} = \sum_{j=0}^{2^k-1} f_{2^k y_0 - j, 2^k a}.$$

If j is such that either $2^k y_0 - j + 2^k a \le 0$ or $2^k y_0 - j - 2^k a \ge 1$, then $f_{2^k y_0 - j, 2^k a} = f_{2^k y_0 - j}$ and

$$\int_0^1 f_{2^k y_0 - j}(y) f_{y_0,a}(y) dy \le \frac{\sqrt{2a}}{|2^k y - j - y_0| \sqrt{|2^k y - j - y_0 - a|}} \le \frac{\sqrt{2}}{2^k a \sqrt{2^k}},$$

which shows that the sum over all such j is less than $\sqrt{2}/(a\sqrt{2^k})$ and gives the first term in the right-hand side of (4.7). Now, if j is such that $2^k y_0 - j + 2^k a \ge 1$ and $2^k y_0 - j - 2^k a \le 0$, then $f_{2^k y_0 - j, 2^k a} = 0$. What is left are whose j, if any, such that $2^k(y_0 - a) - 1 < j < 2^k(y_0 - a)$ or $2^k(y_0 + a) - 1 < j < 2^k(y_0 + a)$ and the corresponding integrals are bounded by the remaining terms in (4.7). From (4.7) it follows that there is a constant $C > 0$ such that

$$\sum_{k=1}^{n-1} \int_0^1 P^k f_{y_0,a}(y) f_{y_0,a}(y) dy \le \frac{C}{\sqrt{a}} \left(\frac{1}{\sqrt{a}} + \log_2 \frac{1}{a} + 1 \right), \quad n \ge 2.$$

To see this observe that for all k such that $2^k a \ge 2$ we have $|y_k^\pm - y_0| \ge 2^{k-1} a$ and for k satisfying $2^k a < 2$ we have $|y_k^- - y_0| \ge 1 - y_0$ or $y_k^- = 2^k y_0 - [2^k y_0] < 1$ and $|y_k^+ - y_0| \ge y_0$ or $y_k^+ = 2^k y_0 - [2^k y_0] > 0$. The number of k such that $y_k^\pm = 2^k y_0 - [2^k y_0] \in (0, 1)$ is finite and the corresponding sum of $1/\sqrt{|y_k^\pm - y_0|}$ or $1/2^k$ does not depend on a. Since

$$\int_0^1 P^k h_{\varepsilon b_n}(y) h_{\varepsilon b_n}(y) dy \le \int_0^1 P^k f_{y_0,(\varepsilon b_n)^{-1}}(y) f_{y_0,(\varepsilon b_n)^{-1}}(y) dy,$$

we conclude that for all sufficiently large n

$$\sum_{k=1}^{n-1} (n-k) \int_0^1 P^k h_{\varepsilon b_n}(y) h_{\varepsilon b_n}(y) dy \le C n \sqrt{\varepsilon b_n} \left(\sqrt{\varepsilon b_n} + \log_2(\varepsilon b_n) + 1 \right),$$

which implies (4.6).

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